A NEW VIEWPOINT ON THEORY OF THE SCALING-LAW HEAT CONDUCTION PROCESS

by

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In this article, we suggest a new model for the heat-conduction problem by using the scaling-law vector calculus with Mandelbrot scaling law. The linear and non-linear scaling-law heat conduction equations are considered as analogues to the work of Fourier, Laplace, and Burgers. The obtained results are considered as typical examples to deal with the Mandelbrots scaling-law phenomena in heat transport system.

Key words: *scaling-law heat conduction equation, fractals, heat conduction, scaling-law vector calculus, Mandelbrots scaling-law phenomena*

Introduction

The classical theory of the heat conduction has played the important role in the real-world problems [1]. However, there are many anomalous phenomena for the heat conduction problems. The scaling law behavior of the heat conduction, as one of the important topics for the anomalous heat transfer, has been studied by many scientists. For example, there exists many materials with the scaling law behaviors for the heat conduction, such as sheared granular materials [2], ion traps [3], carbon nanotube materials [4], non-structural materials [5], carbon nanotubes [6], heterogeneous single-atom catalysts [7], thermal explosion [8], biological media [9], *etc.* These results described by the fractal scaling law have been analyzed from the fractal geometry point of view.

Recently, a scaling-law calculus associated with the Mandelbrots scaling law, which is connected with the fractal geometry and calculus, was proposed by author in 2020 to scaling-law telegraph equations [10]. Based on it, the fractal scaling-law vector calculus was established in 2021 by author, Yang *et al*. [11] to describe the Mandelbrots scaling-law behavior of the theory of the elasticity. The heat conduct problem with the Mandelbrots scaling law is an important topic for us to investigate the heat transport process in the micro-structure with the Mandelbrots-scaling-law behavior. The Mandelbrots-scaling-law heat-conduction problems have not developed based on the fractal scaling-law vector calculus. This main target of the paper is to propose the theory of the scaling-law heat conduction problem.

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The theory of the scaling-law calculus within Mandelbrots scaling law

The scaling-law calculus of one variable

Let $\mathbb{R}_+$ and $\mathbb{N}$ be the sets of the positive real numbers and natural numbers, respectively.

Suppose that:

$$\Xi(x)=\left[\Xi\circ\left(\mu x^{1-D}\right)\right](x)=\Xi\left(\mu x^{1-D}\right)$$

where $\mu\in\mathbb{R}_+$, $x\in\mathbb{R}_+$, and $D\in\mathbb{R}_+$ is the fractional dimension.

The Mandelbrot scaling law, denoted by $\phi(x)$, is given [10]:

$$\phi(x)=\mu x^{1-D} \tag{1}$$

where $\mu\in\mathbb{R}_+$, $x\in\mathbb{R}_+$, and $0<D<1$ is the fractional dimension.

The scaling-law derivative of the function $\Xi(x)$ of order n with the Mandelbrot scaling law, denoted by ${}^{\mathrm{MSL}}D_x^{(n)}\Xi(x)$, is given [10]:

$${}^{\mathrm{MSL}}D_x^{(n)}\Xi(x)=\left[\frac{x^D}{(1-D)\mu}\frac{\mathrm{d}}{\mathrm{d}x}\right]^n\Xi(x) \tag{2}$$

where $n\in\mathbb{N}$.

The scaling-law integral of the function $\xi(x)$ with the Mandelbrot scaling law, denoted by ${}^{\mathrm{MSL}}_{\ a}I_x^{(1)}\xi(x)$, is defined [10]:

$${}^{\mathrm{MSL}}_{\ a}I_x^{(1)}\xi(x)=(1-D)\mu\int_a^x\xi(x)x^{-D}\mathrm{d}x \tag{3}$$

