NON-DIFFERENTIABLE SOLUTIONS FOR NON-LINEAR LOCAL FRACTIONAL HEAT CONDUCTION EQUATION

by

Sheng ZHANG [*] ***and Xiaowei ZHENG***

School of Mathematical Sciences, Bohai University, Jinzhou, China

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Fractional calculus has many advantages. Under consideration of this paper is a (2+1)-dimensional non-linear local fractional heat conduction equation with arbitrary degree non-linearity. Backlund transformation of a reduced form of the local fractional heat conduction equation is constructed by Painleve analysis. Based on the Backlund transformation, some exact non-differentiable solutions of the local fractional heat conduction equation are obtained. To gain more insights of the obtained solutions, two solutions are constrained to a Cantor set and then two spatio-temporal fractal structures with profiles of these two solutions are shown. This paper further reveals by local fractional heat conduction equation that fractional calculus plays important role in dealing with non-differentiable problems.

Key words: *non-differentiable solution, fractional calculus, Painleve analysis, local fractional heat conduction equation, Backlund transformation*

Introduction

Fractional calculus, Oldham *et al.* [1], originated in 1695, plays important role in the study of heat transfer characteristics of textiles [2-4]. Both fractional derivatives and fractional integrals have been applied to many fields like physics, chemistry and engineering. This is due to fractional calculus brings advantages over its partner – the integer-order fractional calculus, especially used to model the mechanical and electrical properties possessed by real materials, Fan and Shang [4]. With the development of fractional calculus, fractal derivatives [3-7], a kind of fractional derivatives defined in fractal set, have received a wide range of applications [3-8], such as those used in porous structures [3, 4] and electrochemical arsenic sensor, Li *et al.* [8]. Solving and constructing fractional models has become theoretical/practical research topics [9-17], it is because that fractional solutions can provide helpful understanding the essence behind the phenomenon. Yang *et al.* [18] investigated a local fractional Burgers equation envisaging a non-linear local fractional transport equation with linear non-differentiable diffusion term. It is interesting that Hristov [19] developed the derivation of a transient heat diffusion equation with relaxation term expressed by the Caputo-Fabrizio time-fractional derivative. The aim of this article is to extend a (2+1)-D generalized non-linear heat conduction equation [20] to the following local fractional form:

$$\mathrm{D}_t^{\alpha}u - a\mathrm{D}_x^{2\alpha}(u^n) - a\mathrm{D}_y^{2\alpha}(u^n) - u - u^n = 0, \quad (a>0, \quad n>1, \quad 0<\alpha\le 1) \tag{1}$$

* Corresponding author, e-mail: szhangchina@126.com

and construct its exact non-differentiable solutions. Here u is assumed a non-differentiable function in the sense of classical integer-order derivative, a is constant, while D_t^{α}, $D_x^{2\alpha}$, and $D_y^{2\alpha}$ are the local fractional partial derivative operators, Wen and Zhou [20], with respect to *x*, *y*, and *t*, among them:

$$D_t^{\alpha}u(x,y,t) = \Gamma(1+\alpha)\lim_{\varepsilon\to 0}\frac{u(x,y,t+\varepsilon)-u(x,y,t)}{\varepsilon^{\alpha}} \tag{2}$$

As far as we know, the local fractional heat conduction, eq. (1), has not been studied. Some interesting results on the heat conduction equations are given in [11-15].

Painleve analysis and Backlund transformation

Firstly, we take the dependent variable transformation [21]:

$$u = v^{\frac{1}{1-n}}, \quad v = v(\xi) \tag{3}$$

and the fractional complex transformation [22]:

$$\xi = \frac{kx^{\alpha}}{\Gamma(1+\alpha)} + \frac{ly^{\alpha}}{\Gamma(1+\alpha)} - \frac{ct^{\alpha}}{\Gamma(1+\alpha)} \tag{4}$$

where *k*, *l*, and *c* are constants, then the local fractional heat conduction, eq. (1), becomes:

$$(n-1)^2v^3 + an(2n-1)(k^2+l^2)v'^2 - (n-1)^2v'v^2 - c(n-1)v' - an(n-1)(k^2+l^2)vv'' = 0 \tag{5}$$

Secondly, following the idea of Painleve analysis [23] we suppose:

$$v = \sum_{j=0}^{\infty} v_j\phi^{j-\rho}(\xi), \quad (\rho > 0, v_0 \neq 0) \tag{6}$$

where $v_j\,(j = 0,1,2,\cdots)$ are undetermined functions of ξ, and employ eq. (6) to consider the leading order analysis of the following three terms of eq. (5):

$$an(2n-1)(k^2+l^2)v'^2 = an\rho^2(2n-1)(k^2+l^2)v_0^2\phi'^2\phi^{-2\rho-2} + \cdots \tag{7}$$

$$-(n-1)^2v'v^2 = c\rho(n-1)v_0^3\phi_0'\phi^{-3\rho-1} + \cdots \tag{8}$$

$$-an(n-1)(k^2+l^2)vv'' = -an\rho(n-1)(\rho+1)(k^2+l^2)v_0^2\phi'^2\phi^{-2\rho-2} + \cdots \tag{9}$$

