MULTISTAGE OPTIMAL HOMOTOPY ASYMPTOTIC METHOD FOR THE K(2,2) EQUATION ARISING IN SOLITARY WAVES THEORY

by

***Niaz ALI SHAH*[a]*, Imtiaz AHMAD*[a*]*, Hanaa ABU-ZINADAH*[b]*, Khaled MOHAMED KHEDHER*[c,d]*, and Hijaz AHMAD*[e*]**

[a] Department of Mathematics, University of Swabi, Khyber Pakhtunkhwa, Pakistan
[b] Department of Statistics, College of Science, University of Jeddah, Jeddah, Saudi Arabia
[c] Department of Civil Engineering, College of Engineering, King Khalid University, Abha, Saudi Arabia
[d] Department of Civil Engineering, High Institute of Technological Studies, Mrezgua University Campus, Nabeul, Tunisia
[e] Department of Basic Sciences, University of Engineering and Technology, Peshawar, Pakistan

Original scientific paper
https://doi.org/10.2298/TSCI21S2199A

The paper is concern to the approximate analytical solution of K(2,2) using the multistage homotopy asymptotic method which are used in modern physics and engineering. The suggested algorithm is an accurate, effective, and simple to-utilize semi-analytic tool for non-linear problems, and in this manner the current investigation highlights the efficiency and accuracy of the method for the solution of non-linear PDE for large time span. Numerical comparison with the variational iteration method and with homotopy asymptotic method shows the efficacy and accuracy of the proposed method.

Key words: *multistage optimal homotopy asymptotic method, K(2,2) equation*

Introduction

Non-linear PDE are generally utilized in understanding and modeling of a considerable lot of realism matters show up in applied science and material science. Numerous strategies have been developed by the researchers for the non-linear problems wherein the Perturbation procedures Cole [1] were the well-known techniques which depended on the existence of large or small parameters, to be specific the perturbation quantities. Tragically, numerous non-linear problems in physical sciences do not contain such sort of perturbation quantities by any means. To overcome such types of difficulties some non-perturbative procedures which are free of small parameters are proposed in Adomian [2]. Be that as it may, both perturbative and non-perturbative strategies could not give a straightforward method to adjust or control the rate and region of convergence of approximate series Liao [3]. To defeat this trouble, another analytical technique was proposed by [4, 5], known as the optimal homotopy asymptotic method (OHAM) which has been effectively employed to numerous non-linear problems in heat transfer and fluid mechanics Marinca and Herisanu [4].

* Corresponding authors, e-mail: hijaz555@gmail.com, imtiazkakakhil@gmail.com

The OHAM is an approximate analytical method which can be used with ease and having a built in convergence criteria like to homotopy analysis method (HAM) however with more degree of flexibility. Many authors have shown that the suggested procedure is accurate and reliable, and calculated the solutions of complex problems which have significant applications in science and technology, [4, 5]. In view of our observations, the OHAM solution with the easiest and simplest auxiliary function of the form $H_i(q)$ for the initial value problems is valid for a short time span. Accordingly, to circumvent this limitation, a new modification is made which is based on the standard OHAM and called it the multistage homotopy asymptotic method (MOAHM) which insure the validity of the approximations of large time in easy way.

On the other hand, the K(n,n) equation was first appeared in Rosenau *et al.* [6] which is the fundamental equation for compactons. Compactons are characterized as solutions free of exponential tails or solutions with finite wavelengths [6] in solitary waves theory. The purpose of the current research is to utilized efficiently the MOHAM to find out the approximate analytical solutions for the K(2,2) equations.

Description of MOHAM

This section is devoted to the basic principles of the OHAM as given in [4, 5]. Consider the initial-value problem:

$$L_i[U_i(y,\tau)] + N_i[U_i(y,\tau)] = 0, \quad i = 1,2,...,N \tag{1}$$

with initial condition as:

$$U_i(y,\alpha) = \alpha_i \tag{2}$$

where $U_i(y, \tau)$ represent the unknown function whereas L_i and N_i denote the linear operator, non-linear operator, respectively, additionally, y and τ are the independent variables. A homotopy map $h_i[v_i(y,\tau,q),q]: R\times[0,1]\to R$ which satisfies:

$$(1-q)\{L_i[v_i(y,\tau,q)] - U_{i,0}(\tau)\} = Hi(q,\tau)\{L_i[v_i(y,\tau,q)] + N_i[v_i(y,\tau,q)]\} \tag{3}$$

can be constructed. Here $q\in[0,1]$ and $y,\tau\in R$ where $H_i(q)\neq 0$ is an auxiliary function. The $H_i(0) = 0$ for $q = 0$, and $v_i(y,\tau,q)$ is an unknown function. It is understood that $v_i(y,\tau,0) = U_{i,0}(\tau)$ holds for $q = 0$ and $v_i(y,\tau,0) = U_{i,0}(\tau)$ holds for $q = 1$. In the same manner q changes from 0 to 1, the solution $v_i(y,\tau,q)$ changes from $U_{i,0}(y,t)$ to $U_i(y,\tau)$ where $U_{i,0}(y,\tau)$ is the initial guess which is known and calculated from eq. (2) for $q = 0$:

$$L_i[U_{i,0}(y,\tau)] = 0 \tag{4}$$

Now, the auxiliary function $H_i(q)$ has been chosen in the following manner:

$$H_i(q) = C_{1,j}q + C_{2,j}q^2 + C_{3,j}q^3 + \ldots \quad \text{or} \quad H_i(q,\tau) = C_{1,j}q + C_{2,j}\tau q^2 + C_{3,j}\tau^2 q^3 + \ldots \tag{5}$$

where $C_{1,j}, C_{2,j}, C_{3,j},\ldots$ denote convergence control parameters (CCP) and can be find out later. In order to find the required approximate solution, the Taylor's series are utilized in the accompanying form to expand $v_i(y,\tau,q,C_k)$ about q:

$$v_i(y,\tau,q,C_k) = U_0(y,\tau) + \sum_{k=1}^{\infty} U_{i,k}(y,\tau,C_1,C_2,\ldots,C_k)q^k \tag{6}$$

Define the vectors:

$$\vec{C}_i = \{C_1, C_2, \ldots, C_i\}, \ \vec{U}_{i,s} = \{U_{i,0}(y,\tau), U_{i,1}(y,\tau,C_1), \ldots, U_{i,s}(y,\tau,\vec{C}_s)\}$$

where $s = 1, 2, 3, \ldots,$ setting eq. (6) into eq. (3) and to the linear equations which are given below, we proceed by comparing coefficient q, Also, the zeroth-order problem is given by eq. (4) whereas the first- and second-order problems are:

$$L_i[U_{i,1}(y,\tau)] = C_1 N_0(\vec{U}_{i,0}), \quad U_{i,1}(a) = 0, \quad \text{and} \tag{7}$$

$$L_i[U_{i,2}(y,\tau)] - L_i[U_{i,1}(\tau)] = C_2 N_{i,0}(\vec{U}_{i,0}) + C_{1,j}\{L_i[U_{i,1}(y,\tau)] + N_{i,1}(\vec{U}_{i,1})\}, \quad U_{i,2}(a) = 0$$

The general equations for $U_{i,k}(\tau)$ are:

$$L_i[U_{i,k}(y,\tau)] - L_i[U_{i,k-1}(y,\tau)] = C_{k,j} N_{i,0}[U_{i,0}(\tau)] + \sum_{m=1}^{k-1} C_{i,m}\{L_i[U_{i,k-m}(y,\tau)] +$$

$$+ N_{i,1}(\vec{U}_{i,k-1})\}, \quad U_{i,k}(a) = 0 \tag{8}$$

where $k = 2, 3, \ldots$ and $N_{i,m}[U_0(y,\tau), U_{i,1}(y,\tau), \ldots, U_{i,m}(y,\tau)]$ is the coefficient of q^m in the expansion of $N_i[v_i(y,\tau,q)]$ about q which is known as embedding parameter:

$$N_i[v_i(y,\tau,q)] = N_{i,0}[U_{i,0}(y,\tau)] + \sum_{m=1}^{\infty} N_{i,m}(\vec{U}_{i,m}) q^m \tag{9}$$

As it is notice that the convergence of the series given in eq. (9) heavily depends on the convergence control parameters $C_1, C_2, C_3, \ldots,$ if it is convergent at $q = 1$, then:

$$v_i(y,\tau,C_k) = U_{i,0}(\tau) + \sum_{k=1}^{\infty} U_k(y,\tau,C_1,C_2,\ldots,C_k). \tag{10}$$

