ON THE APPROXIMATE NUMERICAL SOLUTIONS OF FRACTIONAL HEAT EQUATION WITH HEAT SOURCE AND HEAT LOSS

by

Hami GUNDOGDU * **and Omer Faruk GOZUKIZIL**

Department of Mathematics, Faculty of Arts and Sciences, Sakarya University, Sakarya, Turkey

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In this paper, we are interested in obtaining an approximate numerical solution of the fractional heat equation where the fractional derivative is in Caputo sense. We also consider the heat equation with a heat source and heat loss. The fractional Laplace-Adomian decomposition method is applied to gain the approximate numerical solutions of these equations. We give the graphical representations of the solutions depending on the order of fractional derivatives. Maximum absolute error between the exact solutions and approximate solutions depending on the fractional-order are given. For the last thing, we draw a comparison between our results and found ones in the literature.

Key words: *fractional Laplace-Adomian decomposition method, heat loss, fractional heat equation, heat source*

Introduction

Scientists have been used linear or non-linear PDE (N/LPDE), to model physical phenomena arising in science and engineering. Not only N/LPDE but also fractional ones have been used for modelling real-life problems. Therefore, fractional-order partial differential equations (FPDE) play a significant role as much as ordinary ones. Fractional calculus allows us to go further in fractional derivatives and integration. So, fractional calculus has been an essential subject in mathematical and physical analysis. The main superiority of fractional calculus is that the fractional derivatives provide an outstanding tool for explaining memory and hereditary features of many materials and processes. Many influential mathematicians, especially Riemann, Liouville, Caputo, and He, have made considerable contributions to this topic. Scientists from other branches of science have improved the theory and applications of fractional calculus, as well.

The FPDE have taken place in many areas such as viscoelasticity, biology, electronic, signal processing, genetics algorithms, robotic technology, traffic systems, telecommunication, chemistry, physics, economics and finance [1-4]. Having many applications in these areas has drawn scientists' attention acquiring the solutions of FPDE. Furthermore, this motivates the scientists to establish methods for solving FPDE. Some of the most famous methods are the modified Bernoulli sub-equation function method [5], homotopy perturbation transform method [6], the Aboodh decomposition method [7], the q-homotopy analysis method [8], perturbation-iteration method [9], fractional power series scheme [10], variational iteration method [11], the double Laplace decomposition method [12], the Laplace-Adomian decomposition method (LADM) [13-16], and so on.

* Corresponding author, e-mail: hamigundogdu@sakarya.edu.tr

One of the important PDE is the heat equation developed by Fourier [17]. It is related to the work of Brownian motion solved by Brown. The 1-D heat equation is given:

$$\frac{\partial u}{\partial t}-k\frac{\partial^2 u}{\partial x^2}=0 \tag{1}$$

where $u(x, t)$ is the temperature at a point x and at time t and $k > 0$ is the heat conductivity. This equation describes the distribution of heat in a region over time in one dimension.

The heat equation given in eq. (1) can be obtained from the conservation of the energy [18]. Furthermore, it plays significant roles in varying fields of science such as mathematics, chemistry and physics. It represents a typical parabolic PDE in mathematics. In the branch of financial mathematics, the Black-Scholes equation can be solved by using the heat equation [19]. It also provides favourable information in Brownian motion by the Fokker-Planck equation in probability theory [20]. It is of a role in the Second law of thermodynamics and provides the information that heat moves to colder regions from hotter ones [21]. For more information about the heat equation, see [22-24].

Considering the aforementioned information about the heat equation provides us with a motive for studying the fractional heat equation. This helps us to work on the heat equation in better detail and understanding. To be specific, we can calculate the fractional differentiation of the temperature. Compared to traditional differential equations, fractional derivative provides us with more comprehensive information. In addition integer order, it allows us to find the solutions to the fractional heat equation. For any rational number between two integers, we can obtain the solutions of the heat equation with the given fractional order. In traditional differential equations, we reach the solutions for just integer orders. This study, therefore, emphasizes the significance of the fractional heat equation.

In the light of the aforementioned, we are motivated to study the fractional heat equation. For this purpose, we put the fractional derivative instead of the ordinary one in the heat eq. (1). That is, we consider the fractional heat equation:

$$D_t^\gamma u - D_x^2 u = 0,\ \ a < x < b,\ \ 0 < \gamma \le 1,\ \ t > 0 \tag{2}$$

with

$$u(x,0) = f(x) \tag{3}$$

and

$$u(a,t) = g(t),\ \ u(b,t) = h(t) \tag{4}$$

where $D_t^\gamma u$ is the γ-order fractional derivative, *i.e.*

$$D_t^\gamma u = \frac{\partial^\gamma u}{\partial t^\gamma}\ \text{ and }\ D_x^2 u = \frac{\partial^2 u}{\partial x^2}$$

In addition to the eq. (2), we examine the heat equation includes a heat source, heat loss and both of them. The heat equation with heat source is given:

$$D_t^\gamma u - D_x^2 u = F(x,t),\ \ a < x < b,\ \ 0 < \gamma \le 1,\ \ t > 0 \tag{5}$$

where $F(x, t)$ is the heat source.

