NEW TYPES OF EXACT SOLUTIONS OF HIGH-FREQUENCY WAVES MODEL IN THE RELAXATION MEDIUM

by

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In this article, based on the extended fan-expansion method, novel soliton wave solutions of the Vakhnenko-Parkes equation are constructed. The stable property of the obtained analytical solutions is tested by implementing the Hamiltonian system's characterizations. The applied method is effective and applicable for many problems of non-linear PDE in mathematical physics.

Key words: *Vakhnenko-Parkes equation, extended fan-expansion method, stable property, soliton wave solutions*

Introduction

The study of the non-linear PDE (NLPDE) occupies the thinking of many researchers. Much of their research has been done to determine the exact solutions of the non-linear evolution equations (NLEE). The investigations of exact solutions of NLEE have a great deal to know the structure, provide better information and its applications. Therefore, to calculate the exact and solitary solutions of NLPD, the researchers introduced many methods. Such as inverse scattering transform method, Darboux transformation method, Hirota’s bilinear method, homogeneous balance method, solitary wave ansatz method, Jacobi elliptic function expansion method, the tanh function method, F-expansion method, projective Ricatti equation method [1-14], and so on. Among them is the extended Fan-expansion method [15-18], a powerful mathematical tool to investigate the exact solutions for NLEE. We will employ this method for solving the Vakhnenko-Parkes equation [19-22].

In this paper is the following strategy was applied:

- Firstly, toinvestigate the analytical solutions of the Vakhnenko-Parkes equation.
- Secondly, to study stability property of the obtained analytical solutions based on the Hamiltonian system’s characterizations [24, 25]
- Finally, present general conclusions.

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Application

In this part, we apply the extended fan-expansion method to the considered model then studying the stability property of the obtained analytical solutions.

Solitary wave solutions

Consider Vakhnenko-Parkes equation in the following formula:

$$u u_{\mathrm{xxt}} - u_x u_{\mathrm{xt}} + u^2 u_t = 0 \tag{1}$$

where $u = u(x, t)$ describes high-frequency waves in the relaxation medium. Applying the next wave transformation $u(x,t) = \phi(\zeta), \zeta = x + ct,$ then integrating the results, convert the system (1) into:

$$\phi^3 + 3\phi\phi'' - 3(\phi')^2 + s = 0 \tag{2}$$

where s is the integration constant. Using the homogenous balance principles and generalized form of solution based on the suggested scheme get the next general solutions:

$$\phi(\zeta) = \sum_{i=0}^{n} a_i\,[\mu + \phi(\zeta)]^i = a_0 + a_1[\mu + \phi(\zeta)] + a_2[\mu + \phi(\zeta)]^2 \tag{3}$$

where μ, a_0, a_1, a_2 are arbitrary constants to be evaluated later. Additionally, $\phi(\zeta)$ satisfies $\phi'(\zeta) = [\varrho + \phi^2(\zeta)]$, where ϱ is arbitrary constant to be evaluated later. Employing the suggested method's steps, get the following values of the previously shown parameters.

Case I

$$a_0 = -6(\mu^2 + \varrho), \quad a_1 = 12\mu, \quad a_2 = -6, \quad s = 0$$

Case II

$$a_0 = -2(3\mu^2 + \varrho), \quad a_1 = 12\mu, \quad a_2 = -6, \quad s = -64\varrho^3$$

Thus, we deduce the exact traveling wave solution of studied model are given as follows.

For $\varrho < 0$, we get:

$$u_I^1(x,t) = -6\varrho \mathrm{Sec}[(ct + x)\sqrt{\varrho}]^2 \tag{4}$$

$$u_I^2(x,t) = -6\varrho \mathrm{Csc}[(ct + x)\sqrt{\varrho}]^2 \tag{5}$$

$$u_{II}^1(x,t) = -2\varrho(1 + 3\mathrm{Tan}[(ct + x)\sqrt{\varrho}]^2) \tag{6}$$

$$u_{II}^2(x,t) = 2\varrho\left\{2 - 3\mathrm{Csc}\left[\left(ct + x\right)\sqrt{\varrho}\right]^2\right\} \tag{7}$$

For $\varrho < 0$, we get:

$$u_I^3(x,t) = -6\varrho \mathrm{Sec}[(ct + x)\sqrt{\varrho}]^2 \tag{8}$$

$$u_I^4(x,t) = -6\varrho \mathrm{Csc}[(ct + x)\sqrt{\varrho}]^2 \tag{9}$$

$$u_{II}^{3}(x,t)=-2\left\{\varrho+3\varrho\operatorname{Tan}\left[(ct+x)\sqrt{\varrho}\right]^{2}\right\} \tag{10}$$

$$u_{II}^{4}(x,t)=-2\left\{\varrho+3\varrho\operatorname{Cot}\left[(ct+x)\sqrt{\varrho}\right]^{2}\right\} \tag{11}$$

