NEW INSIGHTS ON THE J AND Y FUNCTIONS IN THE HEAT TRANSFER

by

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In this article, we propose the integral and differential operators within the kernel of the Y function for the first time. We study the properties of the J and Y functions. We also present the some new applications of the heat transfer and present the new representation for the solution of the heat equation in the 1-D case.

Key words: *the J function, the Y function, special function, heat equation, heat transfer*

Introduction

The special functions [1] have been played the important roles in the fields of mathematical physics and heat engineering. The connected work of the special function in mathematical physics was presented in [2]. The special functions related to the Bessel's functions were used in the theory of the heat exchanger process [3]. The special functions in the terms of the series of modified Bessel functions were considered to solve the heat exchanger problem [4]. The wright function and Mittage-Leffler function were proposed to solve the PDE in the heat and mass transfer [5].

Recently, the Y and J functions with their families were proposed by the author [6] to give the representation of the solution of the fractional diffusion problem in the fractional time and space and to connect with the H function [7], G function [8], Wright generalized hypergeometric function [9], and Clausen hypergeometric function [10].

The main target of the paper is to propose new integral and differential operators within the kernel of the Y function, to give some applications of the heat transfer, and to connect with the Mittag-Leffler [11], Wiman [12], Prabhakar [13], and Kohlrausch [14], and Williams and Watts [15] functions.

The J and Y functions: definitions and properties

Let $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, and $\mathbb{R}_+$ be the sets of the real, complex, integer, positive real numbers, respectively, and let $\Re(s)$ be the real part of a complex variable $s \in \mathbb{C}$.

Suppose that $x \in \mathbb{R}$, $y \in \mathbb{C}$, $z \in \mathbb{C}$, $q \geq 1$, $0 \leq n \leq p$, $0 \leq m \leq q$, $\{\gamma_j, \theta_j\} \in \mathbb{C}$, and $\{\rho_j, \theta_j\} \in \mathbb{R}_+$:

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$$\mathbf{A}_1(s)=\prod_{j=1}^{m}\Gamma\left(\theta_j-\varpi_j s\right),\ \ \mathbf{B}_1(s)=\prod_{j=1}^{n}\Gamma\left(1-\gamma_j+\rho_j s\right) \tag{1}$$

and

$$\mathbf{C}_1(s)=\prod_{j=m+1}^{q}\Gamma\left(1-\theta_j+\varpi_j s\right),\ \ \mathbf{D}_1(s)=\prod_{j=n+1}^{p}\Gamma\left(\gamma_j-\rho_j s\right) \tag{2}$$

They have the poles:

$$k_j^{\ell}=-\frac{\theta_j+\ell}{\beta_j}\left(j=1,n;\ell\in\mathbb{N}\cup\{0\}\right) \tag{3}$$

and

$$k_j^{\hbar}=\frac{1-a_j+\hbar}{\alpha_j}\left(j=1,n;\hbar\in\mathbb{N}\cup\{0\}\right) \tag{4}$$

The J function:

$$\mathbb{J}_{p,q}^{m,n}(x)=\mathbb{J}_{p,q}^{m,n}\left[x\middle|\begin{matrix}(\gamma_1,\rho_1),(\gamma_2,\rho_2),\cdots,(\gamma_p,\rho_p)\\(\theta_1,\varpi_1),(\theta_2,\varpi_2),\cdots,(\theta_q,\varpi_q)\end{matrix}\right]=\mathbb{J}_{p,q}^{m,n}\left[x;\begin{matrix}\{\gamma_j,\rho_j\}_1^p\\\{\theta_j,\varpi_j\}_1^q\end{matrix}\right] \tag{5}$$

is defined [6]:

$$\mathbb{J}_{p,q}^{m,n}(x)=\frac{1}{2\pi i}\int_L\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}s^x\mathrm{d}s \tag{6}$$

provided the integral exists, where L is a suitable contour in the complex plane.