The improper scaling-law integrals of the function $\xi(x)$ with the Mandelbrot scaling law are defined:

$$\lim_{a\to-\infty}{}^{\mathrm{MSL}}_{\ a}I_x^{(1)}\xi(x)=(1-D)\mu\int_{-\infty}^x\xi(x)x^{-D}\mathrm{d}x \tag{4}$$

$$\lim_{b\to\infty}{}^{\mathrm{MSL}}_{\ x}I_b^{(1)}\xi(x)=(1-D)\mu\int_x^{\infty}\xi(x)x^{-D}\mathrm{d}x \tag{5}$$

and

$$\lim_{\substack{b\to\infty\\a\to-\infty}}{}^{\mathrm{MSL}}_{\ a}I_b^{(1)}\xi(x)=(1-D)\mu\int_{-\infty}^{\infty}\xi(x)x^{-D}\mathrm{d}x \tag{6}$$

For the more details of the background and theory of the scaling-law calculus, see [10-12].

The scaling-law vector calculus

Let:

$$\Xi(x,y,z,t)=\left[\Xi\circ\left(\mu_0t^{1-D_0},\mu_1x^{1-D_1},\mu_2y^{1-D_2},\mu_3z^{1-D_3}\right)\right](x,y,z,t)$$

The scaling-law partial derivatives of the function $\Xi(x, y, z, t)$ with the Mandelbrot scaling law is defined [11]:

$${}^{\mathrm{MSL}}\partial_x^{(1)}\Xi(x,y,z,t)=\frac{x^{D_1}}{\mu_1(1-D_1)}\frac{\partial\Xi(x,y,z,t)}{\partial x} \tag{7}$$

$$ {}^{\mathrm{MSL}}\partial_y^{(1)}\Xi(x,y,z,t)=\frac{y^{D_2}}{\mu_2(1-D_2)}\frac{\partial\Xi(x,y,z,t)}{\partial y} \tag{8}$$

$$ {}^{\mathrm{MSL}}\partial_z^{(1)}\Xi(x,y,z,t)=\frac{z^{D_3}}{\mu_3(1-D_3)}\frac{\partial\Xi(x,y,z,t)}{\partial z} \tag{9}$$

$$ {}^{\mathrm{MSL}}\partial_{xy}^{(2)}\Xi(x,y,z,t)=\left[\frac{y^{D_2}}{\mu_2(1-D_2)}\frac{x^{D_1}}{\mu_1(1-D_1)}\right]\frac{\partial^2\Xi(x,y,z,t)}{\partial y\partial x} \tag{10}$$

$$ {}^{\mathrm{MSL}}\partial_{zy}^{(2)}\Xi(x,y,z,t)=\left[\frac{y^{D_2}}{\mu_2(1-D_2)}\frac{z^{D_3}}{\mu_3(1-D_3)}\right]\frac{\partial^2\Xi(x,y,z,t)}{\partial y\partial z} \tag{11}$$

$$ {}^{\mathrm{MSL}}\partial_{zx}^{(2)}\Xi(x,y,z,t)=\left[\frac{x^{D_1}}{\mu_1(1-D_1)}\frac{z^{D_3}}{\mu_3(1-D_3)}\right]\frac{\partial^2\Xi(x,y,z,t)}{\partial x\partial z} \tag{12}$$

$$ {}^{\mathrm{MSL}}\partial_x^{(2)}\Xi(x,y,z,t)=\frac{x^{2D_1}}{\mu_1^2(1-D_1)^2}\frac{\partial^2\Xi(x,y,z,t)}{\partial x^2} \tag{13}$$

$$ {}^{\mathrm{MSL}}\partial_y^{(2)}\Xi(x,y,z,t)=\frac{y^{2D_2}}{\mu_2^2(1-D_2)^2}\frac{\partial^2\Xi(x,y,z,t)}{\partial y^2} \tag{14}$$

$$ {}^{\mathrm{MSL}}\partial_z^{(2)}\Xi(x,y,z,t)=\frac{z^{2D_3}}{\mu_3^2(1-D_3)^2}\frac{\partial^2\Xi(x,y,z,t)}{\partial z^2} \tag{15}$$

