DISPERSION OF GENERALIZED RAYLEIGH WAVES IN THE HALF-PLANE COVERED WITH PRE-STRETCHED TWO LAYERS UNDER COMPLETE CONTACT

by

Muslum OZISIK *

Department of Mathematical Engineering, Yildiz Technical University, Istanbul, Turkey

Original scientific paper
https://doi.org/10.2298/TSCI21S2247O

In this paper the dispersion of the generalized Rayleigh wave propagation in the non-prestressed half-plane covered with pre-stretched two layers under complete contact conditions is investigated by 3-D linearized theory of elasticity. The layers and the half-plane are assumed that elastic, homogeneous, isotropic, and the complete contact conditions are existed. The inter phase zone between the upper layer and half-plane is modeled by this second layer. The purpose of the investigation is the determination on the effect of the existence of the second layer to the considered generalized Rayleigh wave propagation velocity.

For this purpose, firstly the same materials were selected for both layers and the results obtained in previous studies for a single layer in the literature were verified, the accuracy of the modeling was shown, and then the effect of the second layer on the considered problem was shown by selecting the different materials and applying different initial pre-stresses.

Consequently, the present study can be considered as the investigation of the existence of the inter phase zone which is characteristic one for the composite materials to the dispersion of the generalized Rayleigh wave propagation. Numerical results obtained and discussed.

Key words: *dispersion, initial stresses, propagation, generalized Rayleigh wave, complete contact conditions*

Introduction

The theory of propagation of waves in half-planes covered with layers, although an old topic found very important scientific and engineering applications in the classical sense in the last few decades. Considering the areas such as earth and earthquake science, underground exploration, composite materials, smart materials, acoustics, non-destructive testing, damage detection, and the effect of pre-stresses, it is seen that their application areas are current, very common and important. The many investigations, [1-4], refer to the effect of the pre-stresses on the propagation and dispersion. The most important studies in this regard the effect of the contact conditions and the pre-stresses.

The 3-D linearized theory of elasticity (3-DLTE) is utilized in order to investigate the elastic waves in pre-stressed materials. With this theory, there is no need for any additional hypotheses, considering the condition that the pre-stresses are much larger than the stresses due to wave propagation, the precise non-linear 3-D equations of continuous media mechanics are

* Author's, e-mail: ozisik@yildiz.edu.tr

linearized and wave propagation models are made. The 3-DLTE has been elaborated in many investigations such as [5-9] and the others.

There are many investigations have been made in the fields of waves and generalized Rayleigh wave (GRW) depending on both contact conditions and pre-stresses [10-14].

Consequently, the aim of the investigation is to determine the dispersion of GRW propagation (GRWP) in the half-plane, which is covered with two pre-stressed layers and the effect of pre-stresses on this dispersion. In addition to these studies mentioned before, in this study, the half-space was covered with two different layers and firstly the results obtained in studies [12, 15]. Then, the dispersion of GRW which for different initial pre-stresses was examined and interpreted by obtaining results.

The formulation of the considered problem

According to [8, 15] the covered half-planes were considered as isotropic, homogeneous and elastic, fig. 1. It is considered that the there is an initial normal stresses in the direction of Ox_1. In the following formulations for the layers and half-plane will be expressed as the upper (1), (3) and (2) and the values for the stresses, strains and displacements are denoted by $u_i^{(r),0}$, $\varepsilon_{ij}^{(r),0}$, and $\sigma_{ij}^{(r),0}$ (i, j = 1, 2 and r = 1, 2, 3), respectively.

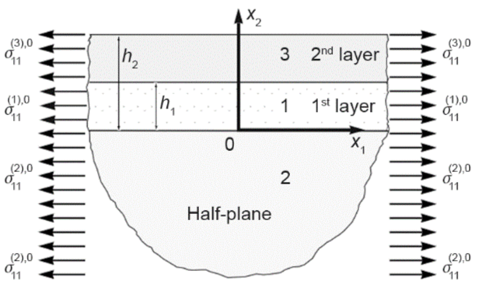

Figure 1. The pre-stressed two-layered half-plane

The geometry of the considered problem is shown in fig. 1. It is accepted that the half-plane covered by the layers with thickness h_1, $h_2 - h_1$ and we associate the Lagrangian coordinates $Ox_1x_2x_3$ which in the natural state coincide with the cartesian co-ordinates.

