THE SERIES REPRESENTATIONS FOR THE J AND H FUNCTIONS APPLIED IN THE HEAT-DIFFUSION EQUATION

by

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In this article the theory of the supertrigonometric and superhyperbolic functions associated with the J and H functions are proposed for the first time. The series representation for the heat-diffusion equations are also given by using the J and H functions. The results are efficient and accurate for the description for the solutions of the PDE in mathematical physics.

Key words: *heat-diffusion equation, H function, superhyperbolic functions, J function, super trigonometric functions,*

Introduction

It is well known that the special functions [1, 2] can be structured by using the power series, infinite products, generating functions, differential, difference, integral, and functional equations, integral representations, repeated differentiation, trigonometric series, or other series in the orthogonal functions, having the important applications not only in the field of pure and applied mathematics but also in the field of mathematical physics [3].

Let us recall the H and J functions, which are defined by means of the Mellin-Barnes type integrals. In order to introduce the H and J function, we now denote $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$ as the sets of the complex, real number, positive real number, and integer numbers, respectively. Suppose that $x \in \mathbb{C}\backslash\{0\}$, $i = (-1)^{1/2}$, $q \geq 1$, $0 \leq n \leq p$, $0 \leq m \leq q$, $\{a_j, b_j\} \in \mathbb{C}$, and $\{\alpha_j, \beta_j\} \in \mathbb{R}_+$.

Let

$$\Lambda(s) = \frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}, \text{ where } \mathbf{A}_1(s) = \prod_{j=1}^{m}\Gamma\left(b_j - \beta_j s\right), \ \mathbf{B}_1(s) = \prod_{j=1}^{n}\Gamma\left(1 - a_j + \alpha_j s\right)$$

$$\mathbf{C}_1(s) = \prod_{j=m+1}^{q}\Gamma\left(1 - b_j + \beta_j s\right), \text{ and } \mathbf{D}_1(s) = \prod_{j=n+1}^{p}\Gamma\left(a_j - \alpha_j s\right)$$

The Fox H function $\mathbb{H}_{p,q}^{m,n}(x)$ proposed by Fox [4], is defined by the Mellin-Barnes type integral [5]:

$$\mathbb{H}_{p,q}^{m,n}(x) = \mathbb{H}_{p,q}^{m,n}\left[x; \begin{matrix}\{\mathbf{a}, \boldsymbol{\alpha}\}\\ \{\mathbf{b}, \boldsymbol{\beta}\}\end{matrix}\right] = \mathbb{H}_{p,q}^{m,n}\left[x; \begin{matrix}\{a_j, \alpha_j\}_1^p\\ \{b_j, \beta_j\}_1^q\end{matrix}\right] = \frac{1}{2\pi i}\int_L \Lambda(s) x^s \mathrm{d}s \tag{1}$$

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where L is the infinite contour in the complex plane, (a certain contour separating the poles of the two factors in the numerator) with the poles [5]:

$$k_j^{\ell} = -\frac{b_j + \ell}{\beta_j}\left(j = 1,\ n;\ \ell \in \mathbb{N} \cup \{0\}\right) \tag{2}$$

and

$$k_j^{\lambda} = \frac{1 - a_j + \lambda}{\alpha_j}\left(j = 1,\ n;\ \lambda \in \mathbb{N} \cup \{0\}\right) \tag{3}$$

The J function $\mathbb{J}_{p,q}^{m,n}(x)$, proposed by Yang [6], is defined by the Mellin-Barnes type integral [6]:

$$\mathbb{J}_{p,q}^{m,n}(x) = \mathbb{J}_{p,q}^{m,n}\left[x; \begin{matrix}\{\mathbf{a}, \boldsymbol{\alpha}\} \\ \{\mathbf{b}, \boldsymbol{\beta}\}\end{matrix}\right] = \mathbb{J}_{p,q}^{m,n}\left[x; \begin{matrix}\{a_j, \alpha_j\}_1^p \\ \{b_j, \beta_j\}_1^q\end{matrix}\right] = \frac{1}{2\pi i}\int_L \Lambda(s) s^x \mathrm{d}s \tag{4}$$

where L is a certain contour separating the poles of the two factors in the numerator with the poles, given by eqs. (2) and (3) [6].

For $l \in \mathbb{N} \cup \{0\}$, the connection with the H and J functions, investigated by author in [6], is given:

$$\mathbb{H}_{p,q}^{m,n}\left(e^{-x}\right) = \sum_{l=0}^{\infty}(-1)^l \mathbb{J}_{p,q}^{m,n}(l)\frac{x^l}{\Gamma(1+l)} \tag{5}$$

and

$$\mathbb{H}_{p,q}^{m,n}\left(\frac{1}{x}\right) = \sum_{l=0}^{\infty}(-1)^l \mathbb{J}_{p,q}^{m,n}(l)\frac{(\log x)^l}{\Gamma(1+l)} \tag{6}$$

since

$$\mathbb{H}_{p,q}^{m,n}(x) = \sum_{l=0}^{\infty}\mathbb{J}_{p,q}^{m,n}(l)\frac{(\log x)^l}{l!} \tag{7}$$

and

$$\mathbb{H}_{p,q}^{m,n}\left(e^{x}\right) = \sum_{l=0}^{\infty}\mathbb{J}_{p,q}^{m,n}(l)\frac{x^l}{\Gamma(1+l)} \tag{8}$$

where L is a certain contour separating the poles of the two factors in the numerator.

