SOLITON SOLUTIONS FOR NON-LINEAR KUDRYASHOV'S EQUATION VIA THREE INTEGRATING SCHEMES

by

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Original scientific paper
https://doi.org/10.2298/TSCI21S2157A

This paper considers the non-linear Kudryashov's equation, that is an extension of the well-known dual-power law of refractive index and is analog to the generalized version of anti-cubic non-linearity. The model is considered in the presence of full non-linearity. The main objective of this paper is to extract soliton solutions of the proposed model. Three state-of-the-art integration schemes, namely modified auxiliary equation method, the sine-Gordon expansion method and the tanh-coth expansion method have been employed for obtaining the desired soliton solutions.

Key words: *soliton solution, Kudryashov's equation, analytical methods*

Introduction

In this paper, the non-linear Kudryashov's equation has been discussed for extraction of soliton solutions. Kudryashov himself discussed the Kudryashov equation in [1]. The dimensionless form of Kudryashov's equation (KE) is:

$$ic_t + ac_{xx} + (d_1 |c|^{-2n} + d_2 |c|^{-n} + d_3 |c|^n + d_4 |c|^{2n})c = 0 \tag{1}$$

In eq. (1), the first term represents the linear temporal evolution. The second term accounts for group velocity dispersion. The coefficients $a, d_1, d_2, d_3,$ and d_4 are real parameters and n is an arbitrary degree of non-linearity. The $d_1, d_2, d_3,$ and d_4 are the coefficients of non-linear terms which occur from the law of refractive index of an optical fiber and give self-phase modulation to the model. When $n = 1$ and $d_1 = d_2 = d_3 = 0,$ then eq. (1) reduces to non-linear Schrödinger equation (NLSE). When $n = 2$ and $d_3 = 0,$ then eq. (1) is the NLSE with anti-cubic non-linearity. If $d_1 = d_2 = 0$, eq. (1) collapses to dual-power law of refractive index. If

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$d_1 = d_2 = d_3 = 0,$ then eq. (1) collapses to power law of refractive index. Equation (1) is also termed as the generalized version of anti-cubic non-linearity [2]. Recently, eq. (1) has been studied in [1], where the bright, dark and singular optical soliton solutions have been recovered by the aid of first integrals.

Over the past thirty years, a large number of approaches and methods have been proposed to find exact solutions to non-linear differential equations. In our opinion some effective methods for construction exact solutions are given in [3-6]. The purpose of this paper is to find periodic and solitary pulses described by eq. (1) and to study the effect of the non-linearity power *n* on the shape of the pulse. Three state-of-the-art integration schemes, namely modified auxiliary equation (MAE) method [7], the sine-Gordon expansion method [8] and the tanh-coth expansion method [9] have been employed for obtaining the desired soliton solutions.

Proposed model

Equation (1) can be solved using the following traveling wave transformation $c(x,t) = C(\tau)e^{i\varphi(x,t)},\ \tau = x - vt,\ \varphi(x,t) = \mu x - \rho t,$ where $C(\tau)$ represents the shape of the pulse, $\varphi(x,t)$ is the phase component and $v,\ \mu,$ and ρ are the velocity, frequency, and wave number of the soliton, respectively.

On substituting the transformation in eq. (1), we obtain:

$$\begin{gathered} \iota vC^{2n}C' + \rho C^{2n}C + aC^{2n}C'' + 2\iota a\mu C^{2n}C' - a\mu^2 C^{2n}y + \\ + d_1 C + d_2 C^n C + d_3 C^{3n} C + d_4 C^{4n} C = 0 \end{gathered} \tag{2}$$

Applying the transformation $C = V^{1/n},$ to eq. (2), the closed form solution is obtained. Separating real and imaginary parts, gives the following expressions:

$$anVV'' - a(n-1)V'^2 + n^2(\rho - a\mu^2)V^2 + d_1 n^2 + d_2 n^2 V + d_3 n^2 V^3 + d_4 n^2 V^4 = 0 \tag{3}$$

The imaginary part gives the velocity of the soliton as $v = 2d_1$.

Next, eq. (3) will be solved using the proposed algorithms as under to find soliton solutions for eq. (1).

Applied analytical techniques

Three analytical techniques have been applied on the proposed model to extract soliton solutions.

The modified auxiliary equation method

Applying the balancing principle on eq. (3) gives $r = 1.$ We have:

$$V(\tau) = a_0 + a_1(z^g) + b_1(z^g)^{-1} \tag{4}$$

where a_0, a_1, and b_1 are unknowns to be evaluated.