The covered layers and half-plane occupy the regions $\{-\infty < x_1 < +\infty,\ 0 \le x_2 \le h_1,\ -\infty < x_3 < +\infty\}$, $\{-\infty < x_1 < +\infty,\ h_1 \le x_2 \le h_2,\ -\infty < x_3 < +\infty\}$, and $\{-\infty < x_1 < +\infty,\ -\infty \le x_2 \le 0,\ -\infty < x_3 < +\infty\}$. In the considered case the initial stresses in the layer and half-plane is given:

$$\sigma_{11}^{(r),0} = \text{const}_r \neq 0, \quad r = 1,2,3, \quad \sigma_{ij}^{(r),0} = 0, \quad i = j \neq 1 \tag{1}$$

According to [4] by using the 3-DLTE, the equations are provided separately in each of the regions covered by the layers and the half-plane as given for (r = 1, 2, 3):

$$\begin{aligned} \frac{\partial \sigma_{11}^{(r)}}{\partial x_1} + \frac{\partial \sigma_{12}^{(r)}}{\partial x_2} + \sigma_{11}^{(r),0} \frac{\partial^2 u_1^{(r)}}{\partial x_1^2} &= \rho^{(r)} \frac{\partial^2 u_1^{(r)}}{\partial t^2} \\ \frac{\partial \sigma_{11}^{(r)}}{\partial x_1} + \frac{\partial \sigma_{12}^{(r)}}{\partial x_2} + \sigma_{11}^{(r),0} \frac{\partial^2 u_2^{(r)}}{\partial x_1^2} &= \rho^{(r)} \frac{\partial^2 u_1^{(r)}}{\partial t^2} \end{aligned} \tag{2}$$

where the conventional notation is used and $\sigma_{ij}^{(r)}$ represents stress tensor components, $u_i^{(r)}$ represents displacement vector components, and $\rho^{(r)}$ represents the density of the layers and half-plane material, respectively. The layers and half-plane materials are linear elastic, homogeneous, isotropic and relationships between stress and strain tensor components are given for r = 1, 2, 3:

$$\sigma_{11}^{(r)} = \lambda^{(r)}\theta^{(r)} + 2\mu^{(r)}\varepsilon_{11}^{(r)},\ \ \sigma_{22}^{(r)} = \lambda^{(r)}\theta^{(r)} + 2\mu^{(r)}\varepsilon_{22}^{(r)}$$

$$\sigma_{12}^{(r)} = 2\mu^{(r)}\varepsilon_{12}^{(r)},\ \ \theta^{(r)} = \varepsilon_{11}^{(r)} + \varepsilon_{22}^{(r)} \tag{3}$$

where $\varepsilon_{ij}^{(r)}$ represents strain tensor components, $\lambda^{(r)}$ and $\rho^{(r)}$ are Lame's constants of the layers and the half-plane material, respectively. The relationship between the strain tensor components and displacement vector components is:

$$\varepsilon_{ij}^{(r)} = \frac{1}{2}\left(\frac{\partial u_i^{(r)}}{\partial x_j} + \frac{\partial u_j^{(r)}}{\partial x_i}\right) \quad i, j = 1, 2 \tag{4}$$

Between the layers and half-plane the complete contact conditions are satisfied as given for i = 1, 2.

$$u_i^{(1)}\,|_{x_2=0} = u_i^{(2)}\,|_{x_2=0},\ \ u_i^{(1)}\,|_{x_2=h_1} = u_i^{(3)}\,|_{x_2=h_1},\ \ \sigma_{i2}^{(1)}\,|_{x_2=0} = \sigma_{i2}^{(2)}\,|_{x_2=0},\ \ \sigma_{i2}^{(1)}\,|_{x_2=h_1} = \sigma_{i2}^{(3)}\,|_{x_2=h_1} \tag{5}$$