Motivated by the Euler formula, given in 1748 Euler [7], and the proposed idea presented in the published monographs [3, 8], we plan to develop the supertrigonometric and superhyperbolic functions associated with the J and H functions. The main target of the paper is to suggest the supertrigonometric and superhyperbolic functions associated with the J and H functions, and to present a potential application in the heat-diffusion equation.

The special functions: the J and H functions

In this section we investigate the series representations for the J and H functions.

In order to study the series representation of the J function, we now introduce the kappa function:

$$\kappa_{p,q}^{m,n}(x;\upsilon) = \kappa_{p,q}^{m,n}\left[x;\upsilon; \begin{matrix}\{\mathbf{a}, \boldsymbol{\alpha}\} \\ \{\mathbf{b}, \boldsymbol{\beta}\}\end{matrix}\right] = \kappa_{p,q}^{m,n}\left[x;\upsilon; \begin{matrix}\{a_j, \alpha_j\}_1^p \\ \{b_j, \beta_j\}_1^q\end{matrix}\right] = \frac{1}{2\pi i}\int_L \Lambda(s)\left[\mu(\upsilon;s)\right]^x \mathrm{d}s \tag{9}$$

where the mu function is defined:

$$\mu(\upsilon;s)=\overbrace{\log\cdots\log}^{\upsilon-\text{times}}s \tag{10}$$

for $s \in \mathbb{C}$ and $\upsilon \in \mathbb{N}$.

There exist the following special cases:

– For $\upsilon = 2$

$$\kappa_{p,q}^{m,n}(x;2)=\sum_{l=0}^{\infty}\kappa_{p,q}^{m,n}(l;3)\frac{x^l}{l!} \tag{11}$$

where

$$\kappa_{p,q}^{m,n}(x;3)=\frac{1}{2\pi i}\int_L \Lambda(s)(\log\log\log s)^x \,\mathrm{d}s \tag{12}$$

– For $\upsilon = 1$, we arrive at

$$\kappa_{p,q}^{m,n}(x;1)=\sum_{l=0}^{\infty}\kappa_{p,q}^{m,n}(l;2)\frac{x^l}{l!} \tag{13}$$

where

$$\kappa_{p,q}^{m,n}(x;2)=\frac{1}{2\pi i}\int_L \Lambda(s)(\log\log s)^x \,\mathrm{d}s \tag{14}$$

– There exists

$$\mathbb{J}_{p,q}^{m,n}(x)=\frac{1}{2\pi i}\int_L \Lambda(s)e^{x\log s}\mathrm{d}s=\frac{1}{2\pi i}\int_L \Lambda(s)\left[\sum_{l=0}^{\infty}\frac{(x\log s)^l}{l!}\right]\mathrm{d}s=\sum_{l=0}^{\infty}\kappa_{p,q}^{m,n}(l;1)\frac{x^l}{l!} \tag{15}$$

where

$$\kappa_{p,q}^{m,n}(x;1)=\frac{1}{2\pi i}\int_L \Lambda(s)(\log s)^x \,\mathrm{d}s \tag{16}$$

It is not difficult to find that eq. (15) is the series representation of the J function by using the kappa function.

In a similar way, it is shown:

$$\mathbb{H}_{p,q}^{m,n}(x)=\frac{1}{2\pi i}\int_L \Lambda(s)e^{s\log x}\mathrm{d}s=\frac{1}{2\pi i}\int_L \Lambda(s)\left[\sum_{l=0}^{\infty}\frac{(s\log x)^l}{l!}\right]\mathrm{d}s \tag{17}$$

where L is a certain contour separating the poles of the two factors in the numerator.

This implies:

$$\mathbb{H}_{p,q}^{m,n}(x)=\sum_{l=0}^{\infty}\mathbb{J}_{p,q}^{m,n}(l)\frac{(\log x)^l}{l!} \tag{18}$$

which leads

$$\mathbb{H}_{p,q}^{m,n}(xy)=\sum_{l=0}^{\infty}\mathbb{J}_{p,q}^{m,n}(l)\frac{(\log x+\log y)^l}{l!}$$

and

$$\mathbb{H}_{p,q}^{m,n}\left(\frac{x}{y}\right)=\sum_{l=0}^{\infty}\mathbb{J}_{p,q}^{m,n}(l)\frac{(\log x-\log y)^l}{l!}$$

It is easy to see that eq. (18) is the series representation of the H function by using the J function.

The supertrigonometric functions associated with the J function

Let $\lambda \in \mathbb{C}$ and $\tau \in \mathbb{C}$.

With eq. (15) we show:

$$\mathbb{J}_{p,q}^{m,n}(\lambda x)=\sum_{l=0}^{\infty}\kappa_{p,q}^{m,n}(l;1)\frac{(\lambda x)^l}{l!} \tag{19}$$

This implies:

$$\mathbb{J}_{p,q}^{m,n}(i\tau x)=\sum_{l=0}^{\infty}\kappa_{p,q}^{m,n}(l;1)\frac{(i\tau x)^l}{l!} \tag{20}$$

and

$$\mathbb{J}_{p,q}^{m,n}(-i\tau x)=\sum_{l=0}^{\infty}\kappa_{p,q}^{m,n}(l;1)\frac{(-i\tau x)^l}{l!} \tag{21}$$

The supersine function associated with the J function is defined:

$$\mathbb{J}\sin_{p,q}^{m,n}(x)=\sum_{l=0}^{\infty}(-1)^l\kappa_{p,q}^{m,n}(2l+1;1)\frac{x^{2l+1}}{(2l+1)!} \tag{22}$$