The following sets of solutions have been obtained using MAE method [7].

Set 1

$$a_0 = -\frac{d_3(1+n)n + \sqrt{-ad_4(1+n)}(2+n)\gamma}{2d_4n(2+n)}, \quad a_1 = 0, \quad b_1 = -\frac{\sqrt{-a(1+n)}\delta}{\sqrt{d_4}n}$$

$$\rho = -\frac{3d_3^2n^2(1+n) + ad_4(2+n)^2(n2\mu^2n^2 + \gamma^2 - 4\delta\epsilon)}{2d_4n^2(2+n)^2}$$

Set 2

$$a_0 = -\frac{d_3(1+n)n + \sqrt{-ad_4(1+n)}(n+2\gamma)}{2d_4n(2+n)}, \quad a_1 = \frac{\sqrt{-a(1+n)}\epsilon}{\sqrt{d_4}n}, \quad b_1 = 0$$

$$\rho = -\frac{3d_3^2n^2(1+n) + ad_4(2+n)^2(2\mu^2n^2 + \gamma^2 - 4\delta\epsilon)}{2d_4n^2(2+n)^2}$$

The obtained soliton solutions for Set 1 are given below.

Family 1. When $W = \gamma^2 - 4\delta\epsilon < 0$ and $\epsilon \neq 0$ then:

$$c_1(x,t) = \mathrm{e}^{i\varphi(x,t)}\left\{-\frac{d_3(1+n)n + \sqrt{-ad_4(1+n)}(2+n)\gamma}{2d_4n(2+n)} - \frac{\sqrt{-a(1+n)}\delta}{\sqrt{d_4}n}\left[\frac{-\gamma + \sqrt{-W}\tan\left(\frac{\sqrt{W}\tau}{2}\right)}{2\epsilon}\right]^{-1}\right\}^{\frac{1}{n}} \tag{5}$$

or

$$c_2(x,t) = \mathrm{e}^{i\varphi(x,t)}\left\{-\frac{d_3(1+n)n + \sqrt{-ad_4(1+n)}(2+n)\gamma}{2d_4n(2+n)} - \frac{\sqrt{-a(1+n)}\delta}{\sqrt{d_4}n}\left[-\frac{\gamma + \sqrt{-W}\cot\left(\frac{\sqrt{W}\tau}{2}\right)}{2\epsilon}\right]^{-1}\right\}^{\frac{1}{n}} \tag{6}$$

Family 2. When $W = \gamma^2 - 4\delta\epsilon > 0$ and $\epsilon \neq 0$ then:

$$c_3(x,t) = \mathrm{e}^{i\varphi(x,t)}\left\{-\frac{d_3(1+n)n + \sqrt{-ad_4(1+n)}(2+n)\gamma}{2d_4n(2+n)} - \frac{\sqrt{-a(1+n)}\delta}{\sqrt{d_4}n}\left[-\frac{\gamma + \sqrt{W}\tanh\left(\frac{\sqrt{W}\tau}{2}\right)}{2\epsilon}\right]^{-1}\right\}^{\frac{1}{n}} \tag{7}$$

or

$$c_4(x,t) = e^{i\varphi(x,t)} \left\{ -\frac{d_3(1+n)n + \sqrt{-ad_4(1+n)}(2+n)\gamma}{2d_4 n(2+n)} - \frac{\sqrt{-a(1+n)}\delta}{\sqrt{d_4}n} \left[-\frac{\gamma + \sqrt{W}\coth\left(\frac{\sqrt{W}\tau}{2}\right)}{2\epsilon} \right]^{-1} \right\}^{\frac{1}{n}} \tag{8}$$

Family 3. When $\gamma^2 - 4\delta\epsilon = 0$ and $\epsilon \neq 0$ then:

$$c_5(x,t) = e^{i\varphi(x,t)} \left[-\frac{d_3(1+n)n + \sqrt{-ad_4(1+n)}(2+n)\gamma}{2d_4 n(2+n)} - \frac{\sqrt{-a(1+n)}\delta}{\sqrt{d_4}n} \left(-\frac{2+\gamma\tau}{2\epsilon\tau} \right)^{-1} \right]^{\frac{1}{n}} \tag{9}$$