The result of the m^{th}-order approximation is:

$$\tilde{U}(y,\tau,C_1,C_2,C_3,\ldots,C_k) = U_{y,0}(y,\tau) + \sum_{k=1}^{\infty} U_k(y,\tau,C_1,C_2,\ldots,C_k) \tag{11}$$

Substituting eq. (11) into eq. (1) gives the accompanying residual:

$$R_i(y,\tau,C_{1,j},C_{2,j},C_{3,j},\ldots,C_{m,j}) = L[\tilde{U}_i(y,\tau,C_{1,j},C_{2,j},C_{3,j},\ldots,C_{m,j})] +$$

$$+ N[\tilde{U}_i(y,\tau,C_{1,j},C_{2,j},C_{3,j},\ldots,C_{m,j})] \tag{12}$$

where $U(y,\tau)$ will represent the exact solution when $R_i = 0$. It is noticed that such a case will not happen for non-linear problems, yet we can limit the function:

$$J_i(C_{1,j},C_{2,j},C_{3,j},\ldots,C_{m,j}) = \int_{\tau_j}^{\tau_{j+h}} R_i^2(y,\tau,C_{1,j},C_{2,j},C_{3,j},\ldots,C_{m,j})\,\mathrm{d}\tau \tag{13}$$

where the length and the number of subintervals (τ_j, τ_{j+1}) is denoted by h and $N = T/h$, respectively. Next, changing the initial approximation in each subinterval from the previous one, we can solve eq. (13) at $j = 0,1,\ldots,N.$ For instant, we define $\alpha = U(\tau_j)$ in the subinterval $(\tau_j, \tau_{j+1}).$ The unknown convergence control parameters $C_{i,j}(i = 1, 2, 3, m, j = 1, 2, N)$ can be determined from the solution of the below given system of equations:

$$\frac{\partial J}{\partial C_{1,j}} = \frac{\partial J}{\partial C_{2,j}} = \ldots = \frac{\partial J}{\partial C_{m,j}} = 0 \tag{14}$$

and hence, the approximate analytic solution will be:

$$\tilde{U}(y,\tau) = \begin{cases} U_1(y,\tau), & \tau_0 \le \tau < \tau_1 \\ U_2(y,\tau), & \tau_1 \le \tau < \tau_2 \\ \vdots \\ U_N(y,\tau), & \tau_{N-1} \le \tau < T \end{cases} \tag{15}$$

Proceeding with thusly, we effectively calculate the initial value problems' solution analytically for large value of T. It merits referencing that the MOHAM convert to the standard OHAM when $j = 0$ It is also essential to mention that MOHM gives an easy way to adjust and control the convergence region by means of the auxiliary function $H_i(q)$ involving many CCP $C_{i,j}$'s. Then again, the proposed method overcomes the main difficulty, due to the large computational domain, in calculating the solution of problems.

Implementation of proposed scheme

The suggested MOHAM is implemented in the section to the K(2,2) equation to show the effectiveness and validity of the algorithm, furthermore, the initial-boundary conditions can be computed easily in accordance to the exact solution throughout the paper.

Test Problem 1. Now, we consider an important equation, namely K(2,2) [7]:

$$u_\tau + (u^2)_y + (u^2)_{yyy} = 0 \tag{16}$$

with analytical/exact solution $u(y,\tau) = y/(1+2\tau).$

To solve the problem (16), we select the linear and non-linear operators:

$$L[u(y,\tau,q)] = u_\tau(y,\tau,q), \quad N[u(y,\tau,q)] = u_{yy}^2(y,\tau,q) + u_{yyy}^2(y,\tau,q) \tag{17}$$

The auxiliary function $H_i(q)$ is taken in the form $H_i(q) = (C_{1,j}q + C_{2,j}q^2)$ where $C_{1,j}, C_{2,j}$ are unknown to be computed. Using the producer as described in section *Description of MOHAM* by taking step-size $h = 0.1$ and starting with $\tau_0 = 0$ to $\tau_{10} = T = 1.$ Various order initial value problems and their solutions as follows.

