A NEW INSIGHT TO THE SCALING-LAW FLUID ASSOCIATED WITH THE MANDELBROT SCALING LAW

by

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This paper addresses a non-traditional approach for the scaling-law fluid-flows described by fractal scaling-law vector calculus associated with the Mandelbrot scaling law. Their quantum equations were proposed to control the fluid-flows associated with the Mandelbrot scaling law. This gives a new insight into the descriptions for the scaling-law behaviors of the fluid-flows in the Mandelbrot scaling-law phenomena.

Key words: *scaling-law fluid-flows, fractal scaling-law vector calculus, Mandelbrot scaling law, fractals*

Introduction

Fluid-flows in nature have the behavior of self-similarity, scaling law, and complexity in the different way of the scale measures. There exist some well-known experimental cases of scaling-law fluid-flows intensely turbulent laboratory flow [1], thin fluid jets [2], plane Poiseuille flow [3], quantum turbulence [4], fluid and MHD turbulent flows [5], pipe and channel flows [6], high speed granular flows [7], microbursts [8], and forced quantum turbulence [9]. The scaling-law fluid-flows have become interesting topics.

These complex fluid-flow subject to the scale measure by using the Mandelbrot scaling law [10] can be considered in the different way. Making use of the relation between fractal and power law, the fractional power-law flow with the fractional calculus was investigated in [11]. New connection with the fractal scaling law and power-law flow was considered in [12]. The fractal metric fluid-flow was proposed in [13].

To deal with the scaling-law fluid-flows, we plan to employ the Mandelbrot scaling law [10] and fractal scaling-law vector calculus [14] to study the quantum equations for the fluid-flows associated with the Mandelbrot scaling law. The structure of the letter is designed as follows.

Theory of the scaling-law vector calculus associated with Mandelbrot scaling law

Let $\mathbb{R}_+$ and $\mathbb{N}$ be the sets of the positive real numbers and natural numbers, respectively.

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Suppose that

$$\Lambda(x)=\left[\Lambda\circ\left(\chi x^{1-D}\right)\right](x)=\Lambda\left(\chi x^{1-D}\right)$$

where $\chi\in\mathbb{R}_{+}$, $x\in\mathbb{R}_{+}$, and $0<D<1$ is the fractional dimension.

The Mandelbrot scaling law reads [15]:

$$\aleph(x)=\chi x^{1-D} \tag{1}$$

where $\lambda\in\mathbb{R}_{+}$, $x\in\mathbb{R}_{+}$, and $0<D<1$ is the fractional dimension.

The scaling-law derivative of the function $\Lambda(x)$ of order n with the Mandelbrot scaling law, denoted by ${}^{\mathrm{MSL}}D_x^{(n)}\Lambda(x)$, is given [14, 15]:

$${}^{\mathrm{MSL}}D_x^{(n)}\Lambda(x)=\left[\frac{x^{D}}{(1-D)\chi}\frac{\mathrm{d}}{\mathrm{d}x}\right]^{n}\Lambda(x) \tag{2}$$

where $n\in\mathbb{N}$.

The scaling-law integral of the function $\gamma(x)$ with the Mandelbrot scaling law, denoted by ${}^{\mathrm{MSL}}_{\ a}I_x^{(1)}\gamma(x)$, is given [14, 15]:

$${}^{\mathrm{MSL}}_{\ a}I_x^{(1)}\gamma(x)=(1-D)\chi\int_a^x\gamma(x)x^{-D}\mathrm{d}x \tag{3}$$

Let

$$\mathrm{H}(x,y,z,t)=\left[\mathrm{H}\left(\chi_0t^{1-D_0},\chi_1x^{1-D_1},\chi_2y^{1-D_2},\chi_3z^{1-D_3}\right)\right](x,y,z,t)$$

The scaling-law partial derivatives of the function $\mathrm{H}(x, y, z, t)$ associated with the Mandelbrot scaling law is given [14]:

$${}^{\mathrm{MSL}}\partial_x^{(1)}\mathrm{H}(x,y,z,t)=\frac{x^{D_1}}{\chi_1(1-D_1)}\frac{\partial\mathrm{H}(x,y,z,t)}{\partial x} \tag{4}$$

$${}^{\mathrm{MSL}}\partial_y^{(1)}\mathrm{H}(x,y,z,t)=\frac{y^{D_2}}{\chi_2(1-D_2)}\frac{\partial\mathrm{H}(x,y,z,t)}{\partial y} \tag{5}$$

$${}^{\mathrm{MSL}}\partial_z^{(1)}\mathrm{H}(x,y,z,t)=\frac{z^{D_3}}{\chi_3(1-D_3)}\frac{\partial\mathrm{H}(x,y,z,t)}{\partial z} \tag{6}$$

$${}^{\mathrm{MSL}}\partial_t^{(1)}\mathrm{H}(x,y,z,t)=\frac{t^{D_0}}{(1-D_0)\chi_0}\frac{\partial\mathrm{H}(x,y,z,t)}{\partial t} \tag{7}$$