The supercosine function associated with the J function is defined:

$$\mathbb{J}\cos_{p,q}^{m,n}(x)=\sum_{l=0}^{\infty}(-1)^l\kappa_{p,q}^{m,n}(2l;1)\frac{x^{2l}}{(2l)!} \tag{23}$$

The supertangent function associated with the J function is defined:

$$\mathbb{J}\tan_{p,q}^{m,n}(x)=\frac{\mathbb{J}\cos_{p,q}^{m,n}(x)}{\mathbb{J}\sin_{p,q}^{m,n}(x)},\ \left[\mathbb{J}\sin_{p,q}^{m,n}(x)\neq 0\right] \tag{24}$$

The supercotangent function associated with the J function is defined:

$$\mathbb{J}\text{cotan}_{p,q}^{m,n}(x)=\frac{\mathbb{J}\sin_{p,q}^{m,n}(x)}{\mathbb{J}\cos_{p,q}^{m,n}(x)},\ \left[\mathbb{J}\cos_{p,q}^{m,n}(x)\neq 0\right] \tag{25}$$

The supercosecant function associated with the J function is defined:

$$\mathbb{J}\sec_{p,q}^{m,n}(x)=\frac{1}{\mathbb{J}\cos_{p,q}^{m,n}(x)},\ \left[\mathbb{J}\cos_{p,q}^{m,n}(x)\neq 0\right] \tag{26}$$

The supersecant function associated with the J function is defined:

$$\mathbb{J}\csc_{p,q}^{m,n}(x)=\frac{1}{\mathbb{J}\sin_{p,q}^{m,n}(x)},\ \left[\mathbb{J}\sin_{p,q}^{m,n}(x)\neq 0\right] \tag{27}$$

Thus, it is easy to show:

$$\mathbb{J}\cos_{p,q}^{m,n}(\tau x)=\frac{1}{2}\left[\mathbb{J}_{p,q}^{m,n}(i\tau x)+\mathbb{J}_{p,q}^{m,n}(-i\tau x)\right] \tag{28}$$

$$\mathbb{J}\sin_{p,q}^{m,n}(\tau x)=\frac{1}{2i}\left[\mathbb{J}_{p,q}^{m,n}(i\tau x)-\mathbb{J}_{p,q}^{m,n}(-i\tau x)\right] \tag{29}$$

$$\mathbb{J}_{p,q}^{m,n}\left(i\tau x\right)=\mathbb{J}\cos_{p,q}^{m,n}\left(\tau x\right)+i\mathbb{J}\sin_{p,q}^{m,n}\left(\tau x\right) \tag{30}$$

$$\mathbb{J}_{p,q}^{m,n}\left(-i\tau x\right)=\mathbb{J}\cos_{p,q}^{m,n}\left(\tau x\right)-i\mathbb{J}\sin_{p,q}^{m,n}\left(\tau x\right) \tag{31}$$

$$\mathbb{J}\cos_{p,q}^{m,n}\left(-\tau x\right)=\mathbb{J}\cos_{p,q}^{m,n}\left(\tau x\right) \tag{32}$$

$$\mathbb{J}\sin_{p,q}^{m,n}\left(-\tau x\right)=-\mathbb{J}\sin_{p,q}^{m,n}\left(\tau x\right) \tag{33}$$

The supertrigonometric functions associated with the H function

By using eq. (7), we see:

$$\mathbb{H}_{p,q}^{m,n}\left(e^{\lambda x}\right)=\sum_{l=0}^{\infty}\mathbb{J}_{p,q}^{m,n}\left(l\right)\frac{\left(\lambda x\right)^{l}}{l!} \tag{34}$$

This implies:

$$\mathbb{H}_{p,q}^{m,n}\left(e^{i\tau x}\right)=\sum_{l=0}^{\infty}\mathbb{J}_{p,q}^{m,n}\left(l\right)\frac{\left(i\tau x\right)^{l}}{l!} \tag{35}$$

and

$$\mathbb{H}_{p,q}^{m,n}\left(e^{-i\tau x}\right)=\sum_{l=0}^{\infty}\mathbb{J}_{p,q}^{m,n}\left(l\right)\frac{\left(-i\tau x\right)^{l}}{l!} \tag{36}$$

The supersine function associated with the H function is defined:

$$\mathbb{H}\sin_{p,q}^{m,n}\left(e^{ix}\right)=\sum_{l=0}^{\infty}\left(-1\right)^{l}\mathbb{J}_{p,q}^{m,n}\left(2l+1\right)\frac{x^{2l+1}}{\left(1+2l\right)!} \tag{37}$$

The supercosine function associated with the H function is defined:

$$\mathbb{H}\cos_{p,q}^{m,n}\left(e^{ix}\right)=\sum_{l=0}^{\infty}\left(-1\right)^{l}\mathbb{J}_{p,q}^{m,n}\left(2l\right)\frac{x^{2l}}{\left(2l\right)!} \tag{38}$$

The supertangent function associated with the H function is defined:

$$\mathbb{H}\tan_{p,q}^{m,n}\left(e^{ix}\right)=\frac{\mathbb{H}\cos_{p,q}^{m,n}\left(e^{ix}\right)}{\mathbb{H}\sin_{p,q}^{m,n}\left(e^{ix}\right)},\ \left[\mathbb{H}\sin_{p,q}^{m,n}\left(e^{ix}\right)\neq 0\right] \tag{39}$$