The heat equation with heat loss is known:

$$D_t^\gamma u - D_x^2 u - u = 0,\ \ a < x < b,\ \ 0 < \gamma \le 1,\ \ t > 0 \tag{6}$$

where $-u$ means the heat loss through the lateral sides.

The heat equation, including both the heat source and heat loss:

$$D_t^\gamma u - D_x^2 u - u = F(x,t), \ a < x < b, \ 0 < \gamma \le 1, \ t > 0 \tag{7}$$

where $-u$ is the heat loss and $F(x, t)$ is the heat source.

Our purpose is to gain the solutions of the different types of the fractional heat equations given in eqs. (2)-(7) subjected to the initial and boundary conditions. The LADM is applied to these equations to obtain the solutions satisfying the given conditions. We receive the approximate solutions for each equation in eqs. (2)-(7) and compared the found solutions with the exact solutions. We also give the maximum absolute error between them. We utilize LADM because it is a powerful and effective method for solving FPDE as in ordinary PDE. Furthermore, we compare our findings with the results found in the literature.

Preliminaries

As many scientists have introduced different definitions for fractional derivatives, Yang has organized work on the different types of fractional derivatives. For various explanations for the fractional derivative, see [25]. We give the definitions for some essential fractional derivatives.

First of all, the Riemann-Liouville γ^{th} order fractional derivative of a continuous (but not necessarily differentiable) function $f(x)$ is given:

$$D_x^\gamma f(x) = \frac{1}{\Gamma(n-\gamma)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_0^x \frac{f^{(n)}(s)}{(x-s)^{\gamma+1-n}} \mathrm{d}s, \ n-1 < \gamma \le n \tag{8}$$

where $\Gamma(t)$ is the gamma function:

$$\Gamma(t) = \int_0^\infty e^{-s} s^{t-1} \mathrm{d}s \tag{9}$$

Another well-known fractional derivative is Caputo's one. The γ^{th} order Caputos's fractional derivative of a differentiable function $f(x)$ is defined:

$$D_x^\gamma f(x) = \frac{1}{\Gamma(n-\gamma)} \int_0^x \frac{f(s)}{(x-s)^{\gamma+1-n}} \mathrm{d}s, \ n-1 < \gamma \le n \tag{10}$$

where $\Gamma(t)$ is the gamma function given in eq. (9).

For $0 < \gamma \le 1$, the Caputo's fractional derivative turns into:

$$D_x^\gamma f(x) = \frac{1}{\Gamma(1-\gamma)} \int_0^x \frac{f'(s)}{(x-s)^\gamma} \mathrm{d}s, \ 0 < \gamma \le 1 \tag{11}$$

Some advantageous properties of the Caputo's fractional derivative are given:

$$D_x^\gamma x^m = \frac{\Gamma(1+m)}{\Gamma(1+m-\gamma)} x^{(m-\gamma)} \tag{12}$$

$$D_x^\gamma [cf(x)] = cD_x^\gamma [f(x)] \tag{13}$$

$$D_x^\gamma (c) = 0 \tag{14}$$

$$D_x^\gamma [af(x) + cg(x)] = aD_x^\gamma [f(x)] + cD_x^\gamma [g(x)] \tag{15}$$

where a and c are arbitrary constants. For more information, see [26, 27].

In the definition of Riemann-Liouville fractional derivative, the function can be continuous but not differentiable anywhere. Yet, $D_x^\gamma[f(x)] \neq 0$ if $f(x)$ is a constant function.

To handle the shortcomings of the Riemann-Liouville fractional derivative, Jumarie modified the Riemann-Liouville fractional derivative:

$$D_x^\gamma f(x) = \frac{1}{\Gamma(n-\gamma)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_0^x \frac{f(s)-f(0)}{(x-s)^{\gamma+1-n}} \mathrm{d}s, \;\; n-1 < \gamma \leq n \tag{16}$$

where f is a continuous but not necessarily differentiable function [28]. This derivative meets the previous rules given in eqs. (12)-(15).

