NON-CLASSICAL HEAT EQUATION WITH SINGULAR MEMORY TERM

by

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In this paper, we consider the non-classical heat equation with singular memory term. This equation has many applications in various fields, for example liquids mechanics, solid mechanics, and heat conduction theory first, we prove that the solution exists locally in time. Then we investigate the converegence of the mild solution of non-classical heat equation, and the mild solution of classical heat equation.

Key words: *non-classical heat equation, memory term, local existence, banach fixed point theorem*

Introduction

In this paper, we consider the fractional Sobolev equation:

$$
\begin{gathered}
w_t - \Delta w - k\Delta w_t = F(w) + \int_0^t (t-z)^{-\theta} w(z)\mathrm{d}z,\ (x,t) \in \mathcal{D} \times (0,T) \\
w = 0,\ (x,t) \in \partial\mathcal{D} \times (0,T) \\
w(x,0) = w_0(x)
\end{gathered} \tag{1}
$$

If $k = 0$, problem is called classical heat equation [1-4]. The equation previously described is a special case of the non-classical diffusion equation and has many applications in liquids mechanics, solid mechanics, and heat conduction theory, see for example [5, 9]. Aifantis in [5] showed that the classical reaction-diffusion equation does not include aspects of the reaction problem – diffusion, and it ignores the viscosity, elasticity, and pressure of the environment during solids diffusion, *etc.* He built mathematical models using a variety of concrete examples that could contain elasticity and pressure by the following equation:

$$
w_t - \Delta w - k\Delta w_t = F(w) + g \tag{2}
$$

As we know, in some confounding process, when we study the elasticity of a conductive medium, we need to add fading memory to eq. (2). The important reason that people are interested in studying the equations that contain the term of memory is the speed of energy dissipation for eq. (1) faster than the conventional non-classification diffusion eq. (2). A significant difficulty in examining this equation is the presence of terms $-\Delta u_t$ in eq. (1). Since

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the apperance of this term, it is impossible to apply the compact Sobolev embedding method for eq. (1).

Up to this point, to the best of our knowledge, there has been no mention of this problem (1). The valuable contributions of this paper are described in detail as follows.

– The first result is to prove the existence of local solutions. The main tool is Banach fixed point theorem.
– The second major contribution is the proof that the solution of the problem (1) converges to the solution of the classical heat equation.

The first result is about the existence and unique local solution. The second result is related to the convergence of the solution when the parameter *k* approaches 0.

Preliminaries

For each number $s \geq 0$, we define the following space:

$$\mathbb{H}^s(\mathcal{D}) = \left\{ v = \sum_{j=1}^{\infty} v_j \varphi_j \in L^2(\mathcal{D}) : \|v\|^2_{\mathbb{H}^s(\mathcal{D})} = \sum_{j=1}^{\infty} v_j^2 \lambda_j^{2s} < \infty \right\} \tag{3}$$

By a simple calculation, we get the following ODE with Riemann-Liouville:

$$\frac{\mathrm{d}w_j}{\mathrm{d}t} + \frac{\lambda_j}{1+k\lambda_j} w_j(t) = \frac{1}{1+k\lambda_j} F_j(w) + \frac{1}{1+k\lambda_j} G_j(w)$$

Then we get the following identity:

$$w_j(t) = \exp\left(-\frac{\lambda_j}{1+k\lambda_j} t \right) w_j^0 + \frac{1}{1+k\lambda_j} \int_0^t \exp\left[-\frac{\lambda_j}{1+k\lambda_j}(t-s) \right] F_j(w)(r)\mathrm{d}r + $$
$$+ \frac{1}{1+k\lambda_j} \int_0^t \exp\left[-\frac{\lambda_j}{1+k\lambda_j}(t-s) \right] G_j(w)(r)\mathrm{d}r \tag{4}$$

where

$$v_j(t) = \int_{\mathcal{D}} v(x)\varphi_j(x), \quad F_j(w)(t) = \int_{\mathcal{D}} F[w(t)]\varphi_j(x), G_j(w)(r) = \int_{\mathcal{D}} \int_0^r (r-z)^{-\theta} w(z)\mathrm{d}z \varphi_j(x)$$