The obtained soliton solutions for Set 2 are given below.
Family 1. When $W = \gamma^2 - 4\delta\epsilon < 0$ and $\epsilon \neq 0$ then:

$$c_6(x,t) = e^{i\varphi(x,t)} \left\{ -\frac{d_3(1+n)n + \sqrt{-ad_4(1+n)}(n+2\gamma)}{2d_4 n(2+n)} + \frac{\sqrt{-a(1+n)}}{\sqrt{d_4}n} \left[\frac{-\gamma + \sqrt{-W}\tan\left(\frac{\sqrt{W}\tau}{2}\right)}{2} \right] \right\}^{\frac{1}{n}} \tag{10}$$

or

$$c_7(x,t) = e^{i\varphi(x,t)} \left\{ -\frac{d_3(1+n)n + \sqrt{-ad_4(1+n)}(n+2\gamma)}{2d_4 n(2+n)} + \frac{\sqrt{-a(1+n)}}{\sqrt{d_4}n} \left[-\frac{\gamma + \sqrt{-W}\cot\left(\frac{\sqrt{W}\tau}{2}\right)}{2} \right] \right\}^{\frac{1}{n}} \tag{11}$$

Family 2. When $W=\gamma^2-4\delta\epsilon>0$ and $\epsilon\neq 0$ then.

$$c_8(x,t)=\mathrm{e}^{i\varphi(x,t)}\left\{-\frac{d_3(1+n)n+\sqrt{-ad_4(1+n)}(n+2\gamma)}{2d_4n(2+n)}+\frac{\sqrt{-a(1+n)}}{\sqrt{d_4}n}\left[-\frac{\gamma+\sqrt{W}\tanh\left(\dfrac{\sqrt{W}\tau}{2}\right)}{2}\right]\right\}^{\frac{1}{n}} \tag{12}$$

or

$$c_9(x,t)=\mathrm{e}^{i\varphi(x,t)}\left\{-\frac{d_3(1+n)n+\sqrt{-ad_4(1+n)}(n+2\gamma)}{2d_4n(2+n)}+\frac{\sqrt{-a(1+n)}}{\sqrt{d_4}n}\left[-\frac{\gamma+\sqrt{W}\coth\left(\dfrac{\sqrt{W}\tau}{2}\right)}{2}\right]\right\}^{\frac{1}{n}} \tag{13}$$

Family 3. When $W=\gamma^2-4\delta\epsilon=0$ and $\epsilon\neq 0$ then:

$$c_{10}(x,t)=\mathrm{e}^{i\varphi(x,t)}\left[-\frac{d_3(1+n)n+\sqrt{-ad_4(1+n)}(n+2\gamma)}{2d_4n(2+n)}+\frac{\sqrt{-a(1+n)}}{\sqrt{d_4}n}\left(-\frac{2+\gamma\tau}{2\tau}\right)^{-1}\right]^{\frac{1}{n}} \tag{14}$$

These solutions exist if $a<0$ and $d_4(1+n)>0$.

The sine-Gordon expansion method

Balancing VV'' and V^4 in eq. (3) yields the balance number $m=1$ By taking $m=1$ eq. (3) has solution of the form:

$$V(\tau)=a_0+a_1\cos(w)+b_1\sin(w) \tag{15}$$

where $a_0, a_1,$ and b_1 are to be determined. Substituting eq. (15) along with derivatives in eq. (3), a system of equations is obtained by equating the coefficients of same powers of $\sin(w)\cos(w)$, to zero. After solving the system of equations, the following sets of solutions are obtained.

Set 1.

$$a_0=-\frac{d_3(n+1)}{2d_4(n+2)},\quad a_1=0,\quad b_1=\pm\frac{1}{n}\sqrt{\frac{a(n+1)}{d_4}}$$

$$\rho=\frac{3d_3^2n^2(n+1)+2ad_4(n+2)^2(n^2\mu^2-1)}{2d_4n^2(n+2)^2}$$

Set 2.

$$a_0 = -\frac{d_3(n+1)}{2d_4(n+2)}, \quad a_1 = \pm\frac{i}{2n}\sqrt{\frac{a(t+1)}{d_4}}, \quad b_1 = \pm\frac{1}{2n}\sqrt{\frac{a(n+1)}{d_4}}$$

$$\rho = \frac{3d_3^2 n^2(n+1) + ad_4(n+2)^2(2n^2\mu^2+1)}{2d_4 n^2(n+2)^2}$$

The bright soliton solution corresponding to Set 1 is obtained:

$$c_{11}(x,t) = \mathrm{e}^{i\varphi(x,t)}\left[-\frac{d_3(n+1)}{2d_4(n+2)} \pm \frac{1}{n}\sqrt{\frac{a(n+1)}{d_4}}\operatorname{sech}(\tau)\right]^{\frac{1}{n}} \tag{16}$$