He [29] has introduced a new fractional derivative:

$$D_x^\gamma f(x) = \frac{1}{\Gamma(n-\gamma)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_0^x \frac{f_0(s)-f(0)}{(x-s)^{\gamma+1-n}} \mathrm{d}s, \;\; n-1 < \gamma \leq n \tag{17}$$

where $f_0(x)$ is a known function.

Outlines of the fractional LADM method

In this part, we demonstrate how the LADM method works. This method is the combination of two powerful methods, Laplace transform and Adomian decomposition method. Before giving the highlines of the method, we introduce the laplace transform of the Caputo's fractional derivative:

$$\mathcal{L}[D_t^\gamma u(t)] = s^\gamma \mathcal{L}[u(t)] - \sum_{r=0}^{n-1} s^{\gamma-r-1} D_t^r u(0), \;\; n-1 < \gamma \leq n \tag{18}$$

where s is the Laplace domain function.

Let us consider the following time-fractional PDE in operator form:

$$D_t^\gamma u(x,t) = L[u(x,t)] + N[u(x,t)] + f(x,t), \;\; n-1 < \gamma \leq n \tag{19}$$

with

$$D_t^r u(x,0) = h_r(x), \;\; r = 0,1,2,...,n-1 \tag{20}$$

$$D_t^n u(x,0) = 0 \tag{21}$$

where L and N are the linear and non-linear operators, respectively.

The first thing is applying the Laplace transform to both sides of eq. (19) in t-direction. Then, we have:

$$\mathcal{L}[D_t^\gamma u(x,t)] = \mathcal{L}\{L[u(x,t)] + N[u(x,t)] + f(x,t)\} \tag{22}$$

By the linearity and differentiation property of the Laplace transform eq. (18), we get:

$$\mathcal{L}[u(t)] = \sum_{r=0}^{n-1} s^{-r-1} h_r(x) + \frac{1}{s^\gamma} \mathcal{L}\{L[u(x,t)] + N[u(x,t)] + f(x,t)\} \tag{23}$$

To obtain $u(x, t)$, we apply the inverse operator of the Laplace transform to both sides of the eq. (23). Then we have:

$$u(x,t) = \sum_{r=0}^{n-1} \frac{t^r}{\Gamma(r+1)} h_r(x) + \mathcal{L}^{-1}\left(\frac{1}{s^\gamma}\{\mathcal{L}[f(x,t)]\}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^\gamma}\mathcal{L}\{L[u(x,t)] + N[u(x,t])\}\right) \tag{24}$$

In this method, the solution $u(x, t)$ is defined:

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t) \tag{25}$$

and the non-linear term $N[u(x, t)]$ is given:

$$N[u(x,t)]=\sum_{i=0}^{\infty}A_i \tag{26}$$

where A_i's are the Adomian polynomials defined by Adomian in 1988

$$A_n=\frac{1}{n!}\frac{d^n}{d\alpha^n}\left[N\left(\sum_{n=0}^{\infty}\alpha^n u_n\right)\right],\ n=0,1,2,... \tag{27}$$

From this definition, we can write the first few terms:

$$A_0=N(u_0) \tag{28}$$

$$A_1=u_1N'(u_0) \tag{29}$$

$$A_2=u_2N'(u_0)+\frac{1}{2!}u_1^2N''(u_0) \tag{30}$$

and derive the other terms in the same way.

Putting eqs. (25) and (26) into eq. (24) yields:

$$\sum_{i=0}^{\infty}u_i(x,t)=\sum_{r=0}^{n-1}\frac{t^r}{\Gamma(r+1)}h_r(x)+\mathcal{L}^{-1}\left(\frac{1}{s^\gamma}\left\{\mathcal{L}\left[f(x,t)\right]\right\}\right)+\sum_{i=0}^{\infty}\mathcal{L}^{-1}\left(\frac{1}{s^\gamma}\mathcal{L}\left\{L\left[u_i(x,t)\right]+A_i\right\}\right) \tag{31}$$

Comparing the both sides of eq. (31) provides us with:

$$u_0(x,t)=\sum_{r=0}^{n-1}\frac{t^r}{\Gamma(r+1)}h_r(x)+\mathcal{L}^{-1}\left(\frac{1}{s^\gamma}\left\{\mathcal{L}[f(x,t)]\right\}\right) \tag{32}$$

and

$$u_{i+1}(x,t)=\mathcal{L}^{-1}\left(\frac{1}{s^\gamma}\mathcal{L}\left\{L\left[u_i(x,t)\right]+A_i\right\}\right),\ i\ge 0 \tag{33}$$