Definition 1 The function *w* is called a mild solution of problem (1) if it satisfies:

$$w(t) = \mathcal{M}(t)w^0 + \int_0^t \mathcal{M}(t-r)F[w(r)]\mathrm{d}r + \int_0^t \mathcal{M}(t-r) \int_0^r (r-z)^{-\theta} w(z)\mathrm{d}z\mathrm{d}r \tag{5}$$

where

$$\mathcal{M}(t) f = \sum_{j=1}^{\infty} \exp\left(-\frac{\lambda_j}{1+k\lambda_j} t \right) f_j \varphi_j(x) \ \text{ for any } \ f \in L^2(\mathcal{D}).$$

Lemma 1 Let *f* be the function in $\mathbb{H}^m(\mathcal{D})$. Then:

$$\| \mathcal{M}(t) f \|_{\mathbb{H}^m(\mathcal{D})} \leq \overline{C}_\nu t^{-\nu} \| f \|_{\mathbb{H}^m(\mathcal{D})} \tag{6}$$

for any $0 < \nu < 1$.

Proof. Parseval's equality and the inequality $e^{-z} \le C_\nu z^{-\nu}$ for any $\nu > 0$ allow us to confirm:

$$\| \mathcal{M}(t) f \|^2_{\mathbb{H}^m(\mathcal{D})} = \sum_{j=1}^{\infty} \lambda_j^{2m} \exp\left(-\frac{\lambda_j}{1+k\lambda_j} t\right) | f_j |^2 \le | C_\nu |^2 \sum_{j=1}^{\infty} \lambda_j^{2m} \left(\frac{\lambda_j t}{1+k\lambda_j}\right)^{-2\nu} | f_j |^2 \le$$

$$\le (k+\lambda_1^{-1})^{2\nu} | C_\nu |^2 t^{-2\nu} \sum_{j=1}^{\infty} \lambda_j^{2m} | f_j |^2 = | \overline{C}_\nu |^2 t^{-2\nu} \| f \|^2_{\mathbb{H}^m(\mathcal{D})} \tag{7}$$

This inequality gives the desired result.

Theorem 1 The function *F* satisfies the globally Lipschitz condition:

$$\| F(u) - F(v) \|_{\mathbb{H}^m ⊞D)} \le L_f \| u - v \|_{\mathbb{H}^m ⊞D)} \tag{8}$$

Let $w^0 \in \mathbb{H}^m(D)$ and ν be as $0 < \nu < \min(1, 2-\theta)$. Then problem (1) has a local existence $u \in L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]$ where:

$$\nu < d < \min(1, 2-\theta)$$

Proof. Let us define the following function:

$$\mathcal{P}w(t) = \mathcal{M}(t)w^0 + \int_0^t \mathcal{M}(t-\tau)F[w(\tau)]\mathrm{d}\tau + \int_0^t \mathcal{M}(t-r)\int_0^r (r-z)^{-\theta} w(z)\mathrm{d}z\mathrm{d}r \tag{9}$$

We first give the following estimate:

$$t^d \| \mathcal{M}(t)w^0 \|_{\mathbb{H}^m(\mathcal{D})} \le \overline{C}_\nu t^{d-\nu} \| f \|_{\mathbb{H}^m(\mathcal{D})} \tag{10}$$

Let *u*, *v* be two functions which belong to the space $\mathbb{H}^m(\mathcal{D})$. We need to give the estimation for $\| \mathcal{P}_1 u - \mathcal{P}_1 v \|_{\mathbb{H}^m(\mathcal{D})}$. Indeed, by a simple caculation, we find:

$$\| \mathcal{P}_1 u - \mathcal{P}_1 v \|_{\mathbb{H}^m(\mathcal{D})} = \left\| \int_0^t \mathcal{M}(t-\tau)F[u(\tau)]\mathrm{d}\tau - \int_0^t \mathcal{M}(t-\tau)F[v(\tau)]\mathrm{d}\tau \right\|_{\mathbb{H}^m(\mathcal{D})} \le$$

$$\le \overline{C}_\nu \int_0^t (t-\tau)^{-\nu} \| F[w_1(\tau)] - F[w_2(\tau)] \|_{\mathbb{H}^m(\mathcal{D})} \mathrm{d}\tau \le \overline{C}_\nu L_f \int_0^t (t-\tau)^{-\nu} \| w_1(\tau) - w_2(\tau) \|_{\mathbb{H}^m(\mathcal{D})} \mathrm{d}\tau \tag{11}$$