It is assumed that the covering layer 2 and half-plane satisfy the given boundary and damping conditions in eq. (6):

$$\sigma_{12}^{(3)}\,|_{x_2=h_2} = 0,\ \ \sigma_{22}^{(3)}\,|_{x_2=h_2} = 0,\ \ \sigma_{ij}^{(2)}\,|_{x_2\to-\infty} \to 0,\ \ u_i^{(2)}\,|_{x_2\to-\infty} \to 0,\ \ i = 1, 2 \tag{6}$$

Our aim is to examine the surface waves propagating in the direction of the Ox_1 axis in the layered half-plane and specified by the factor $\exp[i(k.x_1 - \omega.t)]$ within the framework of equations and boundary conditions eqs. (1)-(6). In order to make these examinations, all sizes included in eq. (1)-(6) is written:

$$\left\{\sigma_{ij}^{(r)}, \varepsilon_{ij}^{(r)}, u_i^{(r)}\right\} = \left\{\bar{\sigma}_{ij}^{(r)}, \bar{\varepsilon}_{ij}^{(r)}, \bar{u}_i^{(r)}\right\}\ \exp[i(k\,x_1 - \omega\,t)] \tag{7}$$

In eq. (7), k is the wavelength, ω – the frequency, and c – the phase velocity of the wave. In order to make the further operations simple, the representation eq. (7) is replaced with:

$$u_1^{(r} = \bar{u}_1^{(r)}\sin(kx_1 - \omega t),\ \ u_2^{(r)} = \bar{u}_2^{(r)}\cos(kx_1 - \omega t),\ \ \sigma_{11}^{(r)} = \bar{\sigma}_{11}^{(r)}\cos(kx_1 - \omega t)$$

$$\sigma_{22}^{(r)} = \bar{\sigma}_{22}^{(r)}\cos(kx_1 - \omega t),\ \ \sigma_{12}^{(r)} = \bar{\sigma}_{12}^{(r)}\sin(kx_1 - \omega t),\ \ r = 1, 2, 3 \tag{8}$$

According to Helmholtz decomposition rule, the $u_1^{(r)}$ and $u_2^{(r)}$ are displacement fields can be described by $\phi^{(r)}$ (longitudinal wave potential) and $\psi^{(r)}$ (transverse wave potential) in the given form:

$$u_1^{(r)} = \frac{\partial\phi^{(r)}}{\partial x_1} + \frac{\partial\psi^{(r)}}{\partial x_2},\ \ u_2^{(r)} = \frac{\partial\phi^{(r)}}{\partial x_2} - \frac{\partial\psi^{(r)}}{\partial x_1} \tag{9}$$

$\phi^{(r)}$ and $\psi^{(r)}$ satisfy:

$$\Delta\phi^{(r)} = \frac{1}{[c_1^{(r)}]^2}\frac{\partial^2\phi^{(r)}}{\partial t^2},\ \ \Delta\psi^{(r)} = \frac{1}{[c_2^{(r)}]^2}\frac{\partial^2\psi^{(r)}}{\partial t^2} \tag{10}$$

where $c_1^{(r)}$ and $c_2^{(r)}$ refers to the longitudinal and transverse wave velocity in the m^{th} material and represented by:

$$c_1^{(r)} = \left[\frac{\lambda^{(r)} + 2\mu^{(r)}}{\rho^{(r)}}\right]^{1/2}, \; c_2^{(r)} = \left[\frac{\mu^{(r)}}{\rho^{(r)}}\right]^{1/2} \tag{11}$$

Equations (8) and (9) are written:

$$\phi^{(r)} = \phi_0^{(r)}(x_2)\cos(kx_1 - \omega t), \;\; \psi^{(r)} = \psi_0^{(r)}(x_2)\sin(kx_1 - \omega t) \tag{12}$$