From the eqs. (32) and (33), we get the components of the solution $u(x, t)$. We therefore, can obtain the approximate anaytical solution of the eq. (19) subjected to the conditions in eqs. (20) and (21):

$$u(x,t)=\lim_{I\to\infty}\sum_{i=0}^{I}u_i(x,t) \tag{34}$$

The LADM guarantees that the IVP eqs. (2)-(4), where the derivative in Caputo's sense, has the approximate solution:

$$u_0(x,t)=\sum_{r=0}^{n-1}\frac{t^r}{\Gamma(r+1)}h_r(x)+\mathcal{L}^{-1}\left(\frac{1}{s^\gamma}\left\{\mathcal{L}[f(x,t)]\right\}\right),\ i=0 \tag{35}$$

$$u_{i+1}(x,t)=\mathcal{L}^{-1}\left(\frac{1}{s^\gamma}\mathcal{L}\left\{L\left[u_i(x,t)\right]+A_i\right\}\right),\ i\ge 0 \tag{36}$$

It is shown that the recursive relation is found by the procedure from eqs. (18)-(34).

Numerical solutions of the fractional heat equation

This section obtains the approximate numerical solutions of the different types of the fractional heat equation previously mentioned.

The heat equation

The fractional homogeneous heat equation is given by the eqs. (2)-(4). In the literature, there are many studies on this form of the time-fractional heat equation. The approximate solution of this equation is obtained by using the LADM in t-direction.

We want to find the solution of this equation by the help of the LADM in x-direction. For this purpose, we establish the space-fractional heat equation:

$$D_x^\gamma u = D_t u,\ 0 < x < \pi,\ 1 < \gamma \le 2,\ t > 0 \tag{37}$$

with

$$u(0,t) = 0,\ u_x(0,t) = \mathrm{e}^{-t} \tag{38}$$

Using the relations given in (35) and (36) gives:

$$u_0(x,t) = x\mathrm{e}^{-t} \tag{39}$$

$$u_{i+1}(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\gamma}\mathcal{L}[u_{i_t}(x,t)]\right\},\ i \ge 0 \tag{40}$$

From this equalities, some components of the solution are found:

$$u_0(x,t) = x\mathrm{e}^{-t} \tag{41}$$

$$u_1(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\gamma}\mathcal{L}[u_{0_t}(x,t)]\right\} = -\frac{x^{\gamma+1}}{\Gamma(\gamma+2)}\mathrm{e}^{-t} \tag{42}$$

$$u_2(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\gamma}\mathcal{L}[u_{1_t}(x,t)]\right\} = \frac{x^{2\gamma+1}}{\Gamma(2\gamma+2)}\mathrm{e}^{-t} \tag{43}$$

$$u_3(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\gamma}\mathcal{L}[u_{2_t}(x,t)]\right\} = -\frac{x^{3\gamma+1}}{\Gamma(3\gamma+2)}\mathrm{e}^{-t} \tag{44}$$

$$\vdots$$

Putting these into eq. (34) grants:

$$u(x,t) = \lim_{I\to\infty}\sum_{i=0}^{I} u_i(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots \tag{45}$$

$$u(x,t) = \left[x - \frac{x^{\gamma+1}}{\Gamma(\gamma+2)} + \frac{x^{2\gamma+1}}{\Gamma(2\gamma+2)} - \frac{x^{3\gamma+1}}{\Gamma(3\gamma+2)} + \cdots\right]\mathrm{e}^{-t} \tag{46}$$

We, then, gain the closed form of the approximate solution:

$$u(x,t) = \sum_{i=0}^{\infty}\frac{(-1)^i x^{i\gamma+1}}{\Gamma(i\gamma+2)}\mathrm{e}^{-t} \tag{47}$$

If we take $\gamma = 2$ then the equation turns into well-known heat equations. Putting $\gamma = 2$ in the closed form (47) provides:

$$u(x,t) = \mathrm{e}^{-t}\sin x \tag{48}$$

The heat equation with heat source

We consider the following space-fractional equation:

$$D_x^\gamma u = D_t u - \sin t,\ \ 0 < x < \pi,\ \ 0 < \gamma \le 2,\ \ t > 0 \tag{49}$$

subject to the initial conditions

$$u(0,t) = -\cos t,\ \ u_x(0,t) = 1 \tag{50}$$

where $F(x, t) = \sin t$ is the heat source.

