ANALYTICAL AND SEMI ANALYTICAL SOLUTIONS OF THE INTERNAL WAVES OF DEEP-STRATIFIED FLUIDS

by

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In this paper, we study the soliton wave of the fractional Benjamin-Ono equation based on the extended simplest equation method. The accuracy of the obtained soliton wave solutions is investigated by comparing the obtained analytical and semi-analytical solutions. The semi-analytical solutions are constructed by applying the Adomian decomposition method. The semi-analytical method is used based on the constructed initial and boundary conditions from the obtained analytical solutions. Both solutions (analytical and semi-analytical) are plotted through different techniques for explaining the internal waves of deep-stratified fluids.

Key words: *fractional Benjamin-Ono equation, conformable derivative, soliton wave, semi-analytical solutions*

Introduction

Non-linear evolution equations have attracted the attention of many researchers in different fields based on their ability to formulate many complex phenomena [1, 2]. Traveling wave solution is one of the essential branches for studying these phenomena [3-5]. Consequently, many researchers have formulated accurate analytical schemes such as the extended Jacobi elliptic function method, the F-expansion method, the modified simple equation-method, the modified simple equation-method, the (G'/G)-expansion method, the modified $(G'/G, 1/G)$-expansion method, the $\exp[-\phi(\zeta)]$-expansion method, and so on [6-10]. These methods are effective for developing these complex models through studying the analytical and dynamical behavior of each one of them based on the methods' computtaional solutions [11-15].

The strategy of this paper applies the ESE and analytical and semi-analytical schemes to the FBO equation [16-24].

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Application

In this part, we apply the extended simplest equation (ESE) and Adomian decomposition (AD) analytical and semi-analytical schemes to the fractional Benjamin-Ono (FBO) equation. This model is given by:

$$D_t^{\vartheta}\,\mathcal{H}+\upsilon\mathcal{H}_{xx}+\mathcal{H}\,\mathcal{H}_x=0,\quad 0<\vartheta\le 1 \tag{1}$$

where υ the Hilbert Transform Operator which is defined by:

$$\upsilon=\frac{\mathcal{PV}}{\pi}\int_{-\infty}^{\infty}\frac{\mathcal{T}(\mathcal{Y})}{\mathcal{Y}-\mathcal{X}}\mathrm{d}y$$

Employing the next wave transformation $\mathcal{H}(x,t)=\mathcal{Q}(\zeta),\quad \zeta=x+c(t^{\vartheta}/\vartheta)$ then integrating the obtained ODE with respect to ζ and with zero constant of integration, get:

$$\mathcal{Q}+\upsilon\mathcal{Q}'+\frac{1}{2}\mathcal{Q}^2=0 \tag{2}$$

Balancing the highest order derivative term and non-linear term in eq. (2) gets n = 1. Consequently, the general analytical solution of the FBO equation is given by

$$\mathcal{Q}(\zeta)=\sum_{i=-n}^{n}a_i\,f^{\,i}(\zeta)=a_{-1}\,f(\zeta)+a_0+a_1\,f(\zeta) \tag{3}$$

where $f'(\zeta)=\alpha+\lambda\,f(\zeta)+\mu\,f^2(\zeta)$, while λ, μ are arbitrary conatnts.

Analytical solutions

Handling eq. (2) through the ESE method's framework gets thefollowing values of the above-shown parameters.

Family I

$$a_{-1}=0,\quad a_0=-\lambda\upsilon-\sqrt{\lambda^2\upsilon^2-4\alpha\mu\upsilon^2},\quad a_1=-2\mu\upsilon,\quad c\to\sqrt{(\lambda^2-4\alpha\mu)\upsilon^2}$$

Family II

$$a_{-1}=2\alpha\upsilon,\quad a_0=\lambda\upsilon-\sqrt{\lambda^2\upsilon^2-4\alpha\mu\upsilon^2},\quad a_1=0,\quad c=\sqrt{(\lambda^2-4\alpha\mu)\upsilon^2}$$

Thus, the soliton wave solutions of the FBO equation are given by:

– for $\lambda=0,\ \alpha\,\mu<0$, we get:

$$\mathcal{H}_1(x,t)=-2\sqrt{-\alpha\mu\upsilon^2}-2\sqrt{-\alpha\mu}\upsilon\mathrm{Tanh}\left[\sqrt{-\alpha\mu}\left(x+\frac{2t^{\vartheta}\sqrt{-\alpha\mu\upsilon^2}}{\vartheta}\right)\mp\frac{\mathrm{Log}[\Xi]}{2}\right] \tag{4}$$

$$\mathcal{H}_2(x,t)=-2\sqrt{-\alpha\mu\upsilon^2}-2\sqrt{-\alpha\mu}\upsilon\mathrm{Coth}\left[\sqrt{-\alpha\mu}\left(x+\frac{2t^{\vartheta}\sqrt{-\alpha\mu\upsilon^2}}{\vartheta}\right)\mp\frac{\mathrm{Log}[\Xi]}{2}\right] \tag{5}$$