$$ {}^{\mathrm{MSL}}\partial_t^{(1)}\Xi(x,y,z,t)=\frac{t^{D_0}}{(1-D_0)\mu_0}\frac{\partial\Xi(x,y,z,t)}{\partial t} \tag{16}$$

$$ {}^{\mathrm{MSL}}\partial_{tx}^{(2)}\Xi(x,y,z,t)=\left[\frac{t^{D_0}}{(1-D_0)\mu_0}\frac{x^{D_1}}{\mu_1(1-D_1)}\right]\frac{\partial^2\Xi(x,y,z,t)}{\partial x\partial t} \tag{17}$$

$$ {}^{\mathrm{MSL}}\partial_{ty}^{(2)}\Xi(x,y,z,t)=\left[\frac{t^{D_0}}{(1-D_0)\mu_0}\frac{x^{D2}}{\mu_2(1-D_2)}\right]\frac{\partial^2\Xi(x,y,z,t)}{\partial y\partial t} \tag{18}$$

$$ {}^{\mathrm{MSL}}\partial_{tz}^{(2)}\Xi(x,y,z,t)=\left[\frac{t^{D_0}}{(1-D_0)\mu_0}\frac{z^{D_3}}{\mu_3(1-D_3)}\right]\frac{\partial^2\Xi(x,y,z,t)}{\partial z\partial t} \tag{19}$$

and

$$ {}^{\mathrm{MSL}}\partial_t^{(2)}\Xi(x,y,z,t)=\frac{t^{2D_0}}{(1-D_0)^2\mu_0^2}\frac{\partial^2\Xi(x,y,z,t)}{\partial t^2} \tag{20}$$

where μ_0, μ_1, μ_2, and μ_3 are the positive constants, and $0 < D_0$, D_1, D_2, and $D_3 < 1$ are the the fractional dimensions.

The total scaling-law differential of the Mandelbrot-scaling-law scalar field $\Phi = \Phi(x, y, z)$ is defined [11]:

$$\mathrm{d}\Phi = \left[\mu_1\left(1-D_1\right)x^{-D_1}\ {}^{\mathrm{MSL}}\partial_x^{(1)}\Phi\right]\mathrm{d}x + \left[\mu_2\left(1-D_2\right)y^{-D_2}\ {}^{\mathrm{MSL}}\partial_y^{(1)}\Phi\right]\mathrm{d}y + \\ + \left[\mu_3\left(1-D_3\right)y^{-D_3}\ {}^{\mathrm{MSL}}\partial_z^{(1)}\Phi\right]\mathrm{d}z \tag{21}$$

The scaling-law gradient with respect to the Mandelbrot-scaling-law function in a Cartesian co-ordinate system is given [11] :

$$\nabla^{(D_1,D_2,D_3)} = \boldsymbol{i}\left[\mu_1\left(1-D_1\right)x^{-D_1}\right]{}^{\mathrm{MSL}}\partial_x^{(1)} + \boldsymbol{j}\left[\mu_2\left(1-D_2\right)y^{-D_2}\right]{}^{\mathrm{MSL}}\partial_y^{(1)} + \\ + \boldsymbol{k}\left[\mu_3\left(1-D_3\right)y^{-D_3}\right]{}^{\mathrm{MSL}}\partial_z^{(1)} \tag{22}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are unite vectors in a Cartesian co-ordinate system.

Thus, eq. (21) can be re-written [11]:

$$\mathrm{d}\Phi = \nabla^{(D_1,D_2,D_3)}\Phi\boldsymbol{n}\mathrm{d}r = \nabla^{(D_1,D_2,D_3)}\Phi\mathrm{d}\boldsymbol{r} \tag{23}$$

where $\boldsymbol{n}$ is the unit vector, dr – the distance measured along the normal direction:

$$\mathrm{d}\boldsymbol{r} = \boldsymbol{n}\mathrm{d}r = \boldsymbol{i}\mathrm{d}x + \boldsymbol{j}\mathrm{d}y + \boldsymbol{k}\mathrm{d}z \text{ with } \mathrm{d}\boldsymbol{r} = \boldsymbol{n}\mathrm{d}r$$