The supercotangent function associated with the H function is defined:

$$\mathbb{H}\operatorname{cotan}_{p,q}^{m,n}\left(e^{ix}\right)=\frac{\mathbb{H}\sin_{p,q}^{m,n}\left(e^{ix}\right)}{\mathbb{H}\cos_{p,q}^{m,n}\left(e^{ix}\right)},\ \left[\mathbb{H}\cos_{p,q}^{m,n}\left(e^{ix}\right)\neq 0\right] \tag{40}$$

The supercosecant function associated with the H function is defined

$$\mathbb{H}\sec_{p,q}^{m,n}\left(e^{ix}\right)=\frac{1}{\mathbb{H}\cos_{p,q}^{m,n}\left(e^{ix}\right)},\ \left[\mathbb{H}\cos_{p,q}^{m,n}\left(e^{ix}\right)\neq 0\right] \tag{41}$$

The supersecant function associated with the H function is defined:

$$\mathbb{H}\csc_{p,q}^{m,n}\left(e^{ix}\right)=\frac{1}{\mathbb{H}\sin_{p,q}^{m,n}\left(e^{ix}\right)},\ \left[\mathbb{H}\sin_{p,q}^{m,n}\left(e^{ix}\right)\neq 0\right] \tag{42}$$

Thus, we show:

$$\mathbb{H}\cos_{p,q}^{m,n}\left(e^{i\tau x}\right)=\frac{1}{2}\left[\mathbb{H}_{p,q}^{m,n}\left(e^{i\tau x}\right)+\mathbb{H}_{p,q}^{m,n}\left(e^{-i\tau x}\right)\right] \tag{43}$$

$$\mathbb{H}\sin_{p,q}^{m,n}\left(e^{i\tau x}\right)=\frac{1}{2i}\left[\mathbb{H}_{p,q}^{m,n}\left(e^{i\tau x}\right)-\mathbb{H}_{p,q}^{m,n}\left(e^{-i\tau x}\right)\right] \tag{44}$$

$$\mathbb{H}_{p,q}^{m,n}\left(e^{i\tau x}\right)=\mathbb{H}\cos_{p,q}^{m,n}\left(e^{i\tau x}\right)+i\mathbb{H}\sin_{p,q}^{m,n}\left(e^{i\tau x}\right) \tag{45}$$

$$\mathbb{H}_{p,q}^{m,n}\left(\mathrm{e}^{-i\tau x}\right)=\mathbb{H}\cos_{p,q}^{m,n}\left(\mathrm{e}^{i\tau x}\right)-i\mathbb{H}\sin_{p,q}^{m,n}\left(\mathrm{e}^{i\tau x}\right) \tag{46}$$

$$\mathbb{H}\cos_{p,q}^{m,n}\left(e^{-i\tau x}\right)=\mathbb{H}\cos_{p,q}^{m,n}\left(e^{i\tau x}\right) \tag{47}$$

$$\mathbb{H}\sin_{p,q}^{m,n}\left(e^{-i\tau x}\right)=-\mathbb{H}\sin_{p,q}^{m,n}\left(e^{i\tau x}\right) \tag{48}$$

The superhyperbolic functions associated with the J function

With the aid of eq. (15) we get:

$$\mathbb{J}_{p,q}^{m,n}\left(\lambda x\right)=\sum_{l=0}^{\infty}\boldsymbol{\kappa}_{p,q}^{m,n}\left(l;1\right)\frac{\left(\lambda x\right)^{l}}{l!} \tag{49}$$

and

$$\mathbb{J}_{p,q}^{m,n}\left(-\lambda x\right)=\sum_{l=0}^{\infty}\boldsymbol{\kappa}_{p,q}^{m,n}\left(l;1\right)\frac{\left(-\lambda x\right)^{l}}{l!} \tag{50}$$

The superhyperbolic sine function associated with the J function is defined:

$$\mathbb{J}\sinh_{p,q}^{m,n}\left(x\right)=\sum_{l=0}^{\infty}\boldsymbol{\kappa}_{p,q}^{m,n}\left(2l+1;1\right)\frac{x^{2l+1}}{\left(2l+1\right)!} \tag{51}$$

The superhyperbolic cosine function associated with the J function is defined:

$$\mathbb{J}\cosh_{p,q}^{m,n}\left(x\right)=\sum_{l=0}^{\infty}\boldsymbol{\kappa}_{p,q}^{m,n}\left(2l;1\right)\frac{x^{2l}}{\left(2l\right)!} \tag{52}$$

The superhyperbolic tangent function associated with the J function is defined:

$$\mathbb{J}\tanh_{p,q}^{m,n}\left(x\right)=\frac{\mathbb{J}\cosh_{p,q}^{m,n}\left(x\right)}{\mathbb{J}\sinh_{p,q}^{m,n}\left(x\right)},\ \left[\mathbb{J}\sinh_{p,q}^{m,n}\left(x\right)\neq 0\right] \tag{53}$$

The superhyperbolic cotangent function associated with the J function is defined:

$$\mathbb{J}\mathrm{cotanh}_{p,q}^{m,n}\left(x\right)=\frac{\mathbb{J}\sinh_{p,q}^{m,n}\left(x\right)}{\mathbb{J}\cosh_{p,q}^{m,n}\left(x\right)},\ \left[\mathbb{J}\cosh_{p,q}^{m,n}\left(x\right)\neq 0\right] \tag{54}$$