The scaling-law gradient with respect to the Mandelbrot-scaling-law function in a Cartesian co-ordinate system is given [14]:

$$\begin{aligned}\nabla^{(D_1,D_2,D_3)}&=\boldsymbol{i}\left[\chi_1(1-D_1)x^{-D_1}\right]{}^{\mathrm{MSL}}\partial_x^{(1)}+\boldsymbol{j}\left[\chi_2(1-D_2)y^{-D_2}\right]{}^{\mathrm{MSL}}\partial_y^{(1)}+\\&+\boldsymbol{k}\left[\chi_3(1-D_3)y^{-D_3}\right]{}^{\mathrm{MSL}}\partial_z^{(1)}\end{aligned} \tag{8}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are three unite vectors in the Cartesian co-ordinate system.

We now rewrite eq. (21) as [14]:

$$\mathrm{dH}=\nabla^{(D_1,D_2,D_3)}\mathrm{H}\boldsymbol{n}\mathrm{d}r=\nabla^{(D_1,D_2,D_3)}\mathrm{H}\mathrm{d}\boldsymbol{r} \tag{9}$$

where $\boldsymbol{n}$ is the unit vector and dr is a distance measured along the normal direction, $\mathrm{d}\boldsymbol{r} = \boldsymbol{n}\mathrm{d}r = \boldsymbol{i}\mathrm{d}x + \boldsymbol{j}y + \boldsymbol{k}\mathrm{d}z$ with $\mathrm{d}\boldsymbol{r} = \boldsymbol{n}\mathrm{d}r$.

Let $X(x) = \chi_1 x^{1-D_1}$, $Y(y) = \chi_2 y^{1-D_1}$, and $Z(z) = \chi_3 z^{1-D_3}$.

The scaling area integral of the fractal scaling-law scalar field:

$$\Lambda = \Lambda\left(\chi_1 x^{1-D_1}, \chi_2 y^{1-D_2}\right)$$

is given:

$$A(\Lambda) = \iint_S \Lambda \mathrm{d}S = \iint_S \Lambda \mathrm{d}X(x)\mathrm{d}Y(y) \tag{10}$$

which can be re-written [14]:

$$A(\Lambda) = \int_c^d \left\{ \int_a^b \left[\chi_1 (1-D_1) x^{-D_1}\right] \mathrm{d}x \right\} \left[\chi_2 (1-D_2) y^{-D_2}\right] \Lambda \mathrm{d}y =$$
$$= \int_a^b \left\{ \int_c^d \left[\chi_2 (1-D_2) y^{-D_2}\right] \mathrm{d}y \right\} \left[\chi_1 (1-D_1) x^{-D_1}\right] \Lambda \mathrm{d}x \tag{11}$$

where

$$\mathrm{d}S = \mathrm{d}X(x)\mathrm{d}Y(y) = \left\{\left[\chi_1(1-D_1)z^{-D_1}\right]\left[\chi_2(1-D_2)z^{-D_2}\right]\right\}\mathrm{d}x\mathrm{d}y \tag{12}$$

for $x \in [a, b]$ and $y \in [c, d]$.

The scaling-law volume integral of the fractal scaling-law scalar field:

$$\Lambda = \Lambda\left(\chi_1 x^{1-D_1}, \chi_2 y^{1-D_2}, \chi_3 z^{1-D_3}\right)$$

is given:

$$B(\Lambda) = \iiint_\Omega \Lambda \mathrm{d}V \tag{13}$$

which can be re-written [14]:

$$B(\Lambda) = \int_e^f \left[\chi_3(1-D_3)z^{-D_3}\right]\mathrm{d}z \int_c^d \left[\chi_2(1-D_2)y^{-D_2}\right]\mathrm{d}y \int_a^b \Lambda\left[\chi_1(1-D_1)x^{-D_1}\right]\mathrm{d}x =$$
$$= \int_c^d \left[\chi_1(1-D_1)x^{-D_1}\right]\mathrm{d}x \int_a^b \left[\chi_3(1-D_3)z^{-D_3}\right]\mathrm{d}z \int_e^f \Lambda\left[\chi_2(1-D_2)y^{-D_2}\right]\mathrm{d}y =$$
$$= \int_c^d \left[\chi_2(1-D_2)y^{-D_2}\right]\mathrm{d}y \int_a^b \left[\chi_1(1-D_1)x^{-D_1}\right]\mathrm{d}x \int_e^f \Lambda\left[\chi_3(1-D_3)z^{-D_3}\right]\mathrm{d}z =$$
$$= \int_a^b \mathrm{d}X(x) \int_c^d \mathrm{d}Y(y) \int_e^f \Lambda \mathrm{d}Z(z) \tag{14}$$

where

$$\mathrm{d}V = \mathrm{d}X(x)\mathrm{d}Y(y)\mathrm{d}Z(z) =$$
$$= \left\{\left[\chi_1(1-D_1)z^{-D_1}\right]\left[\chi_2(1-D_2)z^{-D_2}\right]\left[\chi_3(1-D_3)z^{-D_3}\right]\right\}\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{15}$$

for $x \in [a, b]$, $y \in [c, d]$, and $z \in [c, d]$.

The scaling-law surface integral of the scaling-law vector field:

$$\boldsymbol{\psi} = \boldsymbol{\psi}\left(\chi_1 x^{1-D_1}, \chi_2 y^{1-D_2}, \chi_3 z^{1-D_3}\right)$$

is given [14]:

$$\Xi(\boldsymbol{\psi}) = \iint_S \boldsymbol{\psi} \mathrm{d}\boldsymbol{S} = \iint_S \boldsymbol{\psi}\boldsymbol{n} \mathrm{d}S \tag{16}$$

where $\boldsymbol{n} = \mathrm{d}\boldsymbol{S}/\mathrm{d}\boldsymbol{S}$ for $\boldsymbol{S} = \boldsymbol{S}(\chi_1 x^{1-D_1}, \chi_2 x^{1-D_2}, \chi_3 x^{1-D_3})$.

Let $\boldsymbol{n} = \mathrm{d}\boldsymbol{S}/|\mathrm{d}\boldsymbol{S}| = \mathrm{d}\boldsymbol{S}/\mathrm{d}S$, $\mathrm{d}S = |\mathrm{d}\boldsymbol{S}|$ and

$$\begin{aligned} \mathrm{d}\boldsymbol{S} &= \mathrm{d}Y(y)\mathrm{d}Z(z)\boldsymbol{i} + \mathrm{d}X(x)\mathrm{d}Z(z)\boldsymbol{j} + \mathrm{d}X(x)\mathrm{d}Y(y)\boldsymbol{k} = \\ &= \boldsymbol{i}\left[\chi_2(1-D_2)z^{-D_2}\right]\left[\chi_3(1-D_3)z^{-D_3}\right]\mathrm{d}y\mathrm{d}z + \\ &+ \boldsymbol{j}\left[\chi_1(1-D_1)z^{-D_1}\right]\left[\chi_3(1-D_3)z^{-D_3}\right]\mathrm{d}x\mathrm{d}z + \\ &+ \boldsymbol{k}\left[\chi_1(1-D_1)z^{-D_1}\right]\left[\chi_2(1-D_2)z^{-D_2}\right]\mathrm{d}x\mathrm{d}y \end{aligned} \tag{17}$$

where [14]:

$$\mathrm{d}Y(y)\mathrm{d}Z(z) = \left\{\left[\chi_2(1-D_2)z^{-D_2}\right]\left[\chi_3(1-D_3)z^{-D_3}\right]\right\}\mathrm{d}y\mathrm{d}z \tag{18}$$

$$\mathrm{d}X(x)\mathrm{d}Z(z) = \left\{\left[\chi_1(1-D_1)z^{-D_1}\right]\left[\chi_3(1-D_3)z^{-D_3}\right]\right\}\mathrm{d}x\mathrm{d}z \tag{19}$$

and

$$\mathrm{d}X(x)\mathrm{d}Y(y) = \left\{\left[\chi_1(1-D_1)z^{-D_1}\right]\left[\chi_2(1-D_2)z^{-D_2}\right]\right\}\mathrm{d}x\mathrm{d}y \tag{20}$$