For $\varrho=0$, we get:

$$u_{I,IIx,t}^{5}(x,t)=-\frac{6}{(ct+x)^{2}}-6\varrho \tag{12}$$

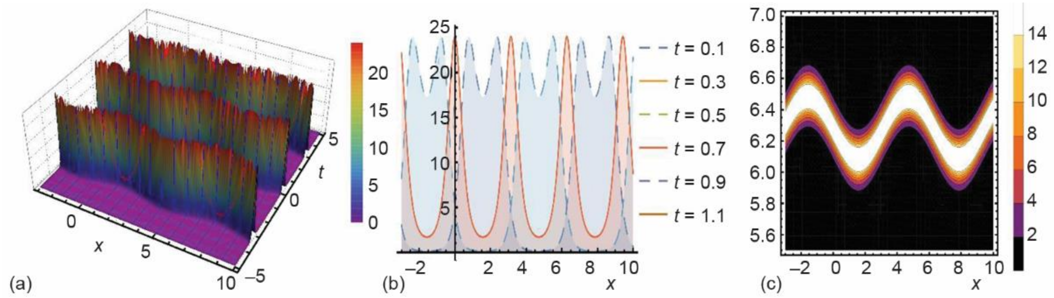

Figure 1. Soliton wave representation in 3-D, 2-D, and contour plots for eq. (4) when $\varrho = -4$ and $c = 5$

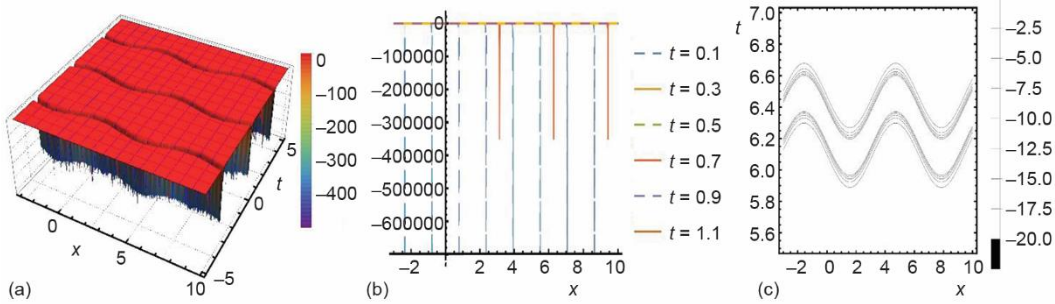

Figure 2. Soliton wave representation in 3-D, 2-D, and contour plots for eq. (5) when $\varrho = -4$ and $c = 5$

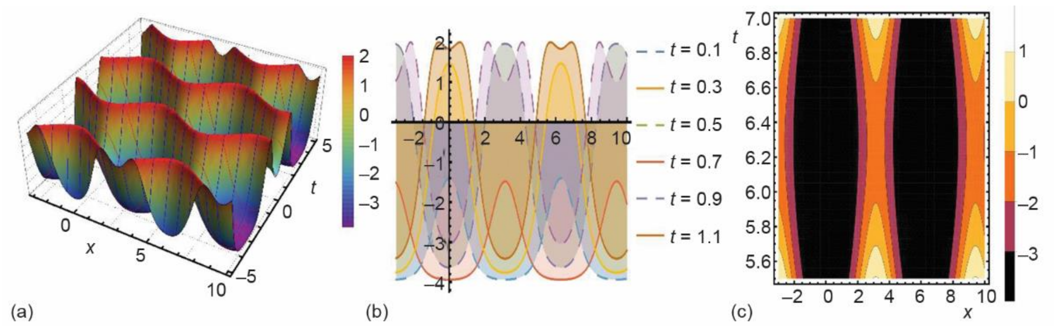

Figure 3. Soliton wave representation in 3-D, 2-D, and contour plots for eq. (6) when $\varrho = -1$ and $c = 2$