Taking $x=0$, we have:

$$\mathbb{J}_{p,q}^{m,n}(0)=\frac{1}{2\pi i}\int_L\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}\mathrm{d}s \tag{7}$$

which is connected with the special value of the Fox H function:

$$\mathbb{H}_{p,q}^{m,n}(1)=\frac{1}{2\pi i}\int_L\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}\mathrm{d}s \tag{8}$$

and the theorem of Barnes [16]:

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(s+\alpha_1)\Gamma(s+\beta_1)\Gamma(\alpha_2-s)\Gamma(\beta_2-s)\mathrm{d}s=\mathbb{J}_{0,0}^{2,2}\left[0\middle|\begin{matrix}(1-\alpha_1,1),(1-\beta_1,1)\\(\alpha_2,1),(\beta_2,1)\end{matrix}\right] \tag{9}$$

where the Fox H function is denoted [17]:

$$\mathbb{H}_{p,q}^{m,n}(x)=\mathbb{H}_{p,q}^{m,n}\left[x\middle|\begin{matrix}(\gamma_1,\rho_1),(\gamma_2,\rho_2),\cdots,(\gamma_p,\rho_p)\\(\theta_1,\varpi_1),(\theta_2,\varpi_2),\cdots,(\theta_q,\varpi_q)\end{matrix}\right]=\mathbb{H}_{p,q}^{m,n}\left[x;\begin{matrix}\{\gamma_j,\rho_j\}_1^p\\\{\theta_j,\varpi_j\}_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}x^s\mathrm{d}s \tag{10}$$

The Y function:

$$\mathbb{Y}_{p,q}^{m,n}[y;z;x]=\mathbb{Y}_{p,q}^{m,n}\left[y;z;x\middle|\begin{matrix}(\gamma_1,\rho_1),(\gamma_2,\rho_2),\cdots,(\gamma_p,\rho_p)\\(\theta_1,\varpi_1),(\theta_2,\varpi_2),\cdots,(\theta_q,\varpi_q)\end{matrix}\right]=\mathbb{Y}_{p,q}^{m,n}\left[x;y;\alpha;\begin{matrix}\{\gamma_j,\rho_j\}_1^p\\\{\theta_j,\varpi_j\}_1^q\end{matrix}\right] \tag{11}$$

is defined:

$$\mathbb{Y}_{p,q}^{m,n}\left[y;z;x\right]=\frac{1}{2\pi i}\int_{L}\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}y^{-s}\mathrm{e}^{sz}s^{x}\mathrm{d}s \tag{12}$$

provided the integral exists, where L is a suitable contour in the complex plane.

The Mellin transform of $f(x)$ has the form [10]:

$$M\left\{f(x)\right\}=F(s)=\int_{0}^{\infty}f(x)x^{s-1}\mathrm{d}x \tag{13}$$

and the inverse Mellin transform is given:

$$M^{-1}\left\{F(s)\right\}=f(x)=\frac{1}{2\pi i}\int_{\Re(s)=r}F(s)x^{-s}\mathrm{d}s \tag{14}$$

where $\Re(s)=r\in\mathbb{R}$.

For $\eta\geq 0$, there exist [6]:

$$\int_{0}^{\infty}\mathbb{Y}_{p,q}^{m,n}\left[\eta y;z;x\right]x^{s-1}\mathrm{d}x=\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}\eta^{-s}\mathrm{e}^{sz}s^{x} \tag{15}$$

such that:

$$\int_{0}^{\infty}y^{\eta}\mathbb{Y}_{p,q}^{m,n}\left[y;z;x\right]x^{s-1}\mathrm{d}x=\frac{\mathbf{A}_1(s+\eta)\mathbf{B}_1(s+\eta)}{\mathbf{C}_1(s+\eta)\mathbf{D}_1(s+\eta)}\mathrm{e}^{(s+\eta)z}(s+\eta)^{x} \tag{16}$$

For $\eta\geq 0$ and $\omega\in\mathbb{R}\setminus\{0\}$, it is not difficult to show that:

$$\int_{0}^{\infty}\mathbb{Y}_{p,q}^{m,n}\left[y^{\omega};z;x\right]x^{s-1}\mathrm{d}x=\frac{1}{|\omega|}\frac{\mathbf{A}_1(s/\omega)\mathbf{B}_1(s/\omega)}{\mathbf{C}_1(s/\omega)\mathbf{D}_1(s/\omega)}\mathrm{e}^{sz/\omega}(s/\omega)^{x} \tag{17}$$