If we follow the aforementioned procedure of the LADM in x-direction, we can write the first few terms:

$$u_0(x,t) = x - \cos t - \frac{x^\gamma}{\Gamma(\gamma+1)}\sin t \tag{51}$$

$$u_1(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\gamma}\mathcal{L}\left[u_{0_t}(x,t)\right]\right\} = \frac{x^\gamma}{\Gamma(\gamma+1)}\sin t - \frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\cos t \tag{52}$$

$$u_2(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\gamma}\mathcal{L}\left[u_{1_t}(x,t)\right]\right\} = \frac{x^{3\gamma}}{\Gamma(3\gamma+1)}\sin t + \frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\cos t \tag{53}$$

$$u_3(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\gamma}\mathcal{L}[u_{2_t}(x,t)]\right\} = -\frac{x^{3\gamma}}{\Gamma(3\gamma+1)}\sin t + \frac{x^{4\gamma}}{\Gamma(4\gamma+1)}\cos t \tag{54}$$

$$\vdots$$

By replacing these terms in the closed form (34), we get the approximate solution:

$$u(x,t) = x - \cos t - \frac{x^\gamma}{\Gamma(\gamma+1)}\sin t + \frac{x^\gamma}{\Gamma(\gamma+1)}\sin t - \frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\cos t +$$
$$+\frac{x^{3\gamma}}{\Gamma(3\gamma+1)}\sin t + \frac{x^{2\gamma}}{\Gamma(2\gamma+1)}\cos t - \frac{x^{3\gamma}}{\Gamma(3\gamma+1)}\sin t + \frac{x^{4\gamma}}{\Gamma(4\gamma+1)}\cos t + \cdots \tag{55}$$

For $i \to \infty$, we have the exact solution:

$$u(x,t) = x - \cos t \tag{56}$$

The heat equation with heat loss

In this part, we have the intention of finding an approximate solution of the heat equation with heat loss. The following equation is examined:

$$D_t^\gamma u = D_x^2 u - u,\ \ 0 < x < \pi,\ \ 0 < \gamma \le 1,\ \ t > 0 \tag{57}$$

expose to the initial condition:

$$u(x,0) = \mathrm{e}^x + 1 \tag{58}$$

Applying the LADM in t-direction gives:

$$u_0(x,t) = \mathrm{e}^x + 1 \tag{59}$$

$$u_1(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\gamma}\mathcal{L}[u_{0_{xx}}(x,t) - u_0(x,t)]\right\} = -\frac{t^\gamma}{\Gamma(\gamma+1)} \tag{60}$$

$$u_2(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\gamma}\mathcal{L}[u_{1_{xx}}(x,t) - u_1(x,t)]\right\} = \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \tag{61}$$

$$u_3(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\gamma}\mathcal{L}[u_{2_{xx}}(x,t) - u_2(x,t)]\right\} = -\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \tag{62}$$

$$\vdots$$

Then we have:

$$u(x,t) = \mathrm{e}^x + 1 - \frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} - \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \cdots \tag{63}$$

We can rewrite this equation

$$u(x,t) = \mathrm{e}^x + \sum_{i=0}^{\infty}\frac{(-1)^i t^{i\gamma}}{\Gamma(i\gamma+1)} \tag{64}$$

For the specific value of $\gamma = 1$, we have the exact solution:

$$u(x,t) = \mathrm{e}^x + \mathrm{e}^{-t} \tag{65}$$

The heat equation including both heat source and heat loss

We finally think of the heat equation contains both heat source and heat loss. To be specific:

$$D_t^\gamma u = D_x^2 u - u + \cos x,\ \ 0 < x < \pi,\ \ 0 < \gamma \le 1,\ \ t > 0 \tag{66}$$

with the initial condition:

$$u(x,0) = x \tag{67}$$

Applying the LADM to this equation provide us:

$$u_0(x,t) = x + \frac{t^\gamma}{\Gamma(\gamma+1)}\cos x \tag{68}$$

$$u_1(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\gamma}\mathcal{L}[u_{0_{xx}}(x,t) - u_0(x,t)]\right\} = -x\frac{t^\gamma}{\Gamma(\gamma+1)} - 2\cos x\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \tag{69}$$

$$u_2(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\gamma}\mathcal{L}[u_{1_{xx}}(x,t) - u_1(x,t)]\right\} = x\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + 4\cos x\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \tag{70}$$

$$u_3(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\gamma}\mathcal{L}[u_{2_{xx}}(x,t) - u_2(x,t)]\right\} = -x\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} - 8\cos x\frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \tag{71}$$

$$\vdots$$

Substitution of these components in eq. (34) gives:

$$u(x,t)=x\left(1-\frac{t^{\gamma}}{\Gamma(\gamma+1)}+\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}-x\frac{t^{3\gamma}}{\Gamma(3\gamma+1)}+\cdots\right)+$$
$$+\cos x\left(\frac{t^{\gamma}}{\Gamma(\gamma+1)}-2\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}+4\frac{t^{3\gamma}}{\Gamma(3\gamma+1)}-8\frac{t^{4\gamma}}{\Gamma(4\gamma+1)}+\cdots\right) \tag{72}$$