The scaling-law volume integral of the fractal scaling-law scalar field:

$$\Phi = \Phi\left(\mu_1 x^{1-D_1}, \mu_2 y^{1-D_2}, \mu_3 z^{1-D_3}\right)$$

is defined [11]:

$$\mathfrak{H}\left(\Phi\right) = \iiint_{\Omega}\Phi\mathrm{d}V \tag{24}$$

where

$$\mathrm{d}V = \left[\mu_1\left(1-D_1\right)x^{-D_1}\right]\left[\mu_2\left(1-D_2\right)y^{-D_2}\right]\left[\mu_3\left(1-D_3\right)z^{-D_3}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z = \\ = \mathrm{d}\ell\left(x\right)\mathrm{d}p\left(y\right)\mathrm{d}q\left(z\right) \tag{25}$$

with $\ell(x) = \lambda_1 x^{1-D_1}$, $p(y) = \lambda_2 y^{1-D_2}$, and $q(z) = \lambda_3 z^{1-D_3}$.

Thus it is easy to see [11]:

$$\iiint_{\Omega}\Phi\mathrm{d}V = \int_{\alpha}^{\beta}\left[\mu_3\left(1-D_3\right)z^{-D_3}\right]\mathrm{d}z\int_{c}^{d}\left[\mu_2\left(1-D_2\right)y^{-D_2}\right]\mathrm{d}y\int_{a}^{b}\Theta\left[\mu_1\left(1-D_1\right)x^{-D_1}\right]\mathrm{d}x = \\ = \int_{c}^{d}\left[\mu_1\left(1-D_1\right)x^{-D_1}\right]\mathrm{d}x\int_{a}^{b}\left[\mu_3\left(1-D_3\right)z^{-D_3}\right]\mathrm{d}z\int_{\alpha}^{\beta}\Theta\left[\mu_2\left(1-D_2\right)y^{-D_2}\right]\mathrm{d}y = \\ = \int_{c}^{d}\left[\mu_2\left(1-D_2\right)y^{-D_2}\right]\mathrm{d}y\int_{a}^{b}\left[\mu_1\left(1-D_1\right)x^{-D_1}\right]\mathrm{d}x\int_{\alpha}^{\beta}\Theta\left[\mu_3\left(1-D_3\right)z^{-D_3}\right]\mathrm{d}z = \\ = \int_{a}^{b}\mathrm{d}\ell\left(x\right)\int_{c}^{d}\mathrm{d}p\left(y\right)\int_{\alpha}^{\beta}\Theta\mathrm{d}q\left(z\right) \tag{26}$$

where $x \in [a, b]$, $y \in [c, d]$, and $z \in [\alpha, \beta]$.

The scaling-law surface integral of the scaling-law vector field [11]:

$$\boldsymbol{H} = \boldsymbol{H}\left(\mu_1 x^{1-D_1}, \mu_2 y^{1-D_2}, \mu_3 z^{1-D_3}\right)$$

is defined [11]:

$$\sum\left(\boldsymbol{H}\right) = \iint_{S}\boldsymbol{H}\mathrm{d}\boldsymbol{S} = \iint_{S}\boldsymbol{H}\boldsymbol{n}\mathrm{d}S \tag{27}$$

where $\boldsymbol{n} = \mathrm{d}\boldsymbol{S}/\mathrm{d}S$ with $\boldsymbol{S} = \boldsymbol{S}(\mu_1 x^{1-D_1}, \mu_2 y^{1-D_2}, \mu_3 z^{1-D_3})$.