$$\int_{0}^{\infty}\mathbb{Y}_{p,q}^{m,n}\left[\eta y^{\omega};z;x\right]x^{s-1}\mathrm{d}x=\frac{1}{|\omega|}\frac{\mathbf{A}_1(s/\omega)\mathbf{B}_1(s/\omega)}{\mathbf{C}_1(s/\omega)\mathbf{D}_1(s/\omega)}\eta^{-s/\omega}\mathrm{e}^{sz/\omega}(s/\omega)^{x} \tag{18}$$

and

$$\int_{0}^{\infty}y^{\eta}\mathbb{Y}_{p,q}^{m,n}\left[y^{\omega};z;x\right]x^{s-1}\mathrm{d}x=\frac{1}{|\omega|}\frac{\mathbf{A}_1\left[(s+\eta)/\omega\right]\mathbf{B}_1\left[(s+\eta)/\omega\right]}{\mathbf{C}_1\left[(s+\eta)/\omega\right]\mathbf{D}_1\left[(s+\eta)/\omega\right]}\mathrm{e}^{\frac{z(s+\eta)}{\omega}}\left(\frac{s+\eta}{\omega}\right)^{x} \tag{19}$$

For $\sigma>0$ and $\Re(s)>0$, we have [18]:

$$M\left\{\varsigma(\sigma-x)\right\}=\frac{\sigma^{s}}{s} \tag{20}$$

such that:

$$\int_{0}^{\infty}\varsigma(\sigma-x)\mathbb{Y}_{p,q}^{m,n}\left[\eta y;z;x\right]x^{s-1}\mathrm{d}x=\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}\eta^{-s}\mathrm{e}^{sz}s^{x-1}\sigma^{s} \tag{21}$$

This implies:

$$\varsigma(\sigma-x)\mathbb{Y}_{p,q}^{m,n}\left[\eta y;z;x\right]=\frac{1}{2\pi i}\int_{L}\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}\eta^{-s}\mathrm{e}^{sz}s^{x-1}\sigma^{s}y^{-s}\mathrm{d}s \tag{22}$$

which leads:

$$\varsigma(1-x)\mathbb{Y}_{p,q}^{m,n}\left[1;0;x\right]=\mathbb{J}_{p,q}^{m,n}(x-1) \tag{23}$$

The Laplace transform of $g(x)$ has the form [1]:

$$L\{g(x)\} = G(s) = \int_0^\infty g(x)\mathrm{e}^{-sx}\mathrm{d}x \tag{24}$$

and the inverse Laplace transform is given:

$$L^{-1}\{G(s)\} = g(x) = \frac{1}{2\pi i}\int_{\Re(s)=r} G(s)\mathrm{e}^{sx}\mathrm{d}s \tag{25}$$

where $\Re(s) = r \in \mathbb{R}$.

Then, the Laplace transform of the Y function reads [6]:

$$\int_0^\infty \mathbb{Y}_{p,q}^{m,n}[y;\eta z;x]\mathrm{e}^{-sz}\mathrm{d}z = \frac{s^x}{y^s\eta^{\alpha+1}}\frac{\mathbf{A}_1(s/\eta)\mathbf{B}_1(s/\eta)}{\mathbf{C}_1(s/\eta)\mathbf{D}_1(s/\eta)} \tag{26}$$

The series representation for the Y function can be given [6]:

$$\mathbb{Y}_{p,q}^{m,n}[y;z;x] = \sum_{h=0}^\infty \frac{(z-\log y)^h}{\Gamma(1+h)}\mathbb{J}_{p,q}^{m,n}(h+x) \tag{27}$$

By using eqs. (23) and (27), we get:

$$\mathbb{Y}_{p,q}^{m,n}[y;z;x] = \sum_{h=0}^\infty \frac{(z-\log y)^h}{\Gamma(1+h)}\varsigma[-(h+x)]\mathbb{Y}_{p,q}^{m,n}[1;0;h+x+1] \tag{28}$$

since

$$\mathbb{J}_{p,q}^{m,n}(h+x) = \varsigma[-(h+x)]\mathbb{Y}_{p,q}^{m,n}[1;0;x+h+x] \tag{29}$$

Let $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbb{C}$. There are the properties:

(I)

$$\mathbb{Y}_{p,q}^{m,n}\left[y;z;x\left|\begin{matrix}(\zeta_1,0),(\gamma_2,\rho_2),\cdots,(\gamma_p,\rho_p)\\(\theta_1,\varpi_1),(\theta_2,\varpi_2),\cdots,(\theta_q,\varpi_q)\end{matrix}\right.\right] = \Gamma(1-\zeta_1)\mathbb{Y}_{p-1,q}^{m,n-1}\left[y;z;x\left|\begin{matrix}(\gamma_2,\rho_2),\cdots,(\gamma_p,\rho_p)\\(\theta_1,\varpi_1),\cdots,(\theta_q,\varpi_q)\end{matrix}\right.\right] \tag{30}$$

where $\Re(\zeta) < 1$ and $n \geq 1$.

(II)

$$\mathbb{Y}_{p,q}^{m,n}\left[y;z;x\left|\begin{matrix}(\gamma_1,\rho_1),(\gamma_2,\rho_2),\cdots,(\gamma_p,\rho_p)\\(\zeta_2,0),(\theta_2,\varpi_2),\cdots,(\theta_q,\varpi_q)\end{matrix}\right.\right] = \Gamma(\zeta_2)\mathbb{Y}_{p,q-1}^{m-1,n}\left[y;z;x\left|\begin{matrix}(\gamma_1,\rho_1),\cdots,(\gamma_p,\rho_p)\\(\theta_2,\varpi_2),\cdots,(\theta_q,\varpi_q)\end{matrix}\right.\right] \tag{31}$$

where $\Re(\zeta_2) > 0$ and $m \geq 1$.

(III)

$$\mathbb{Y}_{p,q}^{m,n}\left[y;z;x\left|\begin{matrix}(\gamma_1,\rho_1),\cdots,(\gamma_{p-1},\rho_{p-1}),(\zeta_3,0)\\(\theta_1,\varpi_1),(\theta_2,\varpi_2),\cdots,(\theta_q,\varpi_q)\end{matrix}\right.\right] = \frac{1}{\Gamma(\zeta_3)}\mathbb{Y}_{p-1,q}^{m,n}\left[y;z;x\left|\begin{matrix}(\gamma_1,\rho_1),\cdots,(\gamma_{p-1},\rho_{p-1})\\(\theta_1,\varpi_1),\cdots,(\theta_q,\varpi_q)\end{matrix}\right.\right] \tag{32}$$

where $\Re(\zeta_3) > 0$ and $p > n$.

(IV)

$$\mathbb{Y}_{p,q}^{m,n}\left[y;z;x\left|\begin{matrix}(\gamma_1,\rho_1),(\gamma_2,\rho_2),\cdots,(\gamma_p,\rho_p)\\(\theta_1,\varpi_1),\cdots,(\theta_{q-1},\varpi_{q-1}),(\zeta_4,0)\end{matrix}\right.\right] = \frac{1}{\Gamma(1-\zeta_4)}\mathbb{Y}_{p,q-1}^{m,n}\left[y;z;x\left|\begin{matrix}(\gamma_1,\rho_1),\cdots,(\gamma_p,\rho_p)\\(\theta_1,\varpi_1),\cdots,(\theta_{q-1},\varpi_{q-1})\end{matrix}\right.\right] \tag{33}$$

where $\Re(\zeta_4) < 1$ and $q > m$.

There are some special cases:

$$\mathbb{Y}_{0,1}^{1,0}\left[-y;0;0\Big|_{(0,1)}^{-}\right]=\mathrm{e}^{y} \tag{34}$$

$$\mathbb{Y}_{0,1}^{1,0}\left[y^{\upsilon};0;0\Big|_{(0,1)}^{-}\right]=\mathrm{e}^{-y^{\upsilon}} \tag{35}$$

$$\mathbb{Y}_{0,1}^{1,0}\left[-\eta y^{\upsilon};0;0\Big|_{(0,1)}^{-}\right]=\mathrm{e}^{\eta y^{\upsilon}} \tag{36}$$

$$\mathbb{Y}_{1,2}^{1,1}\left[-y;0;0\Big|_{(1,0),(1-\upsilon,1)}^{(0,1)}\right]=E_{\upsilon}(y) \tag{37}$$