Multiplying the two sides of the previous inequality by t^d, we get the following estimate:

$$t^d \| \mathcal{P}_1 u - \mathcal{P}_1 v \|_{\mathbb{H}^m(\mathcal{D})} \le \overline{C}_\nu L_f t^d \int_0^t (t-\tau)^{-\nu} \tau^{-d} \tau^d \| w_1(\tau) - w_2(\tau) \|_{\mathbb{H}^m(\mathcal{D})} \mathrm{d}\tau$$

$$\le \overline{C}_{\nu} L_f t^d \left[\int_0^t (t-\tau)^{-\nu} \tau^{-d} \mathrm{d}\tau \right] \| u - v \|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]}$$

$$= \overline{C}_{\nu} L_f t^d t^{1-\nu-d} B(1-\nu, 1-d) \| u - v \|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]}$$

$$\le \overline{C}_{\nu} L_f T^{1-\nu} B(1-\nu, 1-d) \| u - v \|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} \tag{12}$$

where we note that $\nu < 1$ and $d < 1$. The right hand side of (12) is independent of t, so we deduce:

$$\| \mathcal{P}_1 u - \mathcal{P}_1 v \|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} \le \overline{C}_{\nu} L_f T^{1-\nu} B(1-\nu, 1-d) \| u - v \|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} \tag{13}$$

In the following, we continue to show the estimation of $\| \mathcal{P}_2 u - \mathcal{P}_2 v \|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]}$. By a similar previous argumen, we get:

$$\| \mathcal{P}_2 u - \mathcal{P}_2 v \|_{\mathbb{H}^m(\mathcal{D})} =$$

$$= \left\| \int_0^t \mathcal{M}(t-r) \int_0^r (r-z)^{-\theta} u(z) \mathrm{d}z \mathrm{d}r - \int_0^t \mathcal{M}(t-r) \int_0^r (r-z)^{-\theta} v(z) \mathrm{d}z \mathrm{d}r \right\|_{\mathbb{H}^m(\mathcal{D})} \le$$

$$\le \overline{C}_{\nu} \int_0^t (t-r)^{-\nu} \left\| \int_0^r (r-z)^{-\theta} u(z) \mathrm{d}z \mathrm{d}r - \int_0^r (r-z)^{-\theta} v(z) \mathrm{d}z \mathrm{d}r \right\|_{\mathbb{H}^m(\mathcal{D})} \mathrm{d}r \tag{14}$$

It is easy to see that:

$$\left\| \int_0^r (r-z)^{-\theta} u(z) \mathrm{d}z \mathrm{d}r - \int_0^r (r-z)^{-\theta} v(z) \mathrm{d}z \right\|_{\mathbb{H}^m(\mathcal{D})} \le$$

$$\le \int_0^r (r-z)^{-\theta} \| u(z) - v(z) \|_{\mathbb{H}^m(\mathcal{D})} \mathrm{d}z = \int_0^r (r-z)^{-\theta} z^{-d} z^d \| u(z) - v(z) \|_{\mathbb{H}^m(\mathcal{D})} \mathrm{d}z \le$$

$$\le \left[\int_0^r (r-z)^{-\theta} z^{-d} \mathrm{d}z \right] \| u - v \|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} =$$

$$= r^{1-\theta-d} B(1-\theta, 1-d) \| u - v \|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} \tag{15}$$