– for $\alpha = 0,\ \lambda > 0$, we get:

$$\mathcal{H}_3^I(x,t) = -\sqrt{\lambda^2\upsilon^2} + \lambda\left[\upsilon + \frac{2\upsilon}{-1+\mathrm{e}^{\lambda\left(x+\Xi+\frac{t^\vartheta\sqrt{\lambda^2\upsilon^2}}{\vartheta}\right)_{\mu}}}\right] \tag{6}$$

– for $\alpha = 0,\ \lambda < 0$, we get:

$$\mathcal{H}_4^I(x,t) = -\lambda\upsilon + 2\mu\left[1 - \frac{1}{1+\mathrm{e}^{\lambda\left(x+\Xi+\frac{t^\vartheta\sqrt{\lambda^2\upsilon^2}}{\vartheta}\right)_{\mu}}}\right]\upsilon - \sqrt{\lambda^2\upsilon^2} \tag{7}$$

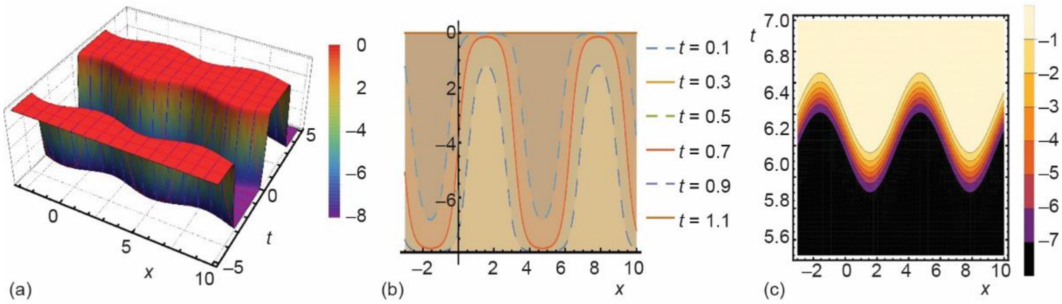

Figure 1. Soliton wave representation in 3-D, 2-D, and contour plots for eq. (4) when α = –1, μ = 4

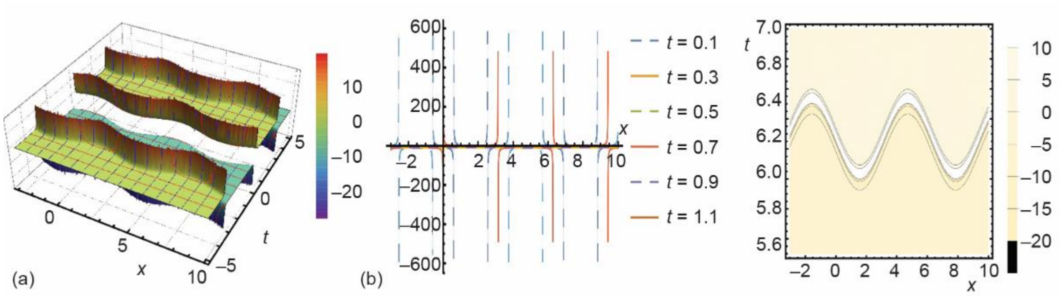

Figure 2. Soliton wave representation in 3-D, 2-D, and contour plots for eq. (5) when α = –1, μ = 4

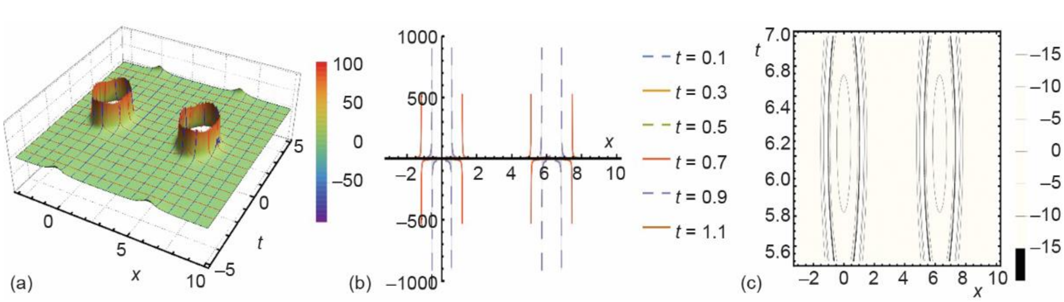

Figure 3. Soliton wave representation in 3-D, 2-D, and contour plots for eq. (6) when α = –1, μ = 4

