STOCHASTIC MODEL FOR MULTI-TERM TIME-FRACTIONAL DIFFUSION EQUATIONS WITH NOISE

by

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This paper studies a spectral collocation approach for evaluating the numerical solution of the stochastic multi-term time-fractional diffusion equations associated with noisy data driven by Brownian motion. This model describes the symmetry breaking in molecular vibrations. The numerical solution of the stochastic multi-term time-fractional diffusion equations is proposed by means of collocation points method based on sixth-kind Chebyshev polynomial approach. For this purpose, the problem under consideration is reduced to a system of linear algebraic equations. Two examples highlight the robustness and accuracy of the proposed numerical approach.

Key words: *stochastic time-fractional heat equation, multi-term time-fractional*

Introduction

More recently, fractional differential equations have appropriately described many complicated phenomena and dynamic processes which can not be explained by classical differential equations [1-4]. Some recent studies reporting multi-fractional equations can better coincide with experimental results and anomalous processes [3-5]. Moreover, stochastic differential equations have attracted the attention of the research community for years due to their uncertainty model being closed with the real world [5-7]. The aim of this paper is to bring together two new areas in fractional, namely, multi-terms fractional integrated with stochastic differential equation which can successfully describe many phenomena in the real world [8-11]. The current work develops a numerical approximation of stochastic multi-term time-fractional diffusion equations (SM-TT-FDE) rising in heat transfer:

$$\mathbb{P}(\mathcal{D}_t) = [\mu + \vartheta \dot{B}(t)]u_{xx}(x,t) + \lambda u_x(x,t) + f(x,t) \tag{1}$$

where $(x,t) \in \Omega \times T$, with the boundary and initial conditions:

$$\begin{gathered} u(x,t) = \varphi(x,t), \quad 0 < t \le T, \quad x \in \partial\Omega, \\ u(x,0) = \eta(x), \quad x \in \Omega \end{gathered} \tag{2}$$

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where $\mathbb{P}(\mathcal{D}_t) = \Sigma_{k=1}^{m} a_k \mathcal{D}_t^{\alpha_k}$, $\mathcal{D}_t^{\alpha_k}$ denotes a fractional differential operator, where α_k denotes $(1 \geq \alpha_m \geq \cdots \geq \alpha_k \geq \cdots \geq \alpha_1 > 0)$ fractional orders. Here λ, μ, and ϑ indicate real constants. We note that coefficients a_k are positive real constants and $\partial\Omega$ denotes the boundary of Ω. Let $\dot{B}(t) := \mathrm{d}B(t)/\mathrm{d}t$ be the time white noise with the function where $B(t)_{t\in[0,T]}$ is the Brownian motion conformed by a filtration $F_B = \{F_t\}_{t\in[0,T]}$ in a probability space (Ω_B, F_B, P_B). Furthermore, the source term $f(x, t)$, $\varphi(x, t)$, and $\eta(x)$ are some stochastic processes defined on (Ω_B, F_B, P_B) and $u(x, t)$ is a known function to be found.

The sixth-kind Chebyshev polynomial collocation

In this section, we address a sixth-kind Chebyshev polynomial (SKCP) approach to archive the numerical solution for the under consideration problem shown in eq. (1) [5, 7]. The $U(x, t)$ can be considered as approximating the solution of eq. (1) by means of SKCP:

$$u(x,t) \simeq U_{N,M}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} \delta_{i,j} \mathcal{T}_i(x) \overline{\mathcal{T}}_j(t) = \mathbf{J}(x)^T \mathbf{C} \overline{\mathbf{J}}(t) \tag{3}$$

where

$$\mathbf{J}(x) = [\mathcal{T}_0(x), ..., \mathcal{T}_i(x), ..., \mathcal{T}_N(x)]^T, \quad \overline{\mathbf{J}}(t) = [\overline{\mathcal{T}}_0(t), ..., \overline{\mathcal{T}}_i(t), ..., \overline{\mathcal{T}}_M(t)]^T \tag{4}$$

where $\mathcal{T}_i(x) = \hat{\mathcal{T}}_i[(2/l)x - 1]$ on the interval [0, I] and $\overline{\mathcal{T}}_j(t) = \hat{\mathcal{T}}_j[(2/l)t - 1]$ on a time interval $[0, T]$ [7]. Here, $\mathbf{C}$ is the matrix with all entries unknown as in the following form:

$$\mathbf{C} = \begin{pmatrix} \delta_{0,0} & \cdots & \delta_{0,M} \\ \vdots & \ddots & \vdots \\ \delta_{N,0} & \cdots & \delta_{N,M} \\ & & \end{pmatrix}_{(N+1)\times(M+1)}$$