The scaling-law divergence of the scaling-law vector field $\boldsymbol{\psi} = \boldsymbol{i}\psi_x + \boldsymbol{j}\psi_y + \boldsymbol{k}\psi_z$ in the Cartesian co-ordinate system can be given [14]:

$$\begin{aligned} \nabla^{(D_1,D_2,D_3)}\boldsymbol{\psi} &= \left[\chi_1(1-D_1)x^{-D_1}\right]^{\mathrm{MSL}}\partial_x^{(1)}\psi_x + \left[\chi_2(1-D_2)y^{-D_2}\right]^{\mathrm{MSL}}\partial_y^{(1)}\psi_y + \\ &+ \left[\chi_3(1-D_3)z^{-D_3}\right]^{\mathrm{MSL}}\partial_z^{(1)}\psi_z \end{aligned} \tag{21}$$

The scaling-law curl of the scaling-law vector field $\boldsymbol{\psi} = \boldsymbol{i}\psi_x + \boldsymbol{j}\psi_y + \boldsymbol{k}\psi_z$ in the Cartesian co-ordinate system can be re-written [14]:

$$\nabla^{(D_1,D_2,D_3)}\boldsymbol{\psi} = \\ = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \left[\chi_1(1-D_1)x^{-D_1}\right]^{\mathrm{MSL}}\partial_x^{(1)}, & \left[\chi_2(1-D_2)y^{-D_2}\right]^{\mathrm{MSL}}\partial_y^{(1)}, & \left[\chi_3(1-D_3)z^{-D_3}\right]^{\mathrm{MSL}}\partial_z^{(1)} \\ \psi_x & \psi_y & \psi_z \end{vmatrix} \tag{22}$$

The Gauss-Ostrogradsky-like theorem for the scaling-law vector calculus states [14]:

$$\iiint_\Omega \nabla^{(D_1,D_2,D_3)}\boldsymbol{\psi}\mathrm{d}V = \oiint_S \boldsymbol{\psi}\mathrm{d}\boldsymbol{S} \tag{23}$$

In this section, we introduce the Leibniz derivative and Riemann-Stieltjes integral, which are calld the calculus with respect to monotone function.

The theory of the fractal scaling-law fluid-flows

The scaling-law material derivative of the scaling-law fluid field

Suppose that

$$\Theta = \Theta\left(\chi_1 x^{1-D_1}, \chi_2 y^{1-D_2}, \chi_3 z^{1-D_3}, \chi_0 t^{1-D_0}\right)$$

be the scaling-law fluid field.

By using eq. (9) we write the scaling-law differential of the scaling-law fluid field:

$$\begin{aligned} \mathrm{d}\Theta = &\boldsymbol{i}\left[\chi_1\left(1-D_1\right)x^{-D_1}\right]^{\mathrm{MSL}}\partial_x^{(1)}\Theta\mathrm{d}x+\boldsymbol{j}\left[\chi_2\left(1-D_2\right)y^{-D_2}\right]^{\mathrm{MSL}}\partial_y^{(1)}\Theta\mathrm{d}y+\\ &+\boldsymbol{k}\left[\chi_3\left(1-D_3\right)y^{-D_3}\right]^{\mathrm{MSL}}\partial_z^{(1)}\Theta\mathrm{d}z+\left[\chi_0\left(1-D_0\right)t^{-D_0}\right]^{\mathrm{MSL}}\partial_t^{(1)}\Theta\mathrm{d}t \end{aligned} \tag{24}$$

which implies:

$$\begin{aligned} \frac{D\Phi}{Dt} = &\boldsymbol{i}\left[\chi_1\left(1-D_1\right)x^{-D_1}\right]^{\mathrm{MSL}}\partial_x^{(1)}\Theta\frac{\mathrm{d}x}{\mathrm{d}t}+\boldsymbol{j}\left[\chi_2\left(1-D_2\right)y^{-D_2}\right]^{\mathrm{MSL}}\partial_y^{(1)}\Theta\frac{\mathrm{d}y}{\mathrm{d}t}+\\ &+\boldsymbol{k}\left[\chi_3\left(1-D_3\right)y^{-D_3}\right]^{\mathrm{MSL}}\partial_z^{(1)}\Theta\frac{\mathrm{d}z}{\mathrm{d}t}+\left[\chi_0\left(1-D_0\right)t^{-D_0}\right]^{\mathrm{MSL}}\partial_t^{(1)}\Theta \end{aligned} \tag{25}$$

Let

$$\boldsymbol{\upsilon} = \left(\partial x/\partial t, \partial y/\partial t, \partial z/\partial t\right) = i\upsilon_x + j\upsilon_y + k\upsilon_z$$

be the vector of the velocity.