Suppose that $\boldsymbol{n} = \mathrm{d}\boldsymbol{S}/|\mathrm{d}\boldsymbol{S}| = \mathrm{d}\boldsymbol{S}/\mathrm{d}S$, $\mathrm{d}S = |\mathrm{d}\boldsymbol{S}|$, and

$$\begin{aligned}\mathrm{d}\boldsymbol{S} &= \mathrm{d}p(y)\mathrm{d}q(z)\boldsymbol{i} + \mathrm{d}\ell(x)\mathrm{d}q(z)\boldsymbol{j} + d\ell(x)\mathrm{d}p(y)\boldsymbol{k} = \\ &= \boldsymbol{i}\left[\mu_2(1-D_2)z^{-D_2}\right]\left[\mu_3(1-D_3)z^{-D_3}\right]\mathrm{d}y\mathrm{d}z + \\ &+ \boldsymbol{j}\left[\mu_1(1-D_1)z^{-D_1}\right]\left[\mu_3(1-D_3)z^{-D_3}\right]\mathrm{d}x\mathrm{d}z + \\ &+ \boldsymbol{k}\left[\mu_1(1-D_1)z^{-D_1}\right]\left[\mu_2(1-D_2)z^{-D_2}\right]\mathrm{d}x\mathrm{d}y\end{aligned} \tag{28}$$

where [11]

$$\mathrm{d}p(y)\mathrm{d}q(z) = \left\{\left[\mu_2(1-D_2)z^{-D_2}\right]\left[\mu_3(1-D_3)z^{-D_3}\right]\right\}\mathrm{d}y\mathrm{d}z \tag{29}$$

$$\mathrm{d}\ell(x)\mathrm{d}q(z) = \left[\left[\mu_1(1-D_1)z^{-D_1}\right]\left[\mu_3(1-D_3)z^{-D_3}\right]\right]\mathrm{d}x\mathrm{d}z \tag{30}$$

and

$$\mathrm{d}\ell(x)\mathrm{d}q(y) = \left[\left[\mu_1(1-D_1)z^{-D_1}\right]\left[\mu_2(1-D_2)z^{-D_2}\right]\right]\mathrm{d}x\mathrm{d}y \tag{31}$$

Thus, we may obtain that [11]:

$$\sum(\boldsymbol{H}) = \boldsymbol{H}\iint_{\boldsymbol{S}} \bullet\mathrm{d}\boldsymbol{S} = \iint_{S} H_x\mathrm{d}p(y)\mathrm{d}q(z) + H_y\mathrm{d}\ell(x)\mathrm{d}q(z) + H_z\mathrm{d}\ell(x)\mathrm{d}p(y) \tag{32}$$

where

$$\boldsymbol{H} = \boldsymbol{H}\left(\mu_1 x^{1-D_1}, \mu_2 y^{1-D_2}, \mu_3 z^{1-D_3}\right) = \boldsymbol{i}H_x + \boldsymbol{j}H_y + \boldsymbol{k}H_z$$

Here, we define [11]:

$$\begin{aligned}\iint_{S} H_z\mathrm{d}\ell(x)\mathrm{d}p(y) &= \int_c^d\left\{\int_a^b\left[\mu_1(1-D_1)x^{-D_1}\right]\mathrm{d}x\right\}\left[\mu_2(1-D_2)y^{-D_2}\right]\mathrm{d}y = \\ &= \int_a^b\left\{\int_c^d\left[\left(\mu_2(1-D_2)y^{-D_2}\right)\mathrm{d}y\right]\right\}\left[\mu_1(1-D_1)x^{-D_1}\right]\mathrm{d}x\end{aligned} \tag{33}$$

Let:

$$\Delta S = \Delta\ell(x)\Delta p(y) = \left\{\left[\mu_1(1-D_1)x^{-D_1}\right]\Delta x\right\}\left\{\left[\mu_2(1-D_2)y^{-D_2}\right]\Delta y\right\} \tag{34}$$

and

$$\begin{aligned}\Delta V &= \Delta\ell(x)\Delta p(y)\Delta q(z) = \\ &= \left\{\left[\mu_1(1-D_1)x^{-D_1}\right]\Delta x\right\}\left\{\left[\mu_2(1-D_2)y^{-D_2}\right]\Delta y\right\}\left\{\left[\left(\mu_3(1-D_3)z^{-D_3}\right)\Delta z\right]\right\}\end{aligned} \tag{35}$$