The 0$^{\text{th}}$ order problem:

$$\frac{\partial u_0}{\partial \tau} = y, \quad u_0(y,0) = y \tag{18}$$

Their solution:

$$u_0(y,\tau) = y + \tau y \tag{19}$$

The 1st order problem:

$$\frac{\partial u_1}{\partial \tau} = \frac{\partial u_0}{\partial \tau} + C_{1,j}\frac{\partial u_0}{\partial \tau} + C_{1,j}\frac{\partial u_0^2}{\partial y} + C_{1,j}\frac{\partial^3 u_0^2}{\partial y^3} - y, \quad u_1(y,0) = 0 \tag{20}$$

with solution:

$$u_1(y,\tau,C_{1,j}) = \frac{1}{3}C_{1,j}(9\tau + 6\tau^2 + 2\tau^3)y \tag{21}$$

The 2nd order problem:

$$\frac{\partial u_2}{\partial \tau} = C_{2,j}\frac{\partial u_0}{\partial \tau} + (1 + C_{1,j})\frac{\partial u_1}{\partial \tau} + C_{2,j}\frac{\partial u_0^2}{\partial y} + C_{2,j}\frac{\partial^3 u_0^2}{\partial y^3} +$$

$$+2C_{1,j}\frac{\partial u_0}{\partial y}\frac{\partial u_1}{\partial y} + 2C_{1,j}\frac{\partial^3 u_0}{\partial y^3}\frac{\partial^3 u_1}{\partial y^3}, u_2(y,0) = 0 \tag{22}$$

Their solution:

$$u_2(y,\tau,C_{1,j},C_{2,j}) = \frac{1}{15}\tau(45C_{1,j}^2\tau + 50C_{1,j}^2\tau^2 + 20C_{1,j}^2\tau^3 + 4C_{1,j}^2\tau^4 +$$

$$+45C_{1,j}y + 45C_{1,j}^2 y + 45C_{2,j}y + 30C_{1,j}\tau y + 30C_{1,j}^2\tau y +$$

$$+30C_{2,j}\tau y + 10C_{1,j}\tau^2 y + 10C_{1,j}^2\tau^2 y + 10C_{2,j}\tau^2 y) \tag{23}$$

Hence, second order MOHAM solution for the first subinterval can be obtain from:

$$\tilde{u}(y,\tau,C_{1,j},C_{2,j}) = u_0(y,\tau) + u_1(y,\tau,C_{1,j}) + u_2(y,\tau,C_{1,j},C_{2,j}) \tag{24}$$

which is:

$$\tilde{u}(y,\tau,C_{1,j},C_{2,j}) = y + \tau y + \frac{1}{3}C_{1,j}(9\tau + 6\tau^2 + 2\tau^3)y + \frac{1}{15}\tau(45C_{1,j}^2\tau + 50C_{1,j}^2\tau^2 +$$

$$+20C_{1,j}^2\tau^3 + 4C_{1,j}^2\tau^4 + 45C_{1,j}y + 45C_{1,j}^2 y + 45C_{2,j}y + 30C_{1,j}\tau y +$$

$$+30C_{1,j}^2\tau y + 30C_{2,j}\tau y + 10C_{1,j}\tau^2 y + 10C_{1,j}^2\tau^2 y + 10C_{2,j}\tau^2 y) \tag{25}$$

Following the procedure given in [8], we obtain the values of the CCP which are tabulated in tab. 1. These values are used in eq. (25) to get second order MOHAM solution of eq. (16) in the first subinterval. Similar procedure is adopted for the remaining subintervals.

Table 1. The CCP $C_{i,j}$ for *Test Problem 1*

j	$C_{1,j}$	$C_{2,j}$
1	0	–0.8309351103910446
2	$4.7685619794052564 \times 10^{-15}$	–0.8545224795933848
3	0	–0.8723413872225191
4	0	–0.8862792717229484
5	0	–0.8974795903194425
6	0	–0.9066763731169916
7	0	–0.9143626709484041
8	0	–0.9208820232404521
9	–1.5934646181046106	–0.3926955845326988
10	0	–0.9327887884711599

A comparison between the solution obtained from second-order MOHAM in term of the absolute error with nine iterations of VIM and with 8th order HAM for various times are tabulated in tab. 2. Table 2 shows that the absolute errors obtained from nine iterations solution of VIM and 8th order solution of HAM grow faster for large time span. On the other hand MOHAM has retained its accuracy evev for large time span. Thus MOHAM is more better than VIM and HAM.