Combining eqs. (14) and (15), we deduce:

$$\| \mathcal{P}_2 u - \mathcal{P}_2 v \|_{\mathbb{H}^m(\mathcal{D})} \le B(1-\theta, 1-d) \overline{C}_{\nu} \left[\int_0^t (t-r)^{-\nu} r^{1-\theta-d} \mathrm{d}r \right] \| u - v \|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} =$$

$$= B(1-\theta, 1-d) \overline{C}_{\nu} B(1-\nu, 2-\theta-d) t^{2-\nu-\theta-d} \| u - v \|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} \tag{16}$$

Hence, noting that $\nu+\theta\le 2$, we find:

$$t^d \|\mathcal{P}_2 u-\mathcal{P}_2 v\|_{\mathbb{H}^m(\mathcal{D})} \le B(1-\theta,1-d)\overline{C}_\nu B(1-\nu,2-\theta-d)t^{2-\nu-\theta}\|u-v\|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} \le$$

$$\le B(1-\theta,1-d)\overline{C}_\nu B(1-\nu,2-\theta-d)T^{2-\nu-\theta}\|u-v\|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} \tag{17}$$

The right hand side of (17) is independent of *t*, so we deduce:

$$\|\mathcal{P}_2 u-\mathcal{P}_2 v\|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} \le$$

$$\le B(1-\theta,1-d)\overline{C}_\nu B(1-\nu,2-\theta-d)T^{2-\nu-\theta}\|u-v\|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} \tag{18}$$

Combining eqs. (13) and (18), we obtain:

$$\|\mathcal{P}u-\mathcal{P}v\|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} \le \|\mathcal{P}_1 u-\mathcal{P}_1 v\|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} + \|\mathcal{P}_2 u-\mathcal{P}_2 v\|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} \le$$

$$\le \overline{C}_\nu L_f T^{1-\nu} B(1-\nu,1-d)\|u-v\|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} +$$

$$+B(1-\theta,1-d)\overline{C}_\nu B(1-\nu,2-\theta-d)T^{2-\nu-\theta}\|u-v\|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} \tag{19}$$

By choosing *T* small enough, we can immediately see that $\mathcal{P}$ is a contraction operator in space $L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]$. By applying Banach fixed point theorem, we can deduce that $\mathcal{P}$ has a fixed point $u\in L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]$. So, problem (1) has a unique solution in the space $L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]$.

Theorem 2 Let $u^{(k)}$ be the solution of:

$$\begin{gathered} w_t-\Delta w-k\Delta w_t = F(w),\ (x,t)\in\mathcal{D}\times(0,T) \\ w=0,\ (x,t)\in\partial\mathcal{D}\times(0,T) \\ w(x,0)=w^0(x) \end{gathered} \tag{20}$$

and u^* be the solution of the following classical heat problem:

$$\begin{gathered} w_t-\Delta w = F(w),\ (x,t)\in\mathcal{D}\times(0,T) \\ w=0,\ (x,t)\in\partial\mathcal{D}\times(0,T) \\ w(x,0)=w^0(x) \end{gathered} \tag{21}$$

Let us assume that $w^0\in H^{m+h}(\mathcal{D})$ and $F(u^*)\in L^\infty[0,T;H^{m+h}(\mathcal{D})]$. Then for *T* enough small then the following estimate holds:

$$\|u^{(k)}-u^*\|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} \le$$

$$\leq \frac{\overline{C}_{k,h,\nu} T^{d+h-\nu} k^{h/2} \| w^0 \|_{H^{m+h}(\mathcal{D})} + \overline{C}_{k,h,\nu} T^{d} k^{h/2} \| F(u^*) \|_{L^\infty[0,T;H^{m+h}(\mathcal{D})]}}{1-\overline{C}_\nu L_f T^{1-\nu} B(1-\nu,1-d)} \tag{22}$$

where $0 < h < \nu < 1$ and $1 > d > \nu - h$.