The superhyperbolic cosecant function associated with the J function is defined:

$$\mathbb{J}\mathrm{sech}_{p,q}^{m,n}\left(x\right)=\frac{1}{\mathbb{J}\cosh_{p,q}^{m,n}\left(x\right)},\ \left[\mathbb{J}\cosh_{p,q}^{m,n}\left(x\right)\neq 0\right] \tag{55}$$

The superhyperbolic secant function associated with the J function is defined:

$$\mathbb{J}\text{csch}\,_{p,q}^{m,n}(x)=\frac{1}{\mathbb{J}\sinh\,_{p,q}^{m,n}(x)},\ \left[\mathbb{J}\sinh\,_{p,q}^{m,n}(x)\neq 0\right] \tag{56}$$

Thus, it is easy to show:

$$\mathbb{J}\cosh\,_{p,q}^{m,n}(\tau x)=\frac{1}{2}\left[\mathbb{J}\,_{p,q}^{m,n}(\tau x)+\mathbb{J}\,_{p,q}^{m,n}(-\tau x)\right] \tag{57}$$

$$\mathbb{J}\sinh\,_{p,q}^{m,n}(\tau x)=\frac{1}{2}\left[\mathbb{J}\,_{p,q}^{m,n}(\tau x)-\mathbb{J}\,_{p,q}^{m,n}(-\tau x)\right] \tag{58}$$

$$\mathbb{J}\,_{p,q}^{m,n}(\tau x)=\mathbb{J}\cosh\,_{p,q}^{m,n}(\tau x)+\mathbb{J}\sinh\,_{p,q}^{m,n}(\tau x) \tag{59}$$

$$\mathbb{J}\,_{p,q}^{m,n}(-\tau x)=\mathbb{J}\cosh\,_{p,q}^{m,n}(\tau x)-\mathbb{J}\sinh\,_{p,q}^{m,n}(\tau x) \tag{60}$$

$$\mathbb{J}\cosh\,_{p,q}^{m,n}(-\tau x)=\mathbb{J}\cosh\,_{p,q}^{m,n}(\tau x) \tag{61}$$

$$\mathbb{J}\sinh\,_{p,q}^{m,n}(-\tau x)=-\mathbb{J}\sinh\,_{p,q}^{m,n}(\tau x) \tag{62}$$

The superhyperbolic functions associated with the H function

By using eq. (7), we see:

$$\mathbb{H}_{p,q}^{m,n}\left(e^{\lambda x}\right)=\sum_{l=0}^{\infty}\mathbb{J}_{p,q}^{m,n}(l)\frac{(\lambda x)^{l}}{l!} \tag{63}$$

and

$$\mathbb{H}_{p,q}^{m,n}\left(e^{-\lambda x}\right)=\sum_{l=0}^{\infty}\mathbb{J}_{p,q}^{m,n}(l)\frac{(-\lambda x)^{l}}{l!} \tag{64}$$

The superhyperbolic sine function associated with the H function is defined:

$$\mathbb{H}\sinh_{p,q}^{m,n}\left(e^{x}\right)=\sum_{l=0}^{\infty}\mathbb{J}_{p,q}^{m,n}(2l+1)\frac{x^{2l+1}}{(1+2l)!} \tag{65}$$

The superhyperbolic cosine function associated with the H function is defined:

$$\mathbb{H}\cosh\,_{p,q}^{m,n}\left(e^{x}\right)=\sum_{l=0}^{\infty}\mathbb{J}_{p,q}^{m,n}(2l)\frac{x^{2l}}{(2l)!} \tag{66}$$

The superhyperbolic tangent function associated with the H function is defined:

$$\mathbb{H}\tanh\,_{p,q}^{m,n}\left(e^{x}\right)=\frac{\mathbb{H}\cosh\,_{p,q}^{m,n}\left(e^{x}\right)}{\mathbb{H}\sinh\,_{p,q}^{m,n}\left(e^{x}\right)},\ \left[\mathbb{H}\sinh\,_{p,q}^{m,n}\left(e^{x}\right)\neq 0\right] \tag{67}$$

The superhyperbolic cotangent function associated with the H function is defined:

$$\mathbb{H}\text{cotanh}\,_{p,q}^{m,n}\left(e^{x}\right)=\frac{\mathbb{H}\sinh\,_{p,q}^{m,n}\left(e^{x}\right)}{\mathbb{H}\cosh\,_{p,q}^{m,n}\left(e^{x}\right)},\ \left[\mathbb{H}\cosh\,_{p,q}^{m,n}\left(e^{x}\right)\neq 0\right] \tag{68}$$

The superhyperbolic cosecant function associated with the H function is defined:

$$\mathbb{H}\operatorname{sech}{}_{p,q}^{m,n}\left(e^{x}\right)=\frac{1}{\mathbb{H}\cosh{}_{p,q}^{m,n}\left(e^{x}\right)},\ \left[\mathbb{H}\cosh{}_{p,q}^{m,n}\left(e^{x}\right)\neq 0\right] \tag{69}$$

The superhyperbolic secant function associated with the H function is defined:

$$\mathbb{H}\operatorname{csch}{}_{p,q}^{m,n}\left(e^{x}\right)=\frac{1}{\mathbb{H}\sinh{}_{p,q}^{m,n}\left(e^{x}\right)},\ \left[\mathbb{H}\sinh{}_{p,q}^{m,n}\left(e^{x}\right)\neq 0\right] \tag{70}$$