The scaling-law material derivative of the scaling-law fluid density Φ reads:

$$\frac{D\Phi}{Dt} = \left[\chi_0\left(1-D_0\right)t^{-D_0}\right]^{\mathrm{MSL}}\partial_t^{(1)}\boldsymbol{\Phi}+\boldsymbol{\upsilon}\nabla^{(D_1,D_2,D_3)}\boldsymbol{\Phi} \tag{26}$$

It is well known that the Stokes material derivative, discovered by Stokes [16] to consider the velocity and further developed [17], is one of the special cases of eq. (26) when $D_1 = D_2 = D_3 = D_0 = 0$ and $\chi_1 = \chi_2 = \chi_3 = \chi_0 = 1$.

The transport theorem for the scaling-law fluid

From eq. (26) we have the transport theorem for the scaling-law fluid:

$$\frac{D}{Dt}\iiint_{\Omega(t)}\Phi\mathrm{d}V = \iiint_{\Omega(t)}\left\{\left[\chi_0\left(1-D_0\right)t^{-D_0}\right]^{\mathrm{MSL}}\partial_t^{(1)}\Phi+\boldsymbol{\upsilon}\nabla^{(D_1,D_2,D_3)}\Phi\right\}\mathrm{d}V \tag{27}$$

which yields that:

$$\frac{D}{Dt}\iiint_{\Omega(t)}\Phi\mathrm{d}V = \iiint_{\Omega(t)}\left[\chi_0\left(1-D_0\right)t^{-D_0}\right]^{\mathrm{MSL}}\partial_t^{(1)}\Phi\mathrm{d}V+\oiint_{\boldsymbol{S}(t)}\Phi\boldsymbol{\upsilon}\mathrm{d}\boldsymbol{S} \tag{28}$$

since the Gauss-Ostrogradsky-like theorem for the scaling-law vector calculus:

$$\iiint_{\Omega(t)}\boldsymbol{\upsilon}\nabla^{(D_1,D_2,D_3)}\Phi\mathrm{d}V = \oiint_{\boldsymbol{S}(t)}\Phi\left(\boldsymbol{\upsilon}\boldsymbol{n}\right)\mathrm{d}S = \oiint_{\boldsymbol{S}(t)}\Phi\boldsymbol{\upsilon}\mathrm{d}\boldsymbol{S} \tag{29}$$

holds where $\boldsymbol{S}(t)$ is the surface of $\Omega(t)$, $\boldsymbol{n}$ is the unit normal to the surface, and $\boldsymbol{\upsilon}$ is the velocity vector.

It is noted that the Reynolds transport theorem, discovered by Reynolds [18], is the special case of eq. (28) when $D_1 = D_2 = D_3 = D_0 = 0$ and $\chi_1 = \chi_2 = \chi_3 = \chi_0 = 1$.

The conservation of the mass of the scaling-law fluid-flow

The conservation of the mass of the scaling-law fluid-flow is given:

$$\left[\chi_0\left(1-D_0\right)t^{-D_0}\right]^{\mathrm{MSL}}\partial_t^{(1)}\rho+\boldsymbol{\upsilon}\nabla^{(D_1,D_2,D_3)}\rho = 0 \tag{30}$$

or alternatively:

$$\left[\chi_0\left(1-D_0\right)t^{-D_0}\right]^{\mathrm{MSL}}\partial_t^{(1)}\rho+\nabla^{(D_1,D_2,D_3)}\left(\boldsymbol{\upsilon}\rho\right) = 0 \tag{31}$$

because

$$\frac{D}{Dt}\iiint_{\Omega(t)}\rho\mathrm{d}V=\iiint_{\Omega(t)}\left\{\left[\chi_0\left(1-D_0\right)t^{-D_0}\right]^{\mathrm{MSL}}\partial_t^{(1)}\rho+\boldsymbol{\upsilon}\nabla^{(D_1,D_2,D_3)}\rho\right\}\mathrm{d}V=0 \tag{32}$$

which is connected with the mass of the scaling-law fluid-flow:

$$\mathrm{M}=\iiint_{\Omega(t)}\rho\mathrm{d}V \tag{33}$$

where ρ and M are the density and mass of the complex fluid-flow, respectively.