namely

$$c\rho(n-1)v_0^3\phi'\phi^{-3\rho-1} + an\rho(k^2+l^2)[1+n(\rho-1)]v_0^2\phi'^2\phi^{-2\rho-2} = 0 \tag{10}$$

which gives:

$$\rho = 1, \quad v_0 = \frac{an(k^2+l^2)}{c(1-n)}\phi' \tag{11}$$

Thus, eq. (6) can be expanded:

$$v = v_0\phi^{-1} + v_1 + v_2\phi + v_3\phi^2 + \cdots \tag{12}$$

In order to construct a Backlund transformation of eq. (5), we thirdly further set $v_j = 0$ for any $j \geq 2$, then eq. (6) is truncated as:

$$v = v_0\phi^{-1} + v_1 \tag{13}$$

Substituting eq. (13) along with eq. (11) into eq. (5) and then comparing every coefficient of the same powers of $\phi^s (s = -4, -3, -2, -1, 0, 1, 2, 3, \cdots)$, we finally reduce eq. (5):

$$(n-1)[a(k^2+l^2) - 2c^2 v_1]\phi' + acn(k^2+l^2)\phi'' = 0 \tag{14}$$

$$-(n-1)\phi'^2\{3an(n-1)(k^2+l^2)v_1 - c^2(n-1)v_1^2 + an(k^2+l^2)[1-n+c(4n-3)v_1']\} -$$
$$-a^2n^2(k^2+l^2)^2(2n-1)\phi''^2 + an(n-1)(k^2+l^2)\phi'[c(3n-1)v_1\phi'' + an(k^2+l^2)\phi^{(3)}] = 0 \tag{15}$$

$$-(n-1)v_1^2[3(n-1)\phi' - c\phi''] - an(k^2+l^2)[2(2n-1)v'_1\phi'' - (n-1)v_1''\phi'] +$$
$$+(n-1)v_1[2\phi'(n-1+cv_1') + an(k^2+l^2)\phi^{(3)}] = 0 \tag{16}$$

$$(n-1)^2 v_1^3 + an(2n-1)(k^2+l^2)v_1'^2 - (n-1)^2 v_1' v_1^2 - c(n-1)v_1' - an(n-1)(k^2+l^2)v_1 v_1'' = 0 \tag{17}$$

Obviously, when v_1 solves eq. (5), eqs. (13)-(16) constitute a Backlund transformation of eq. (5).

Non-differentiable solutions

Selecting a simple seed $v_1 = 0$ and then solving eqs. (14)-(16) yields:

$$v_0 = -\mathrm{e}^{\xi}, \quad a = \frac{(n-1)^2}{n^2(k^2+l^2)}, \quad \phi = \mathrm{e}^{\xi} + \beta \tag{18}$$

$$v_1 = 1, \quad c = \frac{n-1}{n} \quad \text{or} \quad v_1 = 0, \quad c = \frac{1-n}{n} \tag{19}$$

where β is arbitrary constant. We therefore obtain two exact non-differentiable solutions of the local fractional heat conduction eq. (1):

$$u = \left(\frac{\mathrm{e}^{\xi}}{\mathrm{e}^{\xi} + \beta}\right)^{\frac{1}{1-n}}, \quad \xi = \frac{kx^{\alpha}}{\Gamma(1+\alpha)} + \frac{ly^{\alpha}}{\Gamma(1+\alpha)} - \frac{(n-1)t^{\alpha}}{n\Gamma(1+\alpha)} \tag{20}$$

$$u = \left(\frac{\beta}{\mathrm{e}^{\xi} + \beta}\right)^{\frac{1}{1-n}}, \quad \xi = \frac{kx^{\alpha}}{\Gamma(1+\alpha)} + \frac{ly^{\alpha}}{\Gamma(1+\alpha)} - \frac{(1-n)t^{\alpha}}{n\Gamma(1+\alpha)} \tag{21}$$

When $\beta = \pm 1$, eqs. (20) and (21) can be written as the hyperbolic functional solutions:

$$u = (1 \pm \tanh\eta)^{\frac{1}{1-n}}, \quad \eta = \frac{\xi}{2} = \frac{kx^{\alpha}}{2\Gamma(1+\alpha)} + \frac{ly^{\alpha}}{2\Gamma(1+\alpha)} - \frac{(n-1)t^{\alpha}}{2n\Gamma(1+\alpha)} \tag{22}$$

and

$$u=(1\pm\coth\eta)^{\frac{1}{1-n}},\quad \eta=\frac{\xi}{2}=\frac{kx^{\alpha}}{2\Gamma(1+\alpha)}+\frac{ly^{\alpha}}{2\Gamma(1+\alpha)}-\frac{(n-1)t^{\alpha}}{2n\Gamma(1+\alpha)}\tag{23}$$

respectively.