Substituting eq. (12) in eq. (9) it is obtained:

$$\frac{\partial^2 \phi_0^{(r)}}{\partial (kx_2)^2} + \left\{\frac{c^2}{[c_1^{(r)}]^2} - 1\right\}\phi_0^{(r)} = 0, \;\; \frac{\partial^2 \psi_0^{(r)}}{\partial (kx_2)^2} + \left\{\frac{c^2}{[c_2^{(r)}]^2} - 1\right\}\psi_0^{(r)} = 0 \tag{13}$$

The solution of the differential eq. (13) is:

$$\begin{aligned}
&\phi_0^{(1)} = C_1^{(1)} \exp[ikp_1^{(1)} x_2] + C_2^{(1)} \exp[-ikp_1^{(1)} x_2] \\
&\psi_0^{(1)} = C_3^{(1)} \exp[ikp_2^{(1)} x_2] + C_4^{(1)} \exp[-ikp_2^{(1)} x_2] \\
&\phi_0^{(2)} = C_5^{(2)} \exp[kq_1^{(2)} x_2], \quad \psi_0^{(2)} = C_6^{(2)} \exp[kq_2^{(2)} x_2] \\
&\phi_0^{(3)} = C_7^{(3)} \exp[iks_1^{(3)} x_2] + C_8^{(3)} \exp[-iks_1^{(3)} x_2] \\
&\psi_0^{(3)} = C_9^{(3)} \exp[iks_2^{(3)} x_2] + C_{10}^{(3)} \exp[-iks_2^{(3)} x_2]
\end{aligned} \tag{14}$$

In eq. (14), $C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, C_4^{(1)}, C_5^{(2)}, C_6^{(2)}, C_7^{(3)}, C_8^{(3)}, C_9^{(3)}$ and $C_{10}^{(3)}$ are unknown constants and:

$$\begin{gathered}
p_1^{(1)} = \frac{c}{c_1^{(1)}} \quad p_2^{(1)} = \frac{c}{c_2^{(1)}} \quad q_1^{(2)} = \left\{1 - \frac{c^2}{[c_2^{(2)}]^2}\right\}^{1/2}, \quad q_2^{(2)} = \left\{1 - \frac{c^2}{[c_2^{(2)}]^2}\right\}^{1/2} \\
s_1^{(3)} = \frac{c}{c_1^{(3)}} \quad s_2^{(3)} = \frac{c}{c_2^{(3)}}
\end{gathered} \tag{15}$$

Considering eq. (14), if we substitute the expressions obtained from eqs. (3), (4), and (12) under the conditions eqs. (5) and (6), it is obtained a linear homogeneous equation system for the unknown constants $C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, C_4^{(1)}, C_5^{(2)}, C_6^{(2)}, C_7^{(3)}, C_8^{(3)}, C_9^{(3)}$ and $C_{10}^{(3)}$ for the complete contact case. The non-zero solution of this system requires that its coefficients of unknowns determinant equals to be zero. This equality form is the dispersion equation. This determinant is 10 × 10 in size and represented:

$$\det\left\|\alpha_{ij}\right\| = 0, \quad i, j = 1,...,10 \tag{16}$$

Equation (16) is the dispersion equation that must be obtained for the solution of the problem under investigation. In eq. (16), α_{ij} depends on kh_1, kh_2, $c = \omega/k$, , $\sigma_{11}^{(1),0}$, $\sigma_{11}^{(2),0}$, $\sigma_{11}^{(3),0}$, $\lambda^{(1)}$, $\lambda^{(2)}$, $\lambda^{(3)}$, $\mu^{(1)}$, $\mu^{(2)}$, , $\mu^{(3)}$, $\rho^{(1)}$, $\rho^{(2)}$, and $\rho^{(3)}$. The parameters indicate the mechanical properties of the layers and the half-plane and they are previously known except c, kh_1, and kh_2.