Theorem 1 [7] Let $\overline{\mathbf{J}}(t)$ is the shifted SKCP vector as (4), see [7] then:

$$\mathcal{D}_t^{\alpha_k} \overline{\mathbf{J}}(t) = \Phi^{\alpha_k}(t)$$

The Caputo fractional derivative of the vector α_k of $\Phi^{\alpha_k}(t)$ is $\overline{\mathbf{J}}(t)$ and is defined:

$$\Phi^{\alpha_k}(t) = \left[0, \sum_{r=1}^{1} \psi_{r,1}^{\alpha_k}(t), \ldots, \sum_{r=1}^{j} \psi_{r,j}^{\alpha_k}(t), \ldots, \sum_{r=1}^{M} \psi_{r,M}^{\alpha_k}(t) \right]^T$$

where

$$\psi_{r,j}^{\alpha_k}(t) = \frac{\Gamma(r+1)}{T^r \Gamma(r+1-\alpha_k)} \bar{\theta}_{r,j} t^{r-\alpha_k}$$

According to eqs. (1) and (3) and imposed *Theorem 1*, we have:

$$\sum_{k=1}^{m} a_k \mathbf{J}^T(x) \mathbf{C} \Phi^{\alpha_k}(t) = [\mu + \vartheta \dot{B}(t)] \mathbf{J}_{xx}^T(x) \mathbf{C} \overline{\mathbf{J}}(t) + \lambda \mathbf{J}_x^T(x) \mathbf{C} \overline{\mathbf{J}}(t) + f(x,t) \tag{5}$$

where

$$\mathbf{J}_x(x) = [\mathcal{T}_0'(x),...,\mathcal{T}_i'(x),...,\mathcal{T}_N'(x)]^T$$

$$\mathbf{J}_{xx}(x) = [\mathcal{T}_0''(x),...,\mathcal{T}_i''(x),...,\mathcal{T}_N''(x)]^T$$

From the conditions provided in eqs. (2) and (3), we have:

$$\mathbf{J}^T(0)\mathbf{C}\bar{\mathrm{J}}(t) = \varphi(0,t), \quad \mathbf{J}^T(x)\mathbf{C}\bar{\mathrm{J}}(0) = \eta(x), \quad \mathbf{J}^T(x)\mathbf{C}\bar{\mathrm{J}}(0) = \eta(x) \tag{6}$$

Assuming $x_0 = 0$, $x_N = l$, and also considering $x_1, \ldots, x_{N-1}$ as roots of $\mathcal{T}_{N-1}(x)$ and $t_j, j = 1,...,M$, as the roots of $\bar{\mathcal{T}}_M(t)$. By assuming those, Λ, Λ_x, and Λ_{xx}, can be defined:

$$\Lambda = [\mathbf{J}(x_1),...,\mathbf{J}(x_i),...,\mathbf{J}(x_{N-1})]^T \tag{7}$$

$$\Lambda_x = [\mathbf{J}_x(x_1),...,\mathbf{J}_x(x_i),...,\mathbf{J}_x(x_{N-1})]^T \tag{8}$$

$$\Lambda_{xx} = [\mathbf{J}_{xx}(x_1),...,\mathbf{J}_{xx}(x_i),...,\mathbf{J}_{xx}(x_{N-1})]^T$$

where the matrices Λ, Λ_x, and Λ_{xx} are of the order $(N-1)\times(N+1)$ and:

$$\begin{aligned} \Psi &= [\bar{\mathbf{J}}(t_1),...,\bar{\mathbf{J}}(t_j),...,\bar{\mathbf{J}}(t_M)]_{(M+1)\times M} \\ \Psi^{\alpha_k} &= [\Phi^{\alpha_k}(t_1),...,\Phi^{\alpha_k}(t_j),...,\Phi^{\alpha_k}(t_M)]_{(M+1)\times M} \end{aligned} \tag{9}$$