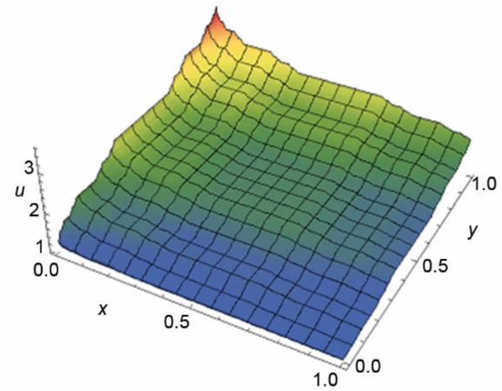

Figure 1. Spatio-temporal fractal structure of non-differentiable solution (20)

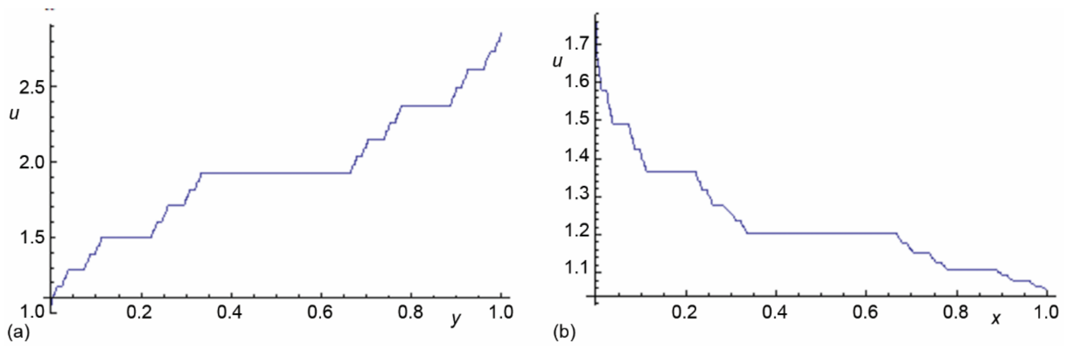

Figure 2. Profiles of non-differentiable solution (20); (a) *x* = 0.1, *t* = 0.5 and (b) *y* = 0.1, *t* = 0.5

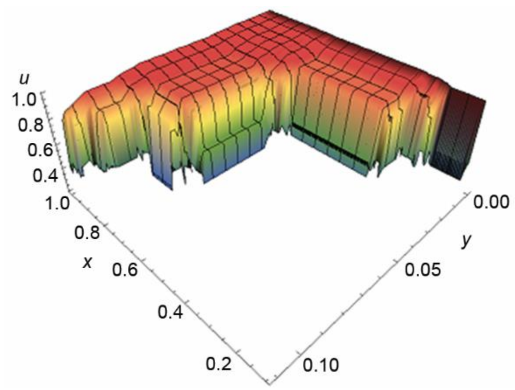

Figure 3. Spatio-temporal fractal structure of non-differentiable solution (21)

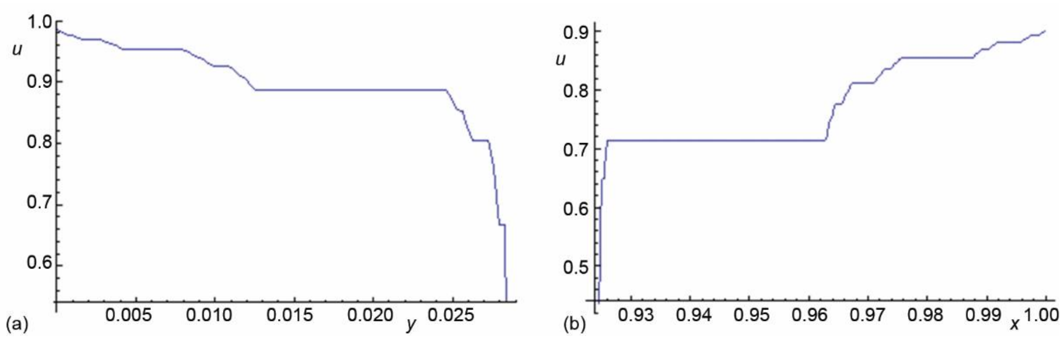

Figure 4. Profiles of non-differentiable solution (21); (a) x = 0.5, t = 0.5 and (b) y = 0.1, t = 0.5

In figs. 1 and 2, the solution (20) constrained into a Cantor set is shown by setting $k=5,\ l=-9,\ c=2,\ \alpha=\ln 2/\ln 3,\ \beta=1,$ and $n=10.$ Selecting the same parameters in figs. 1 and 2 except for $\beta=-1,$ we show in figs. 3 and 4 the solution (21) constrained into a Cantor set.

Conclusion

We have derived a Backlund transformation expressed by eqs. (13)-(17) of eq. (5) reduced from the local fractional heat conduction eq. (1) by Painleve analysis. Then non-differential solutions (17) and (18) are obtained. Constrained into a Cantor set with fractal dimension $\alpha=\ln 2/\ln 3,$ the solutions (20) and (21) are shown by figures. For $\alpha=1,$ if we let u be a differentiable function, then all the obtained solutions (20)-(23) are equivalent to those in [20].

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