The scaling-law divergence of the scaling-law vector field $\boldsymbol{H}$ is defined [11]:

$$\nabla^{(D_1,D_2,D_3)}\boldsymbol{H} = \lim_{\Delta V_m \to 0}\frac{1}{\Delta V_m}\oiint_{\Delta \boldsymbol{S}_m}\boldsymbol{H}\mathrm{d}\boldsymbol{S} \tag{36}$$

where the scaling-law volume, V, is divided into a large number of small sub-volumes, ΔV_m, with the scaling-law surfaces, ΔS_m, and $\mathrm{d}\boldsymbol{S}$ is the the element of the fractal scaling-law surface $\boldsymbol{S}$.

The scaling-law divergence eq. (36) in a Cartesian co-ordinate system can be re-written [11]:

$$\nabla^{(D_1,D_2,D_3)}\bullet\boldsymbol{H}=\left[\mu_1(1-D_1)x^{-D_1}\right]^{\mathrm{MSL}}\partial_x^{(1)}H_x+\left[\mu_2(1-D_2)y^{-D_2}\right]^{\mathrm{MSL}}\partial_y^{(1)}H_y+ \\ +\left[\mu_3(1-D_3)z^{-D_3}\right]^{\mathrm{MSL}}\partial_z^{(1)}H_z \tag{37}$$

where $\boldsymbol{H}=\boldsymbol{i}H_x+\boldsymbol{j}H_y+\boldsymbol{k}H_z$.

The scaling-law curl of the fractal scaling-law vector field $\boldsymbol{H}$ is defined [11]:

$$\nabla^{(D_1,D_2,D_3)}\times\boldsymbol{H}=\lim_{\Delta V_m\to 0}\frac{1}{\Delta V_m}\oiint_{\Delta\boldsymbol{S}_m}\boldsymbol{H}\times\mathrm{d}\boldsymbol{S} \tag{38}$$

where the scaling-law volume, V, is divided into a large number of small sub-volumes, ΔV_m, with the scaling-law surfaces, ΔS_m, and d$\boldsymbol{S}$ is the element of the fractal scaling-law surface $\boldsymbol{S}$.

The scaling-law curl in a Cartesian co-ordinate system can be re-written [11]:

$$\nabla^{(D_1,D_2,D_3)}\times\boldsymbol{H}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \left[\mu_1(1-D_1)x^{-D_1}\right]^{\mathrm{MSL}}\partial_x^{(1)}, & \left[\mu_2(1-D_2)y^{-D_2}\right]^{\mathrm{MSL}}\partial_y^{(1)}, & \left[\mu_3(1-D_3)z^{-D_3}\right]^{\mathrm{MSL}}\partial_z^{(1)}\\ H_x & H_y & H_z\end{vmatrix} \tag{39}$$

where

$$\boldsymbol{H}=\boldsymbol{i}H_x-\boldsymbol{j}H_y+\boldsymbol{k}H_z$$

The Gauss-Ostrogradsky-like theorem for the scaling-law vector calculus states that [11]:

$$\iiint_\Omega\nabla^{(D_1,D_2,D_3)}\bullet\boldsymbol{\psi}\,\mathrm{d}V=\oiint_{\boldsymbol{S}}\boldsymbol{\psi}\bullet\mathrm{d}\boldsymbol{S} \tag{40}$$

where ΔV_m is the element of the fractal scaling-law volume and d$\boldsymbol{S}$ is the element of the fractal scaling-law surface $\boldsymbol{S}$.

Remark. Taking $D_1=D_2=D_3=1$ and $\mu_1-\mu_2-\mu_3-1$ into eq. (40) we get the Gauss-Ostrogradsky theorem [11]. For the more details of the scaling-law vector calculus, see [11].

The scaling-law heat conduct equations in the Mandelbrots scaling-law behavior

Let $T(x, y, z, t)$ be the temperature field at the point $(x, y, z)\in V$ and time $t\in T$.