By evaluating eq. (5) at $(N-1)\times M$ collocation points (x_i, t_i) for $i = 1,\ldots,N-1$ and $j = 1,\ldots,M$, we have:

$$\sum_{k=1}^{m} a_k \Lambda \mathrm{C} \Psi^{\alpha_k} = \Lambda_{xx}\mathrm{C}\Psi\mathcal{B} + \lambda\Lambda_x\mathrm{C}\Psi + \mathcal{F} \tag{10}$$

in which:

$$\mathcal{B} = diag(\mu + \vartheta b_1, ...\mu + \vartheta b_{j1}, ..., \mu + \vartheta b_M)$$

where $b_j = B(t_j) - B(t_{j-1}), t_0 = 0$ and:

$$\mathcal{F} = [f_{i,j}], \quad i = 1,...,N-1, \quad j = 1,...,M$$

To approximate initial and boundary conditions, we impose collocation points t_i in eq. (6) for each collocation points x_i:

$$\mathbf{J}^T(l)\mathrm{C}\Psi = \mathbf{Y}_0, \quad \mathbf{J}^T(l)\mathrm{C}\Psi = \mathbf{Y}_1, \quad \bar{\Lambda}\mathrm{C}\bar{\mathbf{J}}^T(0) = \bar{\mathbf{Y}} \tag{11}$$

where

$$\mathbf{Y}_0 = [\varphi(0,t_1),...,\varphi(0,t_j),...,\varphi(0,t_M)]^T, \quad \mathbf{Y}_l = [\varphi(l,t_1),...,\varphi(l,t_j),...,\varphi(l,t_M)]^T$$
$$\bar{\Lambda} = [\mathbf{J}(x_0),...,\mathbf{J}(x_i),...,\mathbf{J}(x_N)]^T, \quad \bar{\mathbf{Y}} = [\eta(x_0),...,\eta(x_i),...,\eta(x_N)]^T$$

Using the Kronecker product, eq. (10) transforms to:

$$\mathcal{A}\mathcal{X} = \mathcal{T}_{vec} \tag{12}$$

where

$$\mathcal{A}=\sum_{k=1}^{m}a_k(\Psi^{\alpha_k})^T\otimes\Lambda-(\Psi\mathcal{B})^T\otimes\Lambda_{xx}-\lambda\Psi^T\otimes\Lambda_x$$

and $\mathcal{X}=vec(\mathcal{C}),\mathcal{T}_{vec}=vec(\mathcal{F})$. Further, eq. (11) is equivalent to:

$$\overline{E}\mathcal{X}=\overline{\mathbf{Y}},\qquad E_0\mathcal{X}=\mathbf{Y}_0,\qquad E_l\mathcal{X}=\mathbf{Y}_l \tag{13}$$

where

$$\overline{E}=\overline{\mathbf{J}}(0)^T\otimes\overline{\Lambda},\qquad E_0=\Psi^T\otimes\mathbf{J}(0)^T,\qquad E_l=\Psi^T\otimes\mathbf{J}(l)^T$$

Thus, from eqs. (12) and (13), we gain a system of linear equations $\mathbf{A}\mathcal{X}=\mathbf{B}$ in which:

$$\mathbf{A}=[\mathcal{A}^T,\overline{E}^T,E_0^T,E_l^T]^T,\qquad \mathbf{B}=[\mathcal{T}_{vec}^T,\overline{Y}^T,Y_0^T,Y_l^T]^T$$

Solving this system yields an estimate $U_{N,M}(x,t)$ for the solution of eqs. (1) and (2), which has the form eq (3).

Numerical results

Numerical examples of SM-TT-FDE to demonstrate the reliability, efficiency and accuracy of the proposed method are reported in this section. To evaluate the accuracy of the proposed method, we use L_∞- norm for $\overline{P}=80$ and $\overline{P}=100$ separated Brownian paths. Also, the convergence order for various collocation points by defining:

$$CO=\log_{\frac{N_1}{N_2}}\frac{\|E_{N_1}\|_\infty}{\|E_{N_2}\|_\infty}$$

is reported. Numerical calculations are performed in MATLAB software with a desktop computer, Intel(R) i7-10700, 32GB RAM.