$$\mathbb{Y}_{1,2}^{1,1}\left[-y;0;0\Big|_{(1,0),(1-\upsilon,\mu)}^{(0,1)}\right]=E_{\upsilon,\mu}(y) \tag{38}$$

$$\mathbb{Y}_{1,2}^{1,1}\left[-y;0;0\Big|_{(1,0),(1-\upsilon,\mu)}^{(1-\nu,1)}\right]=E_{\upsilon,\mu}^{\nu}(y) \tag{39}$$

$$\mathbb{Y}_{1,2}^{1,1}\left[-y^{\varpi};0;0\Big|_{(1,0),(1-\upsilon,1)}^{(0,1)}\right]=E_{\upsilon}\left(y^{\varpi}\right) \tag{40}$$

$$\mathbb{Y}_{1,2}^{1,1}\left[-y^{\varpi};0;0\Big|_{(1,0),(1-\upsilon,\mu)}^{(0,1)}\right]=E_{\upsilon,\mu}\left(y^{\varpi}\right) \tag{41}$$

$$\mathbb{Y}_{1,2}^{1,1}\left[-y^{\varpi};0;0\Big|_{(1,0),(1-\upsilon,\mu)}^{(1-\nu,1)}\right]=E_{\upsilon,\mu}^{\nu}\left(y^{\varpi}\right) \tag{42}$$

$$y^{\mu-1}\,\mathbb{Y}_{1,2}^{1,1}\left[-y^{\varpi};0;0\Big|_{(1,0),(1-\upsilon,\mu)}^{(0,1)}\right]=e_{\upsilon,\mu}\left(y^{\varpi}\right) \tag{43}$$

$$\frac{y^{\mu-1}}{\Gamma(\nu)}\mathbb{Y}_{1,2}^{1,1}\left[-y^{\varpi};0;0\Big|_{(1,0),(1-\upsilon,\mu)}^{(1-\nu,1)}\right]=\mathrm{e}_{\upsilon,\mu}^{\nu}\left(y^{\varpi}\right) \tag{44}$$

where the Mittag-Leffler function $E_{\upsilon}(y)$ [11]:

$$E_{\upsilon}(y)=\sum_{n=0}^{\infty}\frac{y^{n}}{\Gamma(n\upsilon+1)} \tag{45}$$

the Wiman function $E_{\upsilon,\mu}(y)$ [12]:

$$E_{\upsilon,\mu}(y)=\sum_{n=0}^{\infty}\frac{y^{n}}{\Gamma(n\upsilon+\mu)} \tag{46}$$

the Prabhakar function $E_{\upsilon,\mu}^{\nu}(y)$: [13]

$$E_{\upsilon,\mu}^{\nu}(y)=\sum_{n=0}^{\infty}\frac{\Gamma(\nu+n)}{\Gamma(n+1)\Gamma(\nu)}\frac{y^{n}}{\Gamma(n\upsilon+\mu)} \tag{47}$$

the Kohlrausch-Williams-Watts function $e^{-y^{\upsilon}}$ [14, 15]:

$$\mathrm{e}^{-y^{\upsilon}}=\sum_{n=0}^{\infty}(-1)^n\frac{y^{n\upsilon}}{n!} \tag{48}$$

$$\mathrm{e}_{\upsilon,\mu}\left(y^{\varpi}\right)=y^{\mu-1}\sum_{n=0}^{\infty}\frac{y^{n\varpi}}{\Gamma(n\upsilon+\mu)} \tag{49}$$

and

$$\mathrm{e}_{\upsilon,\mu}^{\nu}\left(y^{\varpi}\right)=y^{\mu-1}\sum_{n=0}^{\infty}\frac{\Gamma(\nu+n)}{\Gamma(n+1)\Gamma(\nu)}\frac{y^{\varpi n}}{\Gamma(n\upsilon+\mu)} \tag{50}$$

For the historical investigations of the previous formulas (45)-(50), see [1].