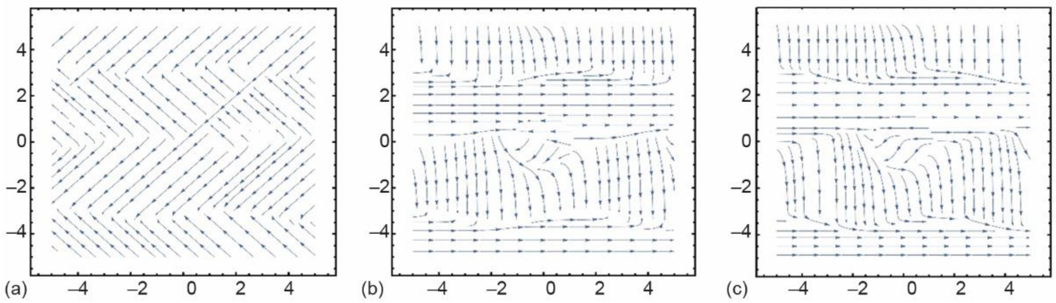

Figure 4. Stream plot representations for eqs. (4)-(6)

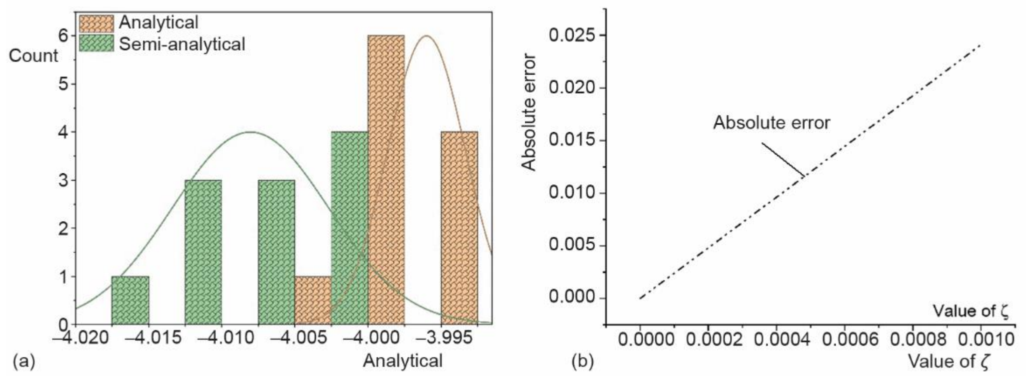

Figure 5. The 2-D plots of analytical and semi-analytical solutions and absolute value of error

Semi-analytical solutions

Applying the AD semi-analytical schemes based on solutions eq. (4) and the following values of the shown-parameters $\alpha = -1, \mu = 4, \upsilon = -1$ get the following semi-analytical solutions of the FBO equation in the following formula:

$$\mathcal{Q}_0(\zeta) = -4 + 8\zeta \tag{8}$$

$$\mathcal{Q}_1(\zeta) = -24\zeta + 32\zeta^2 - \frac{32\zeta^3}{3} \tag{9}$$

$$\mathcal{Q}_2(\zeta) = -96\zeta^2 + \frac{448\zeta^3}{3} - \frac{256\zeta^4}{3} + \frac{256\zeta^5}{15} \tag{10}$$

$$\mathcal{Q}_3(\zeta) = -48\zeta^2 - \frac{352\zeta^3}{3} + \frac{800\zeta^4}{3} - \frac{1024\zeta^5}{5} + \frac{1024\zeta^6}{15} - \frac{512\zeta^7}{63} \tag{11}$$

Thus, the semi-analytical solution is given by:

$$\mathcal{Q}_{Approx.}(\zeta) = -4 - 16\zeta - 112\zeta^2 + \frac{64\zeta^3}{3} + \frac{544\zeta^4}{3} - \frac{2816\zeta^5}{15} + \frac{1024\zeta^6}{15} - \frac{512\zeta^7}{63} \tag{12}$$

Table 1 shows the values of analytical and semi-analytical with different values of ζ.

Table 1. Analytical and semi-analytical solutions' value of the FBO equation along with different value of ζ

Value of ζ	Analytical	Semi-analytical	Absolute error
0	–4	–4	0
0.0001	–3.9992	–4.00160112	0.00240112
0.0002	–3.9984	–4.00320448	0.00480448
0.0003	–3.9976	–4.004810079	0.007210079
0.0004	–3.996800001	–4.006417919	0.009617918
0.0005	–3.996000001	–4.008027997	0.012027996
0.0006	–3.995200002	–4.009640315	0.014440313
0.0007	–3.994400004	–4.011254873	0.016854869
0.0008	–3.993600005	–4.012871669	0.019271664
0.0009	–3.992800008	–4.014490704	0.021690697
0.001	–3.992000011	–4.016111978	0.024111968

Conclusion

This paper applies the ESE and AD analytical and semi-analytical schemes to construct novel accurate soliton solutions of the FBO equation for more explanation of the internal waves of deep-stratified fluids. The obtained solutions have been explained through some different plot techniques to describe the dynamical behavior of the considered model, figs. 1-4. The accuracy of solutions has been checked, fig. 5. The applied methods' framework shows their effectiveness and power for implementing some non-linear evolutions equation with integer-order or fractional order.

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