Thus, we arrive:

$$\mathbb{H}\cosh{}_{p,q}^{m,n}\left(e^{\tau x}\right)=\frac{1}{2}\left[\mathbb{H}{}_{p,q}^{m,n}\left(e^{\tau x}\right)+\mathbb{H}{}_{p,q}^{m,n}\left(e^{-\tau x}\right)\right] \tag{71}$$

$$\mathbb{H}\sinh{}_{p,q}^{m,n}\left(e^{\tau x}\right)=\frac{1}{2}\left[\mathbb{H}{}_{p,q}^{m,n}\left(e^{\tau x}\right)-\mathbb{H}{}_{p,q}^{m,n}\left(e^{-\tau x}\right)\right] \tag{72}$$

$$\mathbb{H}{}_{p,q}^{m,n}\left(e^{i\tau x}\right)=\mathbb{H}\cosh{}_{p,q}^{m,n}\left(e^{\tau x}\right)+\mathbb{H}\sinh{}_{p,q}^{m,n}\left(e^{\tau x}\right) \tag{73}$$

$$\mathbb{H}{}_{p,q}^{m,n}\left(e^{-\tau x}\right)=\mathbb{H}\cosh{}_{p,q}^{m,n}\left(e^{\tau x}\right)-\mathbb{H}\sinh{}_{p,q}^{m,n}\left(e^{\tau x}\right) \tag{74}$$

$$\mathbb{H}\cosh{}_{p,q}^{m,n}\left(e^{-\tau x}\right)=\mathbb{H}\cosh{}_{p,q}^{m,n}\left(e^{\tau x}\right) \tag{75}$$

$$\mathbb{H}\sinh{}_{p,q}^{m,n}\left(e^{-\tau x}\right)=-\mathbb{H}\sinh{}_{p,q}^{m,n}\left(e^{\tau x}\right) \tag{76}$$

The relationships and representations of some special functions

The relationships to the supertrigonometric and superhyperbolic functions associated with the J and H functions can be given:

$$\mathbb{J}\tanh{}_{p,q}^{m,n}\left(ix\right)=i\mathbb{J}\tan{}_{p,q}^{m,n}\left(x\right) \tag{77}$$

$$\mathbb{J}\tanh{}_{p,q}^{m,n}\left(x\right)=-i\mathbb{J}\tan{}_{p,q}^{m,n}\left(ix\right) \tag{78}$$

$$\mathbb{J}\cosh{}_{p,q}^{m,n}\left(x\right)=\mathbb{J}\cos{}_{p,q}^{m,n}\left(ix\right) \tag{79}$$

$$\mathbb{J}\cosh{}_{p,q}^{m,n}\left(ix\right)=\mathbb{J}\cos{}_{p,q}^{m,n}\left(x\right) \tag{80}$$

$$\mathbb{J}\sinh{}_{p,q}^{m,n}\left(x\right)=-i\mathbb{J}\sin{}_{p,q}^{m,n}\left(ix\right) \tag{81}$$

$$\mathbb{J}\sinh{}_{p,q}^{m,n}\left(ix\right)=i\mathbb{J}\sin{}_{p,q}^{m,n}\left(x\right) \tag{82}$$

$$\mathbb{H}\tanh{}_{p,q}^{m,n}\left(e^{ix}\right)=i\mathbb{H}\tan{}_{p,q}^{m,n}\left(e^{ix}\right) \tag{83}$$

$$\mathbb{H}\tanh_{p,q}^{m,n}\left(e^{x}\right)=-i\mathbb{H}\tan_{p,q}^{m,n}\left(e^{-x}\right) \tag{84}$$

$$\mathbb{H}\cosh_{p,q}^{m,n}\left(e^{x}\right)=\mathbb{H}\cos_{p,q}^{m,n}\left(e^{-x}\right) \tag{85}$$

$$\mathbb{H}\cosh_{p,q}^{m,n}\left(e^{ix}\right)=\mathbb{H}\cos_{p,q}^{m,n}\left(e^{ix}\right) \tag{86}$$

$$\mathbb{H}\sinh_{p,q}^{m,n}\left(e^{x}\right)=-i\mathbb{H}\sin_{p,q}^{m,n}\left(e^{-x}\right) \tag{87}$$

and

$$\mathbb{H}\sinh_{p,q}^{m,n}\left(e^{ix}\right)=i\mathbb{H}\sin_{p,q}^{m,n}\left(e^{ix}\right) \tag{88}$$

Let $E_\chi(x)$, $E_{\chi\beta}(x)$, $E^{\zeta}_{\chi\beta}(x)$, and $E^{\zeta\varpi}_{\chi\beta}(x)$ be the Mittag-Leffler [9], Wiman [10], Prabhakar [11], and four-parameter Mittag-Leffler [12] functions, respectively. Then we have [5, 12]:

$$e^{-x}=\mathbb{H}_{0,1}^{1,0}\left[x;\begin{matrix}-\\(0,1)\end{matrix}\right],\quad \left(1-x\right)^{-\vartheta}=\frac{1}{\Gamma(\vartheta)}\mathbb{H}_{1,1}^{1,1}\left[-x;\begin{matrix}(1-\vartheta,1)\\(0,1)\end{matrix}\right]$$

$$E_{\chi}\left(x\right)=\mathbb{H}_{1,2}^{1,1}\left[-x;\begin{matrix}(0,1)\\(0,1),(0,\chi)\end{matrix}\right],\ E_{\chi,\beta}\left(x\right)=\mathbb{H}_{1,2}^{1,1}\left[-x;\begin{matrix}(0,1)\\(0,1),(1-\beta,\chi)\end{matrix}\right]$$