The First law of thermodynamics reads:

$$\Pi_1(x,y,z,t)=\Pi_3(x,y,z,t)+\Pi_2(x,y,z,t) \tag{41}$$

where $\prod_1(x, y, z, t)$ is the heat entering unit time through the scaling-law surface $\boldsymbol{S}$, $\prod_2(x, y, z, t)$ – the energy generation unit time in the scaling-law volume, V, and $\prod_3(x, y, z, t)$ – the change unit time in storage energy in the scaling-law volume, V.

The first term of eq. (41) can be represented:

$$\Pi_1(x,y,z,t)=\oiint_S\boldsymbol{U}(x,y,z,t)\,\mathrm{d}\boldsymbol{S} \tag{42}$$

which is related to the formula analogous to the Fourier law:

$$\boldsymbol{U}(x,y,z,t)=-K\nabla^{(D_1,D_2,D_3)}T(x,y,z,t) \tag{43}$$

where K is the thermal conductivity of the scaling-law materials.

The second term of eq. (41) becomes:

$$\Pi_2(x,y,z,t)=\iiint_V W(x,y,z,t)\,\mathrm{d}V \tag{44}$$

The third term of eq. (41) is equal to:

$$\Pi_3(x,y,z,t)=\iiint_V \rho c\, {}^{\mathrm{MSL}}\partial_t^{(1)} T(x,y,z,t)\,\mathrm{d}V \tag{45}$$

where ρ and c are the density and the specific heat of the scaling-law materials, respectively.

The First law of thermodynamics implies that eq. (41) can be written:

$$\iiint_V \rho c\, {}^{\mathrm{MSL}}\partial_t^{(1)} T(x,y,z,t)\,\mathrm{d}V+\oiint_S \boldsymbol{U}(x,y,z,t)\,\mathrm{d}\boldsymbol{S}=\iiint_V W(x,y,z,t)\,\mathrm{d}V \tag{46}$$

By the Gauss-Ostrogradsky-like theorem eq. (40), it is easy to see:

$$\oiint_S \boldsymbol{U}(x,y,z,t)\,\mathrm{d}\boldsymbol{S}=-\oiint_S K\nabla^{(D_1,D_2,D_3)} T(x,y,z,t)\,\mathrm{d}\boldsymbol{S}=-\iiint_V \nabla^{(D_1,D_2,D_3)}\left[K\nabla^{(D_1,D_2,D_3)}\right] T(x,y,z,t)\,\mathrm{d}V \tag{47}$$

On putting eq (47) into eq. (46), we have:

$$\iiint_V \rho c\, {}^{\mathrm{MSL}}\partial_t^{(1)} T(x,y,z,t)\,\mathrm{d}V=$$

$$=\iiint_V \left\{W(x,y,z,t)+\nabla^{(D_1,D_2,D_3)}\bullet\left[K\nabla^{(D_1,D_2,D_3)}\right] T(x,y,z,t)\right\}\mathrm{d}V \tag{48}$$

such that

$$\rho c\, {}^{\mathrm{MSL}}\partial_t^{(1)} T(x,y,z,t)=\nabla^{(D_1,D_2,D_3)}\bullet\left[K\nabla^{(D_1,D_2,D_3)}\right] T(x,y,z,t)+W(x,y,z,t) \tag{49}$$

Thus, the scaling-law heat conduct equation reads:

$${}^{\mathrm{MSL}}\partial_t^{(1)} T(x,y,z,t)=\frac{1}{\rho c}\nabla^{(D_1,D_2,D_3)}\left[K\nabla^{(D_1,D_2,D_3)}\right] T(x,y,z,t)+\frac{1}{\rho c}W(x,y,z,t) \tag{50}$$

Taking:

$$\nabla^{(D_1,D_2,D_3)}\bullet\left[K\nabla^{(D_1,D_2,D_3)}\right] T(x,y,z,t)=\nabla^{(2D_1,2D_2,2D_3)} T(x,y,z,t) \tag{51}$$

in eq. (50), the scaling-law heat conduct equation reads:

$${}^{\mathrm{MSL}}\partial_t^{(1)} T(x,y,z,t)=\frac{1}{\rho c}\nabla^{(2D_1,2D_2,2D_3)} T(x,y,z,t)+\frac{1}{\rho c}W(x,y,z,t) \tag{52}$$