$$\alpha_{ij} = \alpha_{ij}\left[c, kh_1, kh_2, \sigma_{11}^{(1),0}, \sigma_{11}^{(2),0}, \sigma_{11}^{(3),0}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \mu^{(1)}, \mu^{(2)}, \mu^{(3)}, \rho^{(1)}, \rho^{(2)}, \rho^{(3)}\right] \quad (17)$$

where c is the phase velocity, kh_1 and kh_2 – the dimensionless wavelength, respectively. If the phase velocity is dependent on wavelength, such waves are called dispersive waves. The GRW propagating in the linear elastic and isotropic plane are dispersive waves [8, 15, 16]. Thus, in order to obtain the numerical results the analysed case is where $\sigma_{11}^{(1),0} = \sigma_{11}^{(2),0} = \sigma_{11}^{(3),0} = 0$, investigated in [15, 16] and discussed in [8]. According to [8, 15, 16] for the case $\sigma_{11}^{(1),0} = \sigma_{11}^{(2),0} = \sigma_{11}^{(3),0} = 0$ the function $\psi = c(kh)$ has two branches that correspond to the M_1 and M_2 kind of wave propagation.

Moreover, in [8, 15, 16] it had been shown that the dispersion eq. (17) for the case $\sigma_{11}^{(1),0} = \sigma_{11}^{(2),0} = \sigma_{11}^{(3),0} = 0$ has many modes M_{1z} and M_{2z}, respectively. As can be seen from the mode graphs, there is an increasing (kh) cut-off value with the each increasing mode number.

In order to solve eq. (16), the value of the kh parameter is given initially and the value of the c in the eq. (16) that satisfies this equation is sought. Thus, c phase velocity of the wave is found depending on the kh values. According to the stated definitions, if different c values are found for different kh values, the wave is a dispersive wave (M_{11}: the first branch in the first mode, M_{21}: the second branch in the first mode, M_{12}: the first branch in the second mode, M_{22}: the second branch in the second mode) [8, 15]. The process is repeated for all kh values.

Results and discussion

For the relationship between c and kh previously mentioned, materials for the layers bronze and brass and for the half-plane steel were selected. The mechanical properties of the selected materials are ρ = 7.795 gr/cm^3, λ = 9.26 × 10^4 MPa, μ = 7.75 × 10^4 MPa for the steel, ρ = 7.20 gr/cm^3, λ = 8.16 × 10^4 MPa, μ = 3.84 × 10^4 MPa for the bronze, and ρ = 7.20 gr/cm^3, λ = 9.49 × 10^4 MPa, μ = 4.47 × 10^4 MPa for the brass.

First of all, the dispersion curves obtained in study [12] for the steel-bronze and steel-brass pair of materials in the non-pre-stressed contact conditions and for the single layer system. The layer thickness covering the half-plane was taken as kh. The dispersion values in study [12] coincide with the values given by [15]. Therefore, it is aimed to obtain the dispersion curves similar to in [12] for the two-layered system whose geometry is given in fig. 1 and steel, brass and bronze was selected for the half-plane and the layers materials, respectively.

In order to compare the dispersion curves, the half-plane material was chosen as steel in the same way, and both layer materials were chosen as bronze (brass), and the layer thicknesses were taken as equal kh. First of all, both layer materials were chosen as bronze and their thickness as kh and distribution curves in unstressed condition were obtained. Then, both layer materials were taken as brass and their thickness as kh and distribution curves in unstressed condition were obtained. Finally, for the investigated two-layer system whose geometry was given in fig. 1, half-plane steel, first layer brass, second layer bronze, and layer thicknesses are taken as kh and dispersion curves in unstressed state are obtained and given in fig. 2.

In order to determine the effect of the pre-stresses applied to the layers on the dispersion, different tensile pre-stresses values were applied to the layers where the half-plane is non-pre-stressed. The layer materials were selected as same material, and the examination was made by taking the layer thicknesses equal kh ($kh_1 = kh$, $kh_2 - kh_1 = kh$). In figs. 3-6 is shown with dotted lines for bronze and dashed lines for brass.