$$E_{\chi,\beta}^{\varsigma}\left(x\right)=\frac{1}{\Gamma(\varsigma)}\mathbb{H}_{1,2}^{1,1}\left[-x;\begin{matrix}(1-\varsigma,1)\\(0,1),(1-\beta,\chi)\end{matrix}\right]$$

and

$$E_{\chi,\beta}^{\varsigma,\varpi}\left(x\right)=\frac{1}{\Gamma(\varsigma)}\mathbb{H}_{1,2}^{1,1}\left[-x;\begin{matrix}(1-\varsigma,\varpi)\\(0,1),(1-\beta,\chi)\end{matrix}\right]$$

such that

$$e^{-x}=\sum_{l=0}^{\infty}\mathbb{J}_{0,1}^{1,0}\left[l;\begin{matrix}-\\(0,1)\end{matrix}\right]\frac{(\log x)^{l}}{l!} \tag{89}$$

$$\left(1-x\right)^{-\vartheta}=\frac{1}{\Gamma(\vartheta)}\sum_{l=0}^{\infty}\mathbb{J}_{1,1}^{1,1}\left[-l;\begin{matrix}(1-\vartheta,1)\\(0,1)\end{matrix}\right]\frac{(\log x)^{l}}{l!} \tag{90}$$

$$E_{\chi}\left(x\right)=\sum_{l=0}^{\infty}\mathbb{J}_{1,2}^{1,1}\left[-l;\begin{matrix}(0,1)\\(0,1),(0,\chi)\end{matrix}\right]\frac{(\log x)^{l}}{l!} \tag{91}$$

$$E_{\chi,\beta}\left(x\right)=\sum_{l=0}^{\infty}\mathbb{J}_{1,2}^{1,1}\left[-l;\begin{matrix}(0,1)\\(0,1),(1-\beta,\chi)\end{matrix}\right]\frac{(\log x)^{l}}{l!} \tag{92}$$

$$E_{\chi,\beta}^{\varsigma}\left(x\right)=\frac{1}{\Gamma(\varsigma)}\sum_{l=0}^{\infty}\mathbb{J}_{1,2}^{1,1}\left[-l;\begin{matrix}(1-\varsigma,1)\\(0,1),(1-\beta,\chi)\end{matrix}\right]\frac{(\log x)^{l}}{l!} \tag{93}$$

$$E_{\chi,\beta}^{\varsigma,\varpi}\left(x\right)=\frac{1}{\Gamma\left(\varsigma\right)}\sum_{l=0}^{\infty}\mathbb{J}_{1,2}^{1,1}\left[-l;\begin{matrix}\left(1-\varsigma,\varpi\right)\\ \left(0,1\right),\left(1-\beta,\chi\right)\end{matrix}\right]\frac{\left(\log x\right)^{l}}{l!} \tag{94}$$

These imply that:

$$e^{-e^{x}}=\sum_{l=0}^{\infty}\mathbb{J}_{0,1}^{1,0}\left[l;\begin{matrix}-\\ \left(0,1\right)\end{matrix}\right]\frac{x^{l}}{l!} \tag{95}$$

$$\left(1-e^{x}\right)^{-\vartheta}=\frac{1}{\Gamma\left(\vartheta\right)}\sum_{l=0}^{\infty}\mathbb{J}_{1,1}^{1,1}\left[-l;\begin{matrix}\left(1-\vartheta,1\right)\\ \left(0,1\right)\end{matrix}\right]\frac{x^{l}}{l!} \tag{96}$$

$$E_{\chi}\left(e^{x}\right)=\sum_{l=0}^{\infty}\mathbb{J}_{1,2}^{1,1}\left[-l;\begin{matrix}\left(0,1\right)\\ \left(0,1\right),\left(0,\chi\right)\end{matrix}\right]\frac{x^{l}}{l!} \tag{97}$$

$$E_{\chi,\beta}\left(e^{x}\right)=\sum_{l=0}^{\infty}\mathbb{J}_{1,2}^{1,1}\left[-l;\begin{matrix}\left(0,1\right)\\ \left(0,1\right),\left(1-\beta,\chi\right)\end{matrix}\right]\frac{x^{l}}{l!} \tag{98}$$

$$E_{\chi,\beta}^{\varsigma}\left(e^{x}\right)=\frac{1}{\Gamma\left(\varsigma\right)}\sum_{l=0}^{\infty}\mathbb{J}_{1,2}^{1,1}\left[-l;\begin{matrix}\left(1-\varsigma,1\right)\\ \left(0,1\right),\left(1-\beta,\chi\right)\end{matrix}\right]\frac{x^{l}}{l!} \tag{99}$$

and

$$E_{\chi,\beta}^{\varsigma,\varpi}\left(e^{x}\right)=\frac{1}{\Gamma\left(\varsigma\right)}\sum_{l=0}^{\infty}\mathbb{J}_{1,2}^{1,1}\left[-l;\begin{matrix}\left(1-\varsigma,\varpi\right)\\ \left(0,1\right),\left(1-\beta,\chi\right)\end{matrix}\right]\frac{x^{l}}{l!} \tag{100}$$