Proof. Let us define:

$$u^{(k)}(t) = \mathcal{Q}_k(t) w^0 + \int_0^t \mathcal{Q}_k(t-r) F[u^{(k)}(r)] \mathrm{d}r \tag{23}$$

where

$$\mathcal{Q}_k(t) f = \sum_{j=1}^{\infty} \exp\left(-\frac{\lambda_j}{1+k\lambda_j} t\right) f_j \varphi_j(x),$$

for any $f \in L^2(\mathcal{D})$. The function u^* is defined by:

$$u^*(t) = \mathcal{Q}^0(t) w^0 + \int_0^t \mathcal{Q}^0(t-r) F[u^*(r)] \mathrm{d}r \tag{24}$$

where $\mathcal{Q}^0(t) f = \Sigma_{j=1}^{\infty} \exp(-\lambda_j t) f_j \varphi_j(x)$, for any $f \in L^2(\mathcal{D})$. It is obvious to give a following simple caculation:

$$u^{(k)}(t) - u^*(t) = \mathcal{Q}_k(t) w^0 - \mathcal{Q}^0(t) w^0 + \int_0^t \mathcal{Q}_k(t-r) \{F[u^{(k)}(r)] - F[u^*(r)]\} \mathrm{d}r +$$

$$+ \int_0^t [\mathcal{Q}_k(t-r) - \mathcal{Q}^0(t-r)] F[u^*(r)] \mathrm{d}r \tag{25}$$

Now, using the inequality if $0 < a < b$ then $|e^{-a} - e^{-b}| \leq C_{h,\nu} a^{-\nu} |a-b|^h$ for any $h > 0, \quad \nu > 0$, we can find:

$$\| \mathcal{Q}_k(t) w^0 - \mathcal{Q}^0(t) w^0 \|^2_{\mathbb{H}^m(\mathcal{D})} = \sum_{j=1}^{\infty} \lambda_j^{2m} \left[\exp\left(-\frac{\lambda_j}{1+k\lambda_j} t\right) - \exp(-\lambda_j t) \right]^2 |w_j^0|^2 \leq$$

$$\leq C_{h,\nu}^2 k^{2h} t^{2h-2\nu} \sum_{j=1}^{\infty} \lambda_j^{2m} \left(\frac{\lambda_j}{1+k\lambda_j} \right)^{-2\nu} \left(\frac{\lambda_j^2}{1+k\lambda_j} \right)^{2h} |w_j^0|^2 \leq$$

$$\leq C_{h,\nu}^2 k^{2h} t^{2h-2\nu} \sum_{j=1}^{\infty} \lambda_j^{4h-2\nu+2m} (1+k\lambda_j)^{2\nu-2h} |w_j^0|^2 \tag{26}$$

Since the assumption $h \leq \nu < h+1$ and using the inequality $(c+d)^\sigma \leq c^\sigma + d^\sigma$ for any $0 < \sigma < 1$, we arrive at:

$$(1+k\lambda_j)^{2\nu-2h} \leq (2 + 2k^2 \lambda_j^2)^{\nu-h} \leq (2^{\nu-h} + 2^{\nu-h} k^{2\nu-2h} \lambda_j^{2\nu-2h}) \tag{27}$$

Hence, we find:

$$\sum_{j=1}^{\infty}\lambda_j^{4h-2\nu+2m}(1+k\lambda_j)^{2\nu-2h}|w_j^0|^2\le\sum_{j=1}^{\infty}(2^{\nu-h}\lambda_j^{4h-2\nu+2m}+2^{\nu-h}k^{2\nu-2h}\lambda_j^{4h-2\nu+2m}\lambda_j^{2\nu-2h})|w_j^0|^2\le$$

$$\le C_{k,h,\nu}\sum_{j=1}^{\infty}\lambda_j^{2h+2m}|w_j^0|^2\lesssim\|w^0\|_{H^{m+h}(\mathcal{D})}^2 \tag{28}$$

This implies:

$$\|\mathcal{Q}_k(t)w^0-\mathcal{Q}^0(t)w^0\|_{\mathbb{H}^m(\mathcal{D})}\le\overline{C}_{k,h,\nu}t^{h-\nu}k^{h/2}\|w^0\|_{H^{m+h}(\mathcal{D})} \tag{29}$$