Example 1 Let us consider the SM-TT-FDE in the case of three fractional terms with the analytic solution given by $u(x,t)=(\alpha_1+\alpha_2+\alpha_3)x^5t^2e^{x^2}$.

$$\mathcal{D}_t^{\alpha_1}u(x,t)+\mathcal{D}_t^{\alpha_2}u(x,t)+\mathcal{D}_t^{\alpha_3}u(x,t)=\dot{B}(t)u_{xx}(x,t)+\lambda u_x(x,t)+f(x,t),\quad 0<t\le T$$
$$u(x,0)=u(0,t)=0,\;\; u(1,t)=(\alpha_1+\alpha_2+\alpha_3)et^2$$

and $\lambda = 0.01$ and:

$$f(x,t)=e^{x^2}(\alpha_1+\alpha_2+\alpha_3)\left\{\frac{\Gamma(3)}{\Gamma(3-\alpha_1)}t^{2-\alpha_1}+\frac{\Gamma(3)}{\Gamma(3-\alpha_2)}t^{2-\alpha_2}+\frac{\Gamma(3)}{\Gamma(3-\alpha_3)}t^{2-\alpha_2}-\right.$$
$$\left.-0.01x^4t^2(5+2x^2)-[2\dot{B}(t)][2x^3t^2(10+11x^2+2x^4)]\right\}$$

In tab. 1 we report the error norms L_∞ and convergence order (CO) for α_1 = 0.75, α_2 = 0.1, and α_3 = 0.05, and several values of N. Figures 1(a) and 1(b) show the profile of the exact solution and numerical solution for M = N = 12, respectively, considering α_1 = 0.75, α_2 = 0.1, and α_3 = 0.05.

Table 1. Absolute error at $T = 1$ with $\alpha_1 = 0.75$, $\alpha_2 = 0.1$, and $\alpha_3 = 0.05$ for *Example 1*

$\delta\tau$	$M = N$	L_∞	CO	$N = M$	L_∞	CO
$\frac{1}{M}$	4	1.37716E-01	–	10	1.99980E-05	10.6749
	5	8.85960E-02	9.2300	11	1.15496E-05	12.3413
	6	1.06741E-02	4.3334	12	2.19685E-06	4.0935
	7	1.03037E-02	1.3452	13	6.86568E-08	9.4304
	8	5.73450E-03	6.5349	14	6.48553E-09	4.6052
	9	2.37930E-04	4.4716			

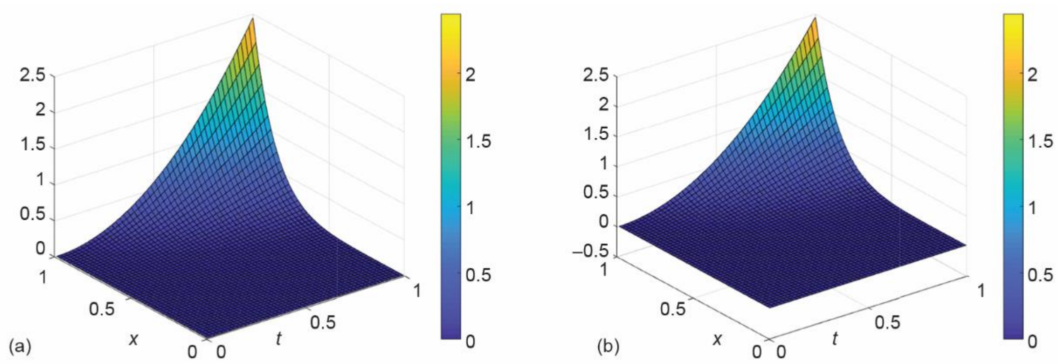

Figure 1. The exact and numerical solution of *Example 1* with $\alpha_1 = 0.75$, $\alpha_2 = 0.1$, and $\alpha_3 = 0.05$; (a) exact solution and (b) approximated solution

Example 2 We assume the following SM-TT-FDE with the exact solution $u(x, t) = (\alpha_1 + \alpha_2)x^5(t^2 - t)\sin(\pi x)$ in computational domain $x = [0, 1]$.