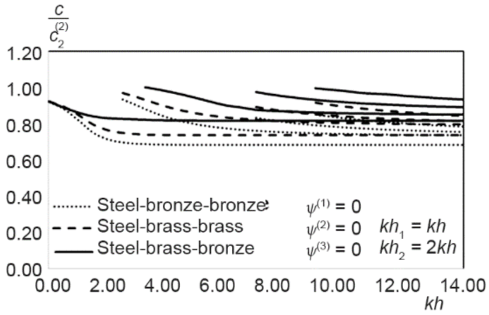

Figure 2. Dispersion curves

Afterwards, the layer materials were selected differently for the problem to be investigated and tensile pre-stresses was applied. In figs. 3-6 are shown with a continuous line. Therefore, figs. 3-6 show the wave propagation velocities obtained when the layer materials are both the same and different with layer thicknesses kh_1 and $kh_2 - kh_1$.

Moreover, figs. 3-6 show the graphs of different modes and branches as expressed in [8, 15, 16] (M_{11} in fig. 3, M_{21} in fig. 4, M_{12} in fig. 5, M_{22} in fig. 6).

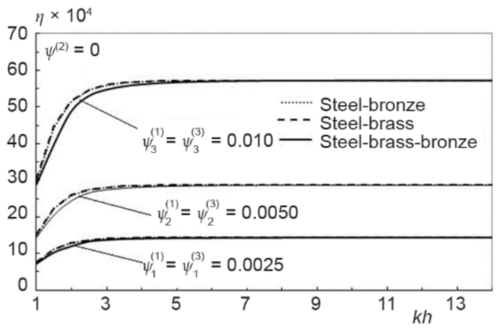

Figure 3. The graph of GRWP for M_{11}

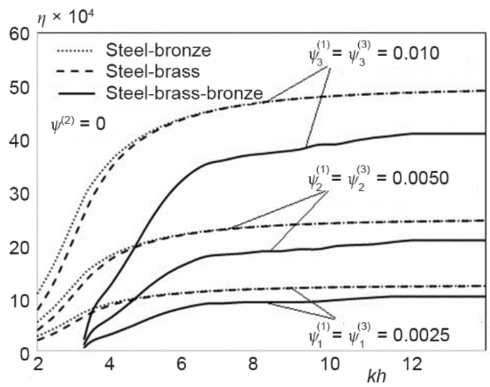

Figure 4. The graph of GRWP for M_{21}

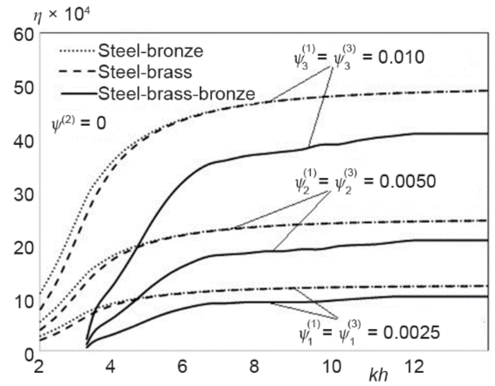

Figure 5. The graph of GRWP for M_{21}

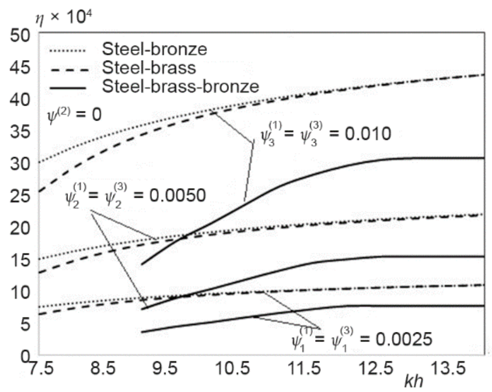

Figure 6. The graph of GRWP for M_{22}

In order to see the effect of prestresses on velocity more clearly in figs. 3-6, it has been taken:

$$\eta = \frac{c\big|_{\psi^{(1)}\neq 0,\psi^{(3)}\neq 0,\psi^{(2)}=0} - c\big|_{\psi^{(1)}=0,\psi^{(3)}=0,\psi^{(2)}=0}}{c\big|_{\psi^{(1)}=0,\psi^{(3)}=0,\psi^{(2)}=0}}, \quad \psi^{(k)} = \frac{\sigma^{(\mathrm{k})}}{\mu^{(\mathrm{k})}} \tag{18}$$

Conclusion

Thus, in the study within the 3-TDLTE the dispersion of the GRWP in the half-plane covered by two pre-stressed layers has been examined. According to the results obtained in figs. 3-6.