We can write:

$$u(x,t)=x\sum_{i=0}^{\infty}\frac{(-1)^{i}t^{i\gamma}}{\Gamma(i\gamma+1)}+\frac{1}{2}\cos x\left[\sum_{i=0}^{\infty}\frac{(-1)^{i}2^{i}(t)^{i\gamma}}{\Gamma(i\gamma+1)}-1\right] \tag{73}$$

For $\gamma = 1$, we have the exact solution:

$$u(x,t)=xe^{-t}+\frac{1}{2}\cos x\left(\mathrm{e}^{-2t}-1\right) \tag{74}$$

For the given eqs. (37), (49), (57), and (66), we obtain the approximate solutions (47), (55), (64), and (73), respectively. Moreover, we reach the smooth solutions (48), (56), as $\gamma \to 2$, and (65), (74) as $\gamma \to 1$. Genreally the fractional differential equations have no smooth solutions since they model unsmooth problems. In this method, the desired solutions depend on the given initial conditions. In this paper, we gain smooth solutions for the given fractional heat equations since the given conditions consist of smooth functions.

Graphs of the solutions

Now, we illustrate the solutions of the given heat equations. Graphical representations demonstrate how the solutions change depending on time t and the fractional derivative γ.

In fig. 1, it is shown that temperature enhances as the value of γ increases from 0-2. Figure 2 represents that increment in the value of γ increases the temperature distribution. It is shown in fig. 3 that a reduction in temperature is prominent as γ increases. It is observed from fig. 4 that enhancement in γ decreases the temperature. As shown in the given figures, various values of γ bring about different behaviours; hence distinct situations can be described. Furthermore, the behaviour of the temperature depending on these situations can be obtained. As a result, it is stated the fractional-order γ has important effects on the modelling of temperature distribution for a given situation.

The previously given figures provides us with physical meanings for the considered problems in this paper. The eqs. (37), (57), and (66) have positive solutions as shown in figs. 1, 3, and 4. The problems related to these equations play significant roles in heating models as the temperature distribution is positive. The eq. (49) takes negative values for some t as there is heat loss in the given problem, fig. 2.

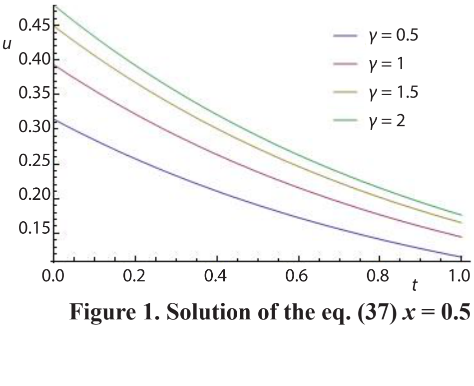

Figure 1. Solution of the eq. (37) $x = 0.5$

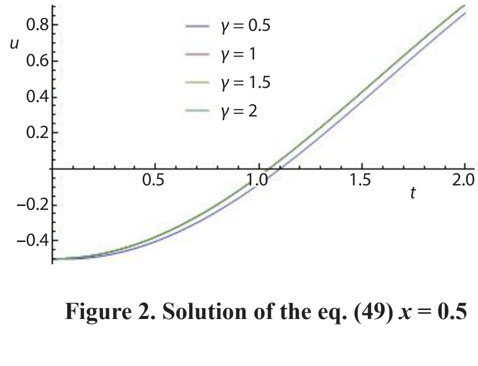

Figure 2. Solution of the eq. (49) $x = 0.5$

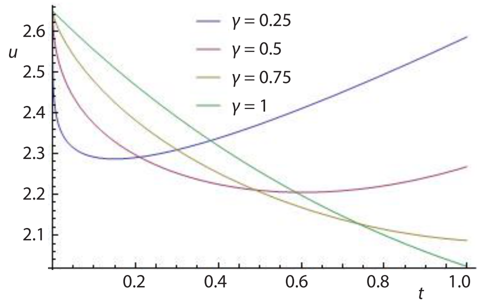

Figure 3. Solution of the eq. (57) x = 0.5

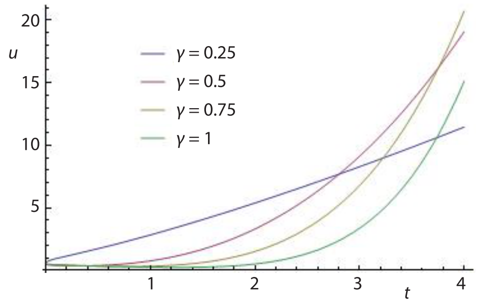

Figure 4. Solution of the eq. (66) x = 0.5

Numerical results

We herein compare the exact solutions with the ones depending on Caputo's fractional derivative. We also calculate the maximum absolute error for the considered eq. (37), (49), (57), and (66) as in tabs. 1-4.