Making use of eqs. (95) and (96), it is easy to show:

$$\frac{1}{2}\left(e^{-e^{ix}}+e^{-e^{-ix}}\right)=\mathbb{H}\cos_{0,1}^{1,0}\left[e^{ix};\begin{matrix}-\\ \left(0,1\right)\end{matrix}\right]=\sum_{l=0}^{\infty}\left(-1\right)^{l}\mathbb{J}_{0,1}^{1,0}\left[2l;\begin{matrix}-\\ \left(0,1\right)\end{matrix}\right]\frac{x^{2l}}{\left(2l\right)!} \tag{101}$$

$$\frac{1}{2i}\left(e^{-e^{ix}}-e^{-e^{-ix}}\right)=\mathbb{H}\sin_{0,1}^{1,0}\left[e^{ix};\begin{matrix}-\\ \left(0,1\right)\end{matrix}\right]=\sum_{l=0}^{\infty}\left(-1\right)^{l}\mathbb{J}_{0,1}^{1,0}\left[2l+1;\begin{matrix}-\\ \left(0,1\right)\end{matrix}\right]\frac{x^{2l+1}}{\left(2l+1\right)!} \tag{102}$$

$$\frac{1}{2}\left(e^{-e^{x}}+e^{-e^{-x}}\right)=\mathbb{H}\cosh_{0,1}^{1,0}\left[e^{x};\begin{matrix}-\\ \left(0,1\right)\end{matrix}\right]=\sum_{l=0}^{\infty}\mathbb{J}_{0,1}^{1,0}\left[2l;\begin{matrix}-\\ \left(0,1\right)\end{matrix}\right]\frac{x^{2l}}{\left(2l\right)!} \tag{103}$$

$$\frac{1}{2}\left(e^{-e^{x}}-e^{-e^{-x}}\right)=\mathbb{H}\sinh_{0,1}^{1,0}\left[e^{x};\begin{matrix}-\\ \left(0,1\right)\end{matrix}\right]=\sum_{l=0}^{\infty}\mathbb{J}_{0,1}^{1,0}\left[2l+1;\begin{matrix}-\\ \left(0,1\right)\end{matrix}\right]\frac{x^{2l+1}}{\left(2l+1\right)!} \tag{104}$$

$$\frac{1}{2}\left[\left(1-e^{ix}\right)^{-\vartheta}+\left(1-e^{-ix}\right)^{-\vartheta}\right]=\frac{1}{\Gamma\left(\vartheta\right)}\sum_{l=0}^{\infty}\left(-1\right)^{l}\mathbb{J}_{1,1}^{1,1}\left[2l;\begin{matrix}\left(1-\vartheta,1\right)\\ \left(0,1\right)\end{matrix}\right]\frac{x^{2l}}{\left(2l\right)!} \tag{105}$$

$$\frac{1}{2}\left[\left(1-e^{x}\right)^{-\vartheta}+\left(1-e^{-x}\right)^{-\vartheta}\right]=\frac{1}{\Gamma\left(\vartheta\right)}\sum_{l=0}^{\infty}\mathbb{J}_{1,1}^{1,1}\left[2l;\begin{matrix}\left(1-\vartheta,1\right)\\ \left(0,1\right)\end{matrix}\right]\frac{x^{2l}}{\left(2l\right)!} \tag{106}$$

and

$$\frac{1}{2}\left[\left(1-e^{x}\right)^{-\vartheta}-\left(1-e^{-x}\right)^{-\vartheta}\right]=\frac{1}{\Gamma(\vartheta)}\sum_{l=0}^{\infty}(-1)^{l}\,\mathbb{J}_{1,1}^{1,1}\left[2l+1;\begin{matrix}(1-\vartheta,1)\\(0,1)\end{matrix}\right]\frac{x^{2l+1}}{(2l+1)!} \tag{107}$$

A typical application representation of the solution for the heat-diffusion problem

We now consider the 1-D heat-diffusion equation [13]:

$$\frac{\partial\Omega(x,t)}{\partial t}=\aleph\frac{\partial^{2}\Omega(x,t)}{\partial x^{2}} \tag{108}$$

with the initial value condition:

$$\Omega(x,0)=\delta(x) \tag{109}$$

where $\aleph$ is the thermal diffusivity and $\Omega(x, t)$ represents the temperature function.

The solution for eq. (108) reads:

$$\Omega(x,t)=\sqrt{\frac{1}{4\aleph\pi t}}\,\mathbb{H}_{0,1}^{1,0}\left[\frac{x^{2}}{4\aleph t};\begin{matrix}-\\(0,1)\end{matrix}\right]=\sqrt{\frac{1}{4\aleph\pi t}}\sum_{l=0}^{\infty}\mathbb{J}_{0,1}^{1,0}\left[l;\begin{matrix}-\\(0,1)\end{matrix}\right]\frac{\left(2\log x-4\aleph\log t\right)^{l}}{l!} \tag{110}$$

since there exists [13]:

$$\Omega(x,t)=\sqrt{\frac{1}{4\aleph\pi t}}\,e^{-\frac{x^{2}}{4\aleph t}} \tag{111}$$

Conclusion

In this work we have suggested the theory of the supertrigonometric and superhyperbolic functions associated with the J and H functions. We gave the series representations for the special functions and the series representation for the heat-diffusion equations. The obtained results are proposed as a useful mathematical tools to present the series solutions for PDE in mathematical physics.

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Nomenclature

t – space co-ordinate, [s]
x – pace co-ordinate, [m]
$\aleph$ – thermal conductivity, [$Wm^{-1}K^{-1}$]

Greek symbol

$\Omega(x, t)$ – temperature function, [K]

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