The dispersion curves obtained in fig. 2 are consistent with the previously studies [8, 15, 16] and have an asymptotic structure for all materials. In figs. 3-6, when the two layers materials are different (brass and bronze) for all modes and branches the velocity decreases compared to the layer materials are the same (bronze-bronze or brass-brass). The pre-stressing the layers affects the velocity for all modes and branches. The wave propagation velocity increases when tensile pre-stresses are applied to layers. The results obtained through the investigations are consistent with the results obtained ones before.

References

[1] Xibing, Li., Ming, T., The İnfluence of İnitial Stress on Wave Propagation and Dynamic Elastic Coefficients.,*Geomechanics and Engineering, 8* (2015), 3, pp. 377-390
[2] Demiray, H., Suhubi, E. H., Small Torsional Oscillations of an Initially Twisted Circular Rubber Cylinder, *International Journal of Eng. Science*, *8* (1970), 1, pp. 19-30
[3] Dowaikh, M. A., Ogden, R. W., Interfacial Waves and Deformations in Pre-Stressed Elastic Media, *Proc. R. Soc. Lond*., *433* (1991), 1888, pp. 313-328
[4] Babich, S. Y., *et al*., Elastic Waves in Bodies with Initial Stresses, *Soviet Applied Mechanics, 15* (1979), 4, pp. 277-271
[5] Lifshitz, E. M., *et al*., *Theory of Elasticity* (3rd ed.), Butterworth-Heinemann, Oxford, UK, 1986, pp. 87-107
[6] Chadwick, P., Jarvis, D. A., Surface Waves in a Pre-Stressed Elastic Body, *Proc. R. Soc. Lond*., *366* (1979) 1727, pp. 517-536
[7] Achenbach, J., *Wave Propagation in Elastic Solids,* North Holland Elsevier, New York, USA, 1973
[8] Eringen, A. C., Suhubi, S. E., *Elastodynamics, Linear Theory,* 1st ed., Academic Press, New York, USA, 1975
[9] Revenko, V. P., Solving the Three-Dimensional Equations of the Linear Theory of Elasticity, *International Applied Mechanics*, *45* (2009), 7, pp. 730-741
[10] Nam, N. T., *et al*., The Effect of Initial Stress on the Propagation of Surface Waves in a Layered Half-Space, *Int. J. of Solids and Structures*, *88* (2016), June, pp. 88-100
[11] Shou-Guo, Y., *et al*., Dispersion Function of Rayleigh Waves in Porous Layered Half-Space System, *App. Geophysics*, *13* (2016), 2, pp. 333-342
[12] Akbarov, S. D., Ozisik, M., The Influence of the Third Order Elastic Constants to the Generalized Rayleigh Wave Dispersion in a Pre-Stressed Stratified Half-Plane, *Int. J. of Eng. Sci*., *41* (2003), 17, pp. 2047-2061
[13] Negin, M., *et al.*, Generalized Rayleigh Wave Dispersion Analysis in a Pre-Stressed Elastic Stratified Half-Space With Imperfectly Bonded Interfaces, *CMC: Computers, Materials & Continua, 42* (2014), 1, pp. 25-61
[14] Bhattacharya, S. N., Rayleigh Wave Dispersion Equation with Real Terms in a Layered Transversely Isotropic Half Space, *Annals of Geophysics*, *60* (2017), Suppl. 6, SE670
[15] Tolstoy, I., Usdin, E., Dispersive Properties of Stratified Elastic and Liquid Media. A Ray Theory, *Geophysics, 18* (1953), 4, pp. 844-870
[16] Achenbach, J. D., Epstein, H. I., Dynamic Interaction of A Layer and a Half-Space, *J. Eng. Mech., ASCE*, *93* (1967), 5, pp. 27-42

Paper submitted: April 5, 2021
Paper revised: April 28, 2021
Paper accepted: May 10, 2021