For the considered problems in this paper, the absolute difference between the exact solutions and Caputo's fractional derivative solutions are obtained for $I = 5$ steps. For the space-fractional eqs. (37) and (49), the absolute error is analyzed by the effect of varying γ ranging from 0-2. For the time-fractional eqs. (57) and (66), it is calculated as γ ranging from 0-1.

Table 1. Maximum absolute error for the eq. (37) with $x = t = 0.5$ and $I = 5$

I/γ	0.4	0.8	1.2	1.6	2.0
1	0.172546	0.091417	0.041979	0.014435	$1.57014 \cdot 10^{-4}$
2	0.068650	0.064503	0.036310	0.013422	$9.36926 \cdot 10^{-7}$
3	0.123108	0.070171	0.036716	0.013444	$3.25712 \cdot 10^{-9}$
4	0.096194	0.069158	0.036694	0.013443	$7.40746 \cdot 10^{-12}$
5	0.108830	0.069316	0.036695	0.013443	$1.18794 \cdot 10^{-14}$

Table 2. Maximum absolute error for the eq. (49) with $x = t = 0.5$ and $I = 5$

I/γ	0.4	0.8	1.2	1.6	2.0
1	$4.09502 \cdot 10^{-1}$	$2.95644 \cdot 10^{-1}$	$1.89401 \cdot 10^{-1}$	$11.0624 \cdot 10^{-2}$	$5.9928 \cdot 10^{-2}$
2	$5.41172 \cdot 10^{-1}$	$2.02497 \cdot 10^{-1}$	$5.5773 \cdot 10^{-2}$	$1.2312 \cdot 10^{-2}$	$2.28537 \cdot 10^{-3}$
3	$1.89401 \cdot 10^{-1}$	$3.0469 \cdot 10^{-2}$	$2.95471 \cdot 10^{-3}$	$2.00999 \cdot 10^{-4}$	$1.04042 \cdot 10^{-5}$
4	$2.02497 \cdot 10^{-1}$	$1.2312 \cdot 10^{-2}$	$3.67926 \cdot 10^{-4}$	$6.74215 \cdot 10^{-6}$	$8.50213 \cdot 10^{-8}$
5	$5.99282 \cdot 10^{-2}$	$1.2485 \cdot 10^{-3}$	$1.04042 \cdot 10^{-5}$	$4.64473 \cdot 10^{-8}$	$1.2902 \cdot 10^{-10}$

Table 3. Maximum absolute error for the eq. (57) with $x = t = 0.5$ and $I = 5$

I/γ	0.2	0.4	0.6	0.8	1.0
1	0.554669	0.460683	0.344911	0.223193	$1.06531 \cdot 10^{-2}$
2	0.299484	0.155979	0.050147	0.007551	$1.84693 \cdot 10^{-2}$
3	0.438897	0.239078	0.121148	0.056002	$2.36399 \cdot 10^{-3}$
4	0.177766	0.008334	0.057595	0.041973	$2.40174 \cdot 10^{-4}$
5	0.322234	0.133334	0.078429	0.044577	$2.02430 \cdot 10^{-5}$

Table 4. Maximum absolute error for the eq. (66) with $x = t = 0.5$ and $I = 5$

I/γ	0.2	0.4	0.6	0.8	1.0
1	0.130617	0.122369	0.112209	0.101528	$9.12895 \cdot 10^{-2}$
2	0.667074	0.285727	0.059512	0.030195	$4.43499 \cdot 10^{-3}$
3	0.235196	0.140939	0.085834	0.064042	$5.79131 \cdot 10^{-3}$
4	0.234661	0.004081	0.003265	0.005102	$5.50432 \cdot 10^{-3}$
5	0.498239	0.146242	0.065082	0.055373	$5.55390 \cdot 10^{-3}$

Convergence and error analysis of ladm

We herein give the sufficient conditions for convergence and error analysis of the method previously mentioned. In the method, the solution is represented:

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) \tag{75}$$

Theorem 1. The series solution of eq. (75) converges if there exists $0 < \epsilon < 1$ such that:

$$\| u_{i+1}(x,t) \| \le \epsilon \| u_i(x,t) \|, \ \forall i \ge i_0 \tag{76}$$

for some i_0.

Proof 1. We take the Banach space $[C(|I|), \|\cdot\|]$ of all continuous functions $f(x, t)$ on the interval I with the usual norm $\|f(x, t)\| = \max_{(x, t) \in I} | f(x, t)|$.