The conservation of the mass for the classical fluid-flow, discovered by Euler [19], is the special case of the conservation of the mass of the scaling-law fluid-flow when $D_1 = D_2 = D_3 = D_0 = 0$ and $\chi_1 = \chi_2 = \chi_3 = \chi_0 = 1$ imply that $X(x) = x$, $Y(y) = \mathrm{y}$, and $Z(z) = z$.

Cauchy-type scaling-law strain tensor, Stokes-type scaling-law strain tensor, and Stokes-type scaling-law velocity gradient tensor

The Cauchy-type scaling-law strain tensor for the scaling-law fluid-flow, denoted by $\sum$, is defined:

$$\sum=\frac{1}{2}\left[\nabla^{(D_1,D_2,D_3)}\boldsymbol{\upsilon}+\boldsymbol{\upsilon}\nabla^{(D_1,D_2,D_3)}\right] \tag{34}$$

The Stokes-type scaling-law strain tensor for the scaling-law fluid-flow, denoted by Λ, is defined:

$$\boldsymbol{\sigma}=\frac{1}{2}\left[\nabla^{(D_1,D_2,D_3)}\boldsymbol{\upsilon}-\boldsymbol{\upsilon}\nabla^{(D_1,D_2,D_3)}\right] \tag{35}$$

The Stokes-type scaling-law velocity gradient tensor for the scaling-law fluid-flow, denoted by $\nabla^{(D_1,D_2,D_3)}\boldsymbol{\upsilon}$, is given:

$$\nabla^{(D_1,D_2,D_3)}\boldsymbol{\upsilon}=\sum+\sigma=\frac{1}{2}\left[\nabla^{(D_1,D_2,D_3)}\boldsymbol{\upsilon}+\boldsymbol{\upsilon}\nabla^{(D_1,D_2,D_3)}\right]+\frac{1}{2}\boldsymbol{\upsilon}\left[\nabla^{(D_1,D_2,D_3)}-\boldsymbol{\upsilon}\nabla^{(D_1,D_2,D_3)}\right] \tag{36}$$

The stress tensor for the scaling-law fluid-flow, denoted by $\boldsymbol{T}$, is given:

$$\boldsymbol{T}=-p\boldsymbol{I}+2\lambda\sum \tag{37}$$

where λ is the shear moduli of the viscosity, $\boldsymbol{I}$ is the unit tensor, and p represents the pressure.

The Cauchy strain tensor by Cauchy [20], Stokes-type strain tensor and Stokes-type velocity gradient tensor by Stocks [21] are the special cases of the scaling-law fluid-flow if $D_1 = D_2 = D_3 = D_0 = 0$ and $\chi_1 = \chi_2 = \chi_3 = \chi_0 = 1$

Conservation of the momentums for the scaling-law fluid-flow

The conservation of the momentums for the scaling-law fluid-flow reads:

$$\frac{D}{Dt}\iiint_{\Omega(t)}\rho\boldsymbol{\upsilon}\mathrm{d}V=\iiint_{\Omega(t)}\boldsymbol{f}\mathrm{d}V+\oiint_{\boldsymbol{S}(t)}\boldsymbol{T}\mathrm{d}\boldsymbol{S} \tag{38}$$

where f is the specific body force.

This implies:

$$\left[\chi_0\left(1-D_0\right)t^{-D_0}\right]^{\mathrm{MSL}}\partial_t^{(1)}\left(\rho\boldsymbol{\upsilon}\right)+\nabla^{(D_1,D_2,D_3)}\left(\boldsymbol{\upsilon}\rho\right)=\nabla^{(D_1,D_2,D_3)}\boldsymbol{T}+\boldsymbol{f} \tag{39}$$

because there exists:

$$\iiint_{\Omega(t)}\left\{\left[\chi_0\left(1-D_0\right)t^{-D_0}\right]^{\text{MSL}}\partial_t^{(1)}\left(\rho\boldsymbol{\upsilon}\right)+\nabla^{\left(D_1,D_2,D_3\right)}\left(\boldsymbol{\upsilon}\rho\right)-\nabla^{\left(D_1,D_2,D_3\right)}\boldsymbol{T}-\boldsymbol{f}\right\}\mathrm{d}V=0 \tag{40}$$