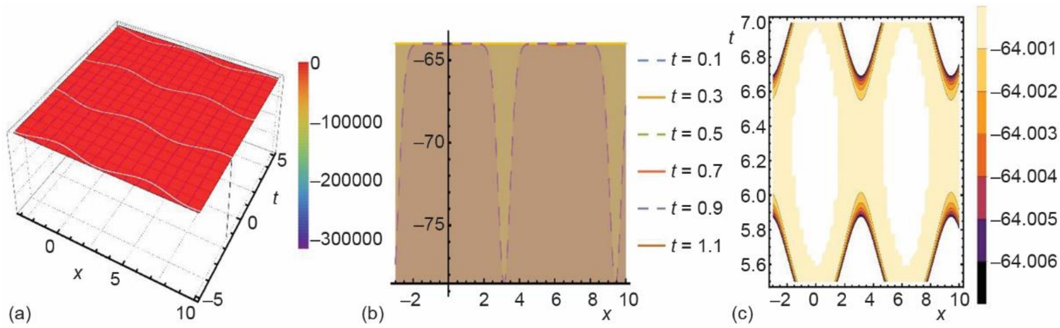

Figure 4. Soliton wave representation in 3-D, 2-D, and contour plots for eq. (7) when $\varrho = -15$ and $c = 5$

Stable characterization

Studying the stability of the previously obtained solutions based on the Hamiltonian system's characterizations through calculating the momentum of these solutions as following:

$$\begin{aligned}M_I^1(x,t) = {}& \frac{1}{c^2}24(\text{Csch}(10-10c)^2 - \text{Csch}[10(1+c)]^2 + 4\text{Log}[\text{Tanh}(10-10c)] - \\ & -2\text{Log}[1-\text{Tanh}(10-10c)^2] - 4\text{Log}\{\text{Tanh}[10(1+c)]] + 2\text{Log}1 - \{\text{Tanh}[10(1+c)]^2\}) - \\ & -\frac{1}{c}24\{20\text{Coth}(10-10c)\text{Csch}(10-10c)^2 + 20\text{Coth}[10(1+c)]\text{Csch}[10(1+c)]^2 - \\ & -40\text{Csch}(10-10c)\text{Sech}(10-10c) - 40\text{Csch}[10(1+c)]\text{Sech}[10(1+c)] - \\ & -\frac{40\text{Sech}(10-10c)^2\text{Tanh}(10-10c)}{1-\text{Tanh}(10-10c)^2} - \frac{40\text{Sech}[10(1+c)]^2\text{Tanh}[10(1+c)]}{1-\text{Tanh}[10(1+c)]^2}\}\end{aligned} \tag{13}$$

$$M_I^2(x,t) = \frac{\begin{aligned}&6(2\text{Log}[1-\text{Tanh}(5-5c)^2] - 2\text{Log}[1-\text{Tanh}[5(1+c)]^2] - \\ &-\text{Tanh}[5-5c]^2 + \text{Tanh}[5(1+c)]^2)\end{aligned}}{c} \tag{14}$$

$$\begin{aligned}M_{II}^1(x,t) = {}& \frac{1}{c}2\{100c + 30c(\text{ArcTanh}[\text{Tanh}(5-5c)] + \text{ArcTanh}\{\text{Tanh}[5(1+c)]\}) - \\ & -6\text{Log}[1-\text{Tanh}(5-5c)^2] + +6\text{Log}\{1-\text{Tanh}[5(1+c)]^2\} - \\ & -3\text{Tanh}(5-5c)^2 + 3\text{Tanh}[5(1+c)]^2)\end{aligned} \tag{15}$$

$$\begin{aligned}M_{II}^2(x,t) = {}& \frac{1}{c}2(400c - 3\text{Csch}(5-5c)^2 + 3\text{Csch}[5(1+c)]^2 + 12\text{Log}[\text{Tanh}(5-5c)] - \\ & -6\text{Log}[1-\text{Tanh}(5-5c)^2] - 12\text{Log}\{1-\text{Tanh}[5(1+c)]\} + 6\text{Log}\{1-\text{Tanh}[5(1+c)]^2\})\end{aligned} \tag{16}$$

Thus, the stability conditions of these solutions are given by:

$$\frac{\partial M_I^1(x,t)}{\partial c}\Big|_{c=2} = -480.0 + 75.398223\text{i} \tag{17}$$

$$\frac{\partial M_I^2(x,t)}{\partial c}\Big|_{c=2} = -59.9999992 \tag{18}$$

$$\frac{\partial M_{II}^1(x,t)}{\partial c}\Big|_{c=2} = 59.9885605 \tag{19}$$

$$\frac{\partial M_{II}^2(x,t)}{\partial c}\Big|_{c=2} = 60.011442 - 18.8495559\text{i} \tag{20}$$

Consequently, eqs. (4), (5) are unstable while eqs. (6), (7) are stable solutions. Using the same technqiue for studying the sable property of the obtained solutions, gives a clear vision of the high-frequency waves in the relaxation medium.

Conclusion

In this paper, the extended fan-expansion method successfully constructs many new solutions for solving the Vakhnenko-Parkes equation. These solutions have been represented through some graphs (figs. 1-4). Additionally, the stability property of the obtained solutions has been investigated through the Hamiltonian system's characterizations

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