Let S_n be a sequence of:

$$S_n = u_0(x,t) + u_1(x,t) + ... + u_n(x,t) \tag{77}$$

We will show that S_n is a Cuachy sequence in the given Banach space. For this aim, we check:

$$\| S_{n+1} - S_n \| = \| u_{n+1} \| \le \epsilon \| u_n \| \le ... \le \epsilon^{n-i_0+1} \| u_{i_0} \| = \epsilon^{n-i_0+1} \max_{(x,t)\in I} | u_{i_0} | \tag{78}$$

Then, we have:

$$\| S_n - S_m \| = \| \sum_{k=m}^{n-1} (S_{k+1} - S_k) \| \le \sum_{k=m}^{n-1} \| (S_{k+1} - S_k) \| \le$$

$$\le \sum_{k=m}^{n-1} \max_{(x,t)\in I} | u_{i_0} | = \frac{1-\epsilon^{n-m}}{1-\epsilon} \epsilon^{m-i_0+1} \max_{(x,t)\in I} | u_{i_0} | \tag{79}$$

for every $n, m \in N$ and $n \ge m > i_0$.

This gives:

$$lim_{n,m\to\infty} \| S_n - S_m \| = 0 \tag{80}$$

due to the fact that $0 < \epsilon < 1$. This completes the proof.

Now, the estimate for maximum absolute truncated error is given by the following theorem.

Theorem 2. The truncated series $\sum_{i=0}^{m} u_i(x, t)$ is considered as a numerical solution $u(x, t)$, then the maximum absolute truncated error is calculated:

$$\| u(x,t) - \sum_{i=0}^{m} u_i(x,t) \| \le \frac{\epsilon^{m+1}}{1-\epsilon} \| u_0(x,t) \| \tag{81}$$

Proof 2.

$$\| u(x,t) - \sum_{i=0}^{m} u_i(x,t) \| = \| \sum_{i=m+1}^{\infty} u_i(x,t) \| \le \sum_{i=m+1}^{\infty} \| u_i(x,t) \| \le$$

$$\le \sum_{i=m+1}^{\infty} \epsilon^i \| u_0(x,t) \| \le \frac{\epsilon^{m+1}}{1-\epsilon} \| u_0(x,t) \| \quad (82)$$

Hence, we concluded the proof.

Comparison of the results

Now, we compare the results found in this paper to others in the literature. In 2020, the initial periodic boundary value problem for the fractional heat equation:

$$D_t^\gamma u = D_x^2 u, \ \ 0 < \gamma \le 1 \quad (83)$$

with

$$u(x,0) = \cos(\pi x) \quad (84)$$

is studied [30]. By using the separation of variables method, the approximate solution for the previous problem is obtained. In that method, the given equation is divided into two equations as a time-fractional and an ordinary differential. Sometimes, it is not possible to solve these equations or to solve these equations is harder than solving the eq. (83). The method, therefore, requires more calculations. Comparison between our method and that one gives that our method is more productive and fertile in solving these types of equations.

In a recent study, the fractional heat equation:

$$D_t^\gamma u = D_x^2 u, \ \ 0 < \gamma \le 1 \quad (85)$$

with

$$u(x,0) = \sin(\pi x) \quad (86)$$

is considered [31]. The q-homotopy analysis transform method is applied to this equation. Some components are obtained, and then the approximate solution is given in the closed-form. Compared to our method, finding the components seems to be more complicated in that method. Moreover, we reach the solution faster.

Conclusions

The main aim of this work is that the fractional heat equation subject to the conditions is investigated. As a further matter, some distinct forms of the heat equation, including a heat source, heat loss, and both are examined, as well. The solutions of these equations previously mentioned are obtained by using LADM. This method is indeed one of the most powerful and fertile methods for finding solutions to any PDE exposed to initial conditions. In addition these features, we exemplify that LADM can be used in t-direction or x-direction. In both ways, one can get the solution of a time or space fractional PDE.

Throughout this work, the fractional derivative is considered as in Caputo's sense. The values of the solutions are given as shown in the figures depending on Caputo's fractional derivative. For the equations (37) and (49), the solutions reach the exact solutions of the PDE as $\gamma \to 2$. Moreover, we achieve the exact solutions the PDE (57) and (66) as $\gamma \to 1$. Our finding shows that this method is trustworthy for Caputo's fractional PDE subject to the initial conditions.

In our future work, we can modify the LADM for various fractional derivatives on the different kinds of fractional heat equations studied in this paper. As a further matter, we can compare the solutions found according to each fractional derivative.

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