We have the following for second term (*II*):

$$\|(II)\|_{\mathbb{H}^m(\mathcal{D})}=\left\|\int_0^t\mathcal{Q}_k(t-r)\{F[u^{(k)}(r)]-F[u^*(r)]\}\mathrm{d}r\right\|_{\mathbb{H}^m(\mathcal{D})}\le$$

$$\le\overline{C}_\nu\int_0^t(t-\tau)^{-\nu}\|F[u^{(k)}(r)]-F[u^*(r)]\|_{\mathbb{H}^m(\mathcal{D})}\,\mathrm{d}\tau\le$$

$$\le\overline{C}_\nu L_f\int_0^t(t-\tau)^{-\nu}\|u^{(k)}(r)-u^*(r)\|_{\mathbb{H}^m(\mathcal{D})}\,\mathrm{d}r \tag{30}$$

Finally, we treat the third term (*III*):

$$\|(III)\|_{\mathbb{H}^m(\mathcal{D})}=\left\|\int_0^t[\mathcal{Q}_k(t-r)-\mathcal{Q}^0(t-r)]F[u^*(r)]\mathrm{d}r\right\|_{\mathbb{H}^m(\mathcal{D})}\le$$

$$\le\overline{C}_{k,h,\nu}k^{h/2}\int_0^t(t-r)^{h-\nu}\|F[u^*(r)]\|_{H^{m+h}(\mathcal{D})}\,\mathrm{d}r\le\overline{C}_{k,h,\nu}k^{h/2}\|F(u^*)\|_{L^\infty[0,T;H^{m+h}(\mathcal{D})]} \tag{31}$$

Combining eqs. (25), (29), (30), and (31), we get:

$$t^d\|u^{(k)}(t)-u^*(t)\|_{\mathbb{H}^m(\mathcal{D})}\le t^d\|(I)\|_{\mathbb{H}^m(\mathcal{D})}+t^d\|(II)\|_{\mathbb{H}^m(\mathcal{D})}+t^d\|(III)\|_{\mathbb{H}^m(\mathcal{D})}\le$$

$$\le\overline{C}_{k,h,\nu}t^{d+h-\nu}k^{h/2}\|w^0\|_{H^{m+h}(\mathcal{D})}+\overline{C}_{k,h,\nu}t^dk^{h/2}\|F(u^*)\|_{L^\infty[0,T;H^{m+h}(\mathcal{D})]}+$$

$$+\overline{C}_\nu L_f t^d\int_0^t(t-\tau)^{-\nu}\|u^{(k)}(r)-u^*(r)\|_{\mathbb{H}^m(\mathcal{D})}\,\mathrm{d}r\le$$

$$\le\overline{C}_{k,h,\nu}T^{d+h-\nu}k^{h/2}\|w^0\|_{H^{m+h}(\mathcal{D})}+\overline{C}_{k,h,\nu}T^dk^{h/2}\|F(u^*)\|_{L^\infty[0,T;H^{m+h}(\mathcal{D})]}+$$

$$+\overline{C}_\nu L_f t^d \int_0^t (t-r)^{-\nu} \| u^{(k)}(r) - u^*(r) \|_{\mathbb{H}^m(\mathcal{D})} \, \mathrm{d}r \tag{32}$$

where we note that $d + h - \nu > 0.$ It is easy to see:

$$\overline{C}_\nu L_f t^d \int_0^t (t-r)^{-\nu} \| u^{(k)}(r) - u^*(r) \|_{\mathbb{H}^m(\mathcal{D})} \, \mathrm{d}r \le \overline{C}_\nu L_f t^d \left[\int_0^t (t-\tau)^{-\nu} \tau^{-d} \mathrm{d}\tau \right] \| u^{(k)} -$$

$$- u^* \|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} \le \overline{C}_\nu L_f T^{1-\nu} B(1-\nu, 1-d) \| u^{(k)} - u^* \|_{L_d^\infty[0,T;\mathbb{H}^m(\mathcal{D})]} \tag{33}$$

Combining (32) and (33) and after some rearrangement, we can get the desired result.

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