Table 2. Results of the second order MOHAM for *Test Problem 1*

τ	y	Exact	MOHAM (present method)	Absolute error (MOHAM)	Absolute error (VIM)	Absolute error (HAM)
0	0	0	0	0	0	0
0.1	0.1	0.071428	0.071465	$3.7 \cdot 10^{-5}$	$1.8 \cdot 10^{-9}$	$4.3 \cdot 10^{-8}$
0.2	0.2	0.142857	0.142932	$7.4 \cdot 10^{-5}$	$5.6 \cdot 10^{-6}$	$3.7 \cdot 10^{-5}$
0.3	0.3	0.187500	0.187640	$1.4 \cdot 10^{-4}$	$6.8 \cdot 10^{-4}$	$1.9 \cdot 10^{-3}$
0.4	0.4	0.222222	0.222428	$2.1 \cdot 10^{-4}$	$1.9 \cdot 10^{-2}$	$2.9 \cdot 10^{-2}$
0.5	0.5	0.250000	0.250266	$2.7 \cdot 10^{-4}$	$2.5 \cdot 10^{-1}$	$2.5 \cdot 10^{-1}$
0.6	0.6	0.272727	0.273047	$3.2 \cdot 10^{-4}$	$2.1 \cdot 10^{00}$	$1.4 \cdot 10^{00}$
0.7	0.7	0.291667	0.292033	$3.7 \cdot 10^{-4}$	$1.2 \cdot 10^{01}$	$6.0 \cdot 10^{00}$
0.8	0.8	0.307692	0.308099	$4.1 \cdot 10^{-4}$	$5.4 \cdot 10^{01}$	$2.1 \cdot 10^{01}$
0.9	0.9	0.321429	0.321882	$4.5 \cdot 10^{-4}$	$2.1 \cdot 10^{02}$	$6.4 \cdot 10^{01}$
1.0	1.0	0.333333	0.333561	$2.3 \cdot 10^{-4}$	$6.8 \cdot 10^{03}$	$7.1 \cdot 10^{01}$

Conclusion

In this paper, the MOHAM is used to obtain the analytical approximate solutions of the K(2,2) equation. The comparison between the proposed MOHAM and VIM was made for the K(2,2) equation and in light of approximate results, it was found that MOHAM is more effective than VIM. One of the beauty of the MOHAM is easy and straightforward calculations and secondly, the reduction in the size of computational domain. Additionally, the suggested method gives a helpful way of controlling the convergence region of the series solution. Accomplished results demonstrate that MOHAM is a accurate and efficient for calculating approximate analytical solution of the partial differential equations which utilized scientific material science and engineering.

Acknowledgment

The Authors extend their thanks to the Deanship of Scientific Research at King Khalid University for funding this work.

References

[1] Cole, J. D., *Perturbation Methods in Applied Mathematics*, Blaisdell Publishing Company, Waltham, Mass., USA, 1968
[2] Adomian, G., *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Boston, Mass., USA, 1994
[3] Liao, S. J., On the Homotopy Analysis Method for Non-linear Problems, *Appl. Math. Comput., 147* (2004), 2, pp. 499-513
[4] Marinca, V., Herisanu, N., Application of Optimal Homotopy Asymptotic Method for Solving Non-linear Equations Arising in Heat Transfer, *Int. Commun. Heat Mass Transf., 35* (2008), 6, pp. 710-715
[5] Marinca, V., *et al.* An Optimal Homotopy Asymptotic Method Applied to the Steady Flow of a Fourth-Grade Fluid Past a Porous Plate, *Appl. Math. Lett., 22* (2009), 2, pp. 245-251
[6] Rosenau, P., Hyman, J. M., Compactons: Solitons with Finite Wavelengths, *Phys. Rev. Lett., 70* (1993), 5, pp. 564-567
[7] Alomari, A. K., *et al.* The Homotopy Analysis Method for the Exact Solutions of the K(2,2), Burgers' and Coupled Burgers Equations, *Appl. Math. Sci., 2* (2008), 40, pp. 1963-1977
[8] Shah, N. A., *et al.* Multistage Optimal Homotopy Asymptotic Method for the Non-linear Riccati Ordinary Differential Equation in Non-linear Physics, *Appl. Math. Inf. Sci. (AMIS), 14* (2020), 6, pp. 1-7

Paper submitted: March 5, 2021
Paper revised: March 26, 2021
Paper accepted: April 5, 2021