New integral and differential operators via the kernel of the Y function

The integral operator in the kernel of the Y function is defined:

$$_0I_\tau\left[\left(\mathbb{Y}_{p,q}^{m,n}[y;\tau;x]\right)f\right](\tau)=\int_0^\tau\mathbb{Y}_{p,q}^{m,n}[y;t-\tau;x]f(t)\mathrm{d}t \tag{51}$$

The Laplace transform of (51) reads:

$$L\left\{_0I_\tau\left[\left(\mathbb{Y}_{p,q}^{m,n}[y;\tau;x]\right)f\right](\tau)\right\}=\int_0^\infty {_0I_\tau}\left[\left(\mathbb{Y}_{p,q}^{m,n}[y;z;x]\right)f\right](\tau)e^{-s\tau}\mathrm{d}\tau=\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}\frac{s^{x-1}f(s)}{y^s} \tag{52}$$

which leads:

$$\frac{1}{\Gamma(\nu)}\int_0^\tau\mathbb{Y}_{p,q}^{m,n}[y;t-\tau;x]t^{\nu-1}\mathrm{d}t=\mathbb{Y}_{p,q}^{m,n}[y;z;x-1-\nu] \tag{53}$$

where $\nu>0$.

The differential operator in the kernel of the Y function is defined:

$$_0D_\tau\left[\left(\mathbb{Y}_{p,q}^{m,n}[y;\tau;x]\right)f\right](\tau)=\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau\mathbb{Y}_{p,q}^{m,n}[y;t-\tau;x]f(t)\mathrm{d}t \tag{54}$$

The Laplace transform of eq. (54) can be written:

$$L\left\{_0D_\tau\left[\left(\mathbb{Y}_{p,q}^{m,n}[y;\tau;x]\right)f\right](\tau)\right\}=\int_0^\infty {_0D_\tau}\left[\left(\mathbb{Y}_{p,q}^{m,n}[y;\tau;x]\right)f\right](\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau=\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}\frac{s^{x+1}f(s)}{y^s} \tag{55}$$

Applications in the heat transfer

In this section, we present three typical examples applied in heat transfer process.

Example 1. We now propose a new mathematical model for the heat transfer:

$$\chi\,{_0D_\tau}\left[\left(\mathbb{Y}_{p,q}^{m,n}[y;\tau;x]\right)U\right](\tau)=\Lambda \tag{56}$$

where χ is the thermal diffusivity, $U(\tau)$ is the temperature function, and Λ represents the heat flux density.

By the Laplace transform of eq. (56), we may find:

$$U(\tau) = \frac{\Lambda}{\chi} \mathbb{Y}_{p,q}^{m,n}[y;\tau;x] \tag{57}$$

Example 2. We now present a new mathematical model for the heat diffusion:

$${}_0D_\tau\left[\left(\mathbb{Y}_{p,q}^{m,n}[y;\tau;x]\right)\Theta\right](u,\tau) = \chi \frac{\partial^2\Theta(u,\tau)}{\partial u^2} \tag{58}$$

where χ is the thermal diffusivity and $\Theta(u, \tau)$ is the temperature function.

Example 3. We now consider the 1-D heat equation [19]:

$$\frac{\partial\Theta(u,\tau)}{\partial\tau} = \chi \frac{\partial^2\Theta(u,\tau)}{\partial u^2} \tag{59}$$

subject to the condition:

$$\Theta(u,0) = \delta(x) \tag{60}$$

where χ is the thermal diffusivity and $\Theta(u, \tau)$ is the temperature function.

Then, by eq. (36), we obtain:

$$\Theta(u,\tau) = \frac{1}{\sqrt{4\pi\chi\tau}} \mathbb{Y}_{0,1}^{1,0}\left[\frac{u^2}{4\chi\tau};0;0\left|\begin{matrix} - \\ (0,1)\end{matrix}\right.\right] \tag{61}$$

which is the alternative representation of the solution of (59) [19].

Conclusion

In our work we had addressed new integral and differential operators within the kernel of the Y function. We have investigated the properties related to the Mittag-Leffler, Wiman, Prabhakar, and Kohlrausch-Williams-Watts functions. We considered two mathematical models for the heat transfer and represented the solution of the 1-D heat equation by using the Y function.

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Nomenclature

u – space co-ordinate, [m]

Greek symbols

$\Theta(u, \tau)$ – temperature function, [K]
τ – time co-ordinate, [s]
χ – thermal conductivity, [$Wm^{-1}K^{-1}$]

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