where

$$\frac{D}{Dt}\iiint_{\Omega(t)}\rho\boldsymbol{\upsilon}\mathrm{d}V=\iiint_{\Omega(t)}\left\{\left[\chi_0\left(1-D_0\right)t^{-D_0}\right]^{\text{MSL}}\partial_t^{(1)}\left(\rho\boldsymbol{\upsilon}\right)+\boldsymbol{\upsilon}\nabla^{\left(D_1,D_2,D_3\right)}\left(\rho\boldsymbol{\upsilon}\right)\right\}\mathrm{d}V \tag{41}$$

and

$$\oiint_{\boldsymbol{S}(t)}\boldsymbol{T}\mathrm{d}\boldsymbol{S}=\iiint_{\Omega(t)}\nabla^{\left(D_1,D_2,D_3\right)}\boldsymbol{T}\mathrm{d}V \tag{42}$$

Navier-Stokes-type equations of the scaling-law fluid-flow

With the aid of eq. (37) we have:

$$\nabla^{\left(D_1,D_2,D_3\right)}\boldsymbol{T}=-\nabla^{\left(D_1,D_2,D_3\right)}p+\gamma\nabla^{\left(2D_1,2D_2,2D_3\right)}\boldsymbol{\upsilon} \tag{43}$$

such that:

$$\left[\chi_0\left(1-D_0\right)t^{-D_0}\right]^{\text{MSL}}\partial_t^{(1)}\left(\rho\boldsymbol{\upsilon}\right)+\boldsymbol{\upsilon}\nabla^{\left(D_1,D_2,D_3\right)}\left(\rho\boldsymbol{\upsilon}\right)=-\nabla^{\left(D_1,D_2,D_3\right)}p+\gamma\nabla^{\left(2D_1,2D_2,2D_3\right)}\boldsymbol{\upsilon}+\boldsymbol{f} \tag{44}$$

where

$$\nabla^{\left(D_1,D_2,D_3\right)}\boldsymbol{\upsilon}=0 \tag{45}$$

$$\boldsymbol{\upsilon}\nabla^{\left(D_1,D_2,D_3\right)}\boldsymbol{\upsilon}=\left[\nabla^{\left(D_1,D_2,D_3\right)}\boldsymbol{\upsilon}\right]\times\boldsymbol{\upsilon}+\frac{1}{2}\nabla^{\left(D_1,D_2,D_3\right)}\boldsymbol{\upsilon}^2 \tag{46}$$

$$\nabla^{\left(2D_1,2D_2,2D_3\right)}=\nabla^{\left(D_1,D_2,D_3\right)}\bullet\nabla^{\left(D_1,D_2,D_3\right)} \tag{47}$$

and γ is dynamic viscosity.

From eq. (44) it is easy to see:

$$\rho\left\{\left[\chi_0\left(1-D_0\right)t^{-D_0}\right]^{\text{MSL}}\partial_t^{(1)}\boldsymbol{\upsilon}+\boldsymbol{\upsilon}\nabla^{\left(D_1,D_2,D_3\right)}\boldsymbol{\upsilon}\right\}=-\nabla^{\left(D_1,D_2,D_3\right)}p+\gamma\nabla^{\left(2D_1,2D_2,2D_3\right)}\boldsymbol{\upsilon}+\boldsymbol{f} \tag{48}$$

We denote that eq. (48) is the incompressible scaling-law Navier-Stokes-type equations of the scaling-law fluid-flow and is the extended version of the Navier-Stokes equations of the complex fluid-flow, proposed by Stokes [16] and by Navier [22] when $D_1 = D_2 = D_3 = D_0 = 0$ and $\chi_1 = \chi_2 = \chi_3 = \chi_0 = 1$

Conclusion

In this work we had reported the mathematical theory of the scaling-law Navier-Stokes type equations of the scaling-law fluid-flow by using the fractal scaling-law vector calculus associated with Mandelbrot scaling law. The conservations of the mass and momentums of the scaling-law fluid-flows have been suggested with the aid of the Gauss-Ostrogradsky like theorem for the scaling law vector calculus. Our technology is as an efficient mathematical tool proposed to study the theory of the scaling-law fluid-flow in the Mandelbrot scaling law phenomena.

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Nomenclature

$\boldsymbol{f}$ – specific body force, [Nm^{-3}]
t – time, [s]
x, y, z – co-ordinates, [m]

Greek symbol

$\boldsymbol{\upsilon}$ – velocity vector, [ms^{-1}]

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