The complexiton soliton solution corresponding to Set 2 is obtained:

$$c_{12}(x,t) = \mathrm{e}^{i\varphi(x,t)}\left[-\frac{d_3(n+1)}{2d_4(n+2)} \pm \frac{i}{2n}\sqrt{\frac{a(n+1)}{d_4}}\tanh(\tau) \pm \frac{1}{2n}\sqrt{\frac{a(n+1)}{d_4}}\operatorname{sech}(\tau)\right]^{\frac{1}{n}} \tag{17}$$

These solutions exist if $a > 0$ and $d_4(1+n) > 0$.

The tanh-coth expansion method

Applying the balancing principle on eq. (3) gives $r = 1$ We have:

$$V(\tau) = f(Y) = a_0 + a_1 Y + b_1 Y^{-1} \tag{18}$$

where a_0, a_1, and b_1 are unknowns to be evaluated. Using the proposed method as discussed in *The tanh-coth expansion method* section. The following sets of solutions have been obtained.

Set 1.

$$a_0 = -\frac{d_3(1+n)}{2d_4(2+n)}, \quad a_1 = -\frac{\sqrt{-a(1+n)}\xi}{\sqrt{d_4}n}, \quad b_1 = 0, \quad \rho = \frac{3d_3^2(1+n)}{2d_4(2+n)^2} + a\left(\mu^2 + \frac{2\xi^2}{n^2}\right)$$

Set 2.

$$a_0 = -\frac{d_3(1+n)}{2d_4(2+n)}, \quad a_1 = 0, \quad b_1 = -\frac{\sqrt{-a(1+n)}\xi}{\sqrt{d_4}n}, \quad \rho = \frac{3d_3^2(1+n)}{2d_4(2+n)^2} + a\left(\mu^2 + \frac{2\xi^2}{n^2}\right)$$

Set 3.

$$a_0 = -\frac{d_3(1+n)}{2d_4(2+n)}, \quad a_1 = -\frac{\sqrt{-a(1+n)}\xi}{\sqrt{d_4}n}, \quad b_1 = -\frac{\sqrt{-a(1+n)}\xi}{\sqrt{d_4}n}$$

$$\rho = \frac{3d_3^2(1+n)}{2d_4(2+n)^2} + a\left(\mu^2 + \frac{8\xi^2}{n^2}\right)$$

The solution corresponding to Set 1 is:

$$c_{13}(x,t)=\mathrm{e}^{i\varphi(x,t)}\left[-\frac{d_3(1+n)}{2d_4(2+n)}-\frac{\sqrt{-a(1+n)}\xi}{\sqrt{d_4}n}\tanh(\xi\tau)\right]^{\frac{1}{n}} \tag{20}$$

The solution corresponding to Set 2 is:

$$c_{14}(x,t)=\mathrm{e}^{i\varphi(x,t)}\left[-\frac{d_3(1+n)}{2d_4(2+n)}-\frac{\sqrt{-a(1+n)}\xi}{\sqrt{d_4}n}\coth(\xi\tau)\right]^{\frac{1}{n}} \tag{21}$$

The solution for Set 3 is:

$$c_{15}(x,t)=\mathrm{e}^{i\varphi(x,t)}\left[-\frac{d_3(1+n)}{2d_4(2+n)}-\frac{\sqrt{-a(1+n)}\xi}{\sqrt{d_4}n}\tanh(\xi\tau)-\frac{\sqrt{-a(1+n)}\xi}{\sqrt{d_4}n}\coth(\xi\tau)\right]^{\frac{1}{n}} \tag{22}$$

These solutions exist if $a<0$ and $d_4(1+n)>0$.

Conclusion

This paper studies the non-linear KE with full non-linearity by utilizing three powerful integration architectures. As a result, many new trigonometric and hyperbolic function solutions, including, bright, dark, singular, periodic as well as rational solutions and complexiton solutions are extracted. The MAE method yields dark soliton, singular soliton, periodic soliton and rational solutions. The sine-Gordon expansion method extracts a bright soliton and complexiton solution of the proposed model. The tanh-coth expansion method extracts dark soliton, singular soliton and dark-singular combo soliton solutions. The results of this manuscript are inspiring and motivating for soliton studies.

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Paper submitted: January 25, 2021
Paper revised: February 25, 2021
Paper accepted: February 27, 2021