$$\mathcal{D}_t^{\alpha_1}u(x,t) + \frac{1}{4}\mathcal{D}_t^{\alpha_2}u(x,t) + \frac{1}{4}\mathcal{D}_t^{\alpha_3}u(x,t) = \dot{B}(t)u_{xx}(x,t) + \frac{1}{2}u_x(x,t) + f(x,t)$$
$$u(x,0) = u(0,t) = 0, \quad u(1,t) = 0, \quad 0 < t \le T$$

and:

$$f(x,t) = (\alpha_1 + \alpha_2 + \alpha_3)\left\{\sin(\pi x)\left[\frac{\Gamma(3)t^{2-\alpha_1}}{\Gamma(3-\alpha_1)} - \frac{\Gamma(2)t^{1-\alpha_1}}{\Gamma(2-\alpha_1)}\right] + \frac{1}{4}\left[\frac{\Gamma(3)t^{2-\alpha_2}}{\Gamma(3-\alpha_2)} - \frac{\Gamma(2)t^{1-\alpha_2}}{\Gamma(2-\alpha_2)}\right] + \right.$$
$$\left. + \frac{1}{4}\left[\frac{\Gamma(3)t^{2-\alpha_3}}{\Gamma(3-\alpha_3)} - \frac{\Gamma(2)t^{1-\alpha_3}}{\Gamma(2-\alpha_3)}\right] + \pi(t^2 - t)\cos(\pi x) + [\pi^2\dot{B}(t)](t^2 - t)\sin(\pi x)\right\}$$

Here, we present the proposed approach for numerical solution of SM-TT-FDE having Browning motion coefficients. Table 2 presents CO and the error norm of proposed method at diverse value of collocation points with $\alpha_1 = 0.9, \alpha_2 = 0.5,$ and $\alpha_3 = 0.3$ for $\overline{P} = 100$ random paths. Initial and boundary conditions imposed in this example are derived from the exact solution. Figure 2(a) illustrates the exact solution compared with the approximation at different points in time. Further, fig. 2(b) shows the contours of error for $M = N = 0$ at $T = 1$.

Table 2. Error absolute L_∞ at various values of $M = N$ by letting $T = 1$ and $\alpha_1 = 0.9$, $\alpha_2 = 0.5$, and $\alpha_3 = 0.3$ for *Example 2*

$\delta\tau$	$M = N$	L_∞	CO	$N = M$	L_∞	CO
$\frac{1}{M}$	2	1.55820E-01	–	10	1.19285E-10	308.3762
	4	1.22736E-01	1.2696	12	1.01443E-10	1.1758
	6	1.93902E-05	6329.8203	14	1.39127E-10	0.7291
	8	3.67847E-0	527.1268			

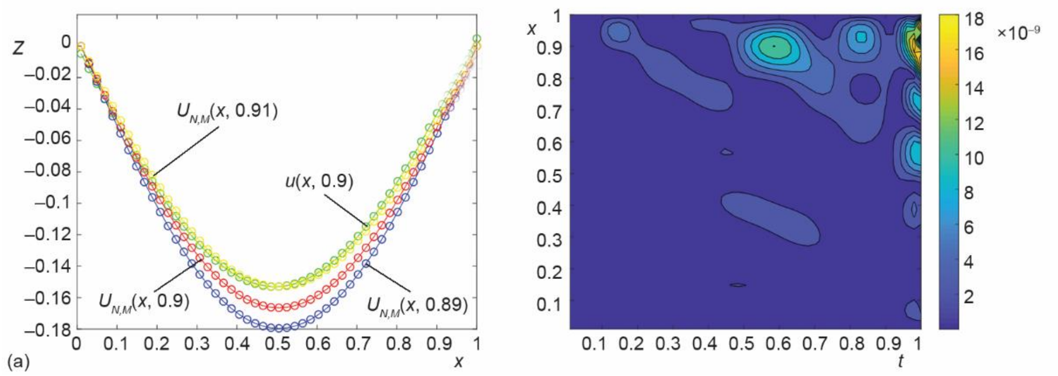

Figure 2. The diagram (a) comparing amount exact solution and approximate solution and (b) error at final time $T = 1$ for *Example 2*

Conclusion

This paper adopted a spectral collocation approach based on SKCP approach for the numerical solution of the SM-TT-FDE associated with noisy data driven by Brownian motion. The time fractional derivatives have been described by means of Caputo sense. To verify the concept of Brownian motion on the purposed method, we applied $\overline{P} = 80$ and $\overline{P} = 100$ random paths for two examples. Numerical results in comparison with the analytical solution demonstrate the accuracy and robustness of the proposed method.

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