Taking $W(x, y, z, t) = 0$, we arrive:

$${}^{\mathrm{MSL}}\partial_t^{(1)} T(x,y,z,t)=\frac{K}{\rho c}\nabla^{(2D_1,2D_2,2D_3)} T(x,y,z,t) \tag{53}$$

Taking:

$${}^{\mathrm{MSL}}\partial_t^{(1)} T(x,y,z,t)=0 \tag{54}$$

we have from eq. (53) that:

$$\nabla^{(2D_1,2D_2,2D_3)} T(x,y,z,t)=0 \tag{55}$$

which is the steady scaling-law heat conduct equation, which is an analogue to the Laplace equation [13].

Some special cases of eq. (53) are given:

– The scaling-law -space and -time heat conduct equation in 1-D case can be given:

$${}^{\mathrm{MSL}}\partial_t^{(1)} T(x,t)=\frac{K}{\rho c}\,{}^{\mathrm{MSL}}\partial_x^{(2)} T(x,t) \tag{56}$$

– The scaling-law space heat conduct equation in 1-D case can be written:

$$\frac{\partial}{\partial t}T(x,t)=\frac{K}{\rho c}{}^{\mathrm{MSL}}\partial_x^{(2)}T(x,t) \tag{57}$$

– The scaling-law time heat conduct equation in 1-D case can be expressed:

$${}^{\mathrm{MSL}}\partial_t^{(1)}T(x,t)=\frac{K}{\rho c}\frac{\partial^2 T(x,t)}{\partial x^2} \tag{58}$$

It is not difficult to show that eq. (43) is analogous of the theory of the Fourier [1] heat conduction in 1822.

Taking $K = K_0T(x, t)$ in eq. (50), we obtain the following special cases.

– The non-linear scaling-law, -space, and -time heat conduct equation in 1-D case can be given:

$${}^{\mathrm{MSL}}\partial_t^{(1)}T(x,t)=\frac{K_0}{\rho c}T(x,t)\,{}^{\mathrm{MSL}}\partial_x^{(1)}T(x,t)+\frac{K_0}{\rho c}{}^{\mathrm{MSL}}\partial_x^{(2)}T(x,t) \tag{59}$$

– The non-linear scaling-law space heat conduct equation in 1-D case can be written:

$$\frac{\partial}{\partial t}T(x,t)=\frac{K_0}{\rho c}T(x,t)\,{}^{\mathrm{MSL}}\partial_x^{(1)}T(x,t)+\frac{K_0}{\rho c}{}^{\mathrm{MSL}}\partial_x^{(2)}T(x,t) \tag{60}$$

– The non-linear scaling-law time heat conduct equation in 1-D case can be expressed:

$${}^{\mathrm{MSL}}\partial_t^{(1)}T(x,t)=\frac{K_0}{\rho c}T(x,t)\frac{\partial T(x,t)}{\partial x}+\frac{K_0}{\rho c}\frac{\partial^2 T(x,t)}{\partial x^2} \tag{61}$$

It is not difficult to show that eqs. (59)-(61) are analogous of the theory of the Burgers diffusion [14].

The 1-D scaling-law heat conduct equation has an initial condition:

$$T(x,0)=h(x) \tag{62}$$

and boundary conditions:

$$T(0,t)=l(t) \tag{63}$$

and

$$\lim_{x\to\infty}T(x,t)=0 \tag{64}$$

Conclusion

In our work we have proposed the mathematical model for the heat conduction problem giving thought to the behavior of the Mandelbrots scaling law. We also suggested the scaling-law equations analogues to the work of Fourier, Laplace, and Burgers. The analytic, approximate, numerical and exact solutions for the aforementioned results are still open. These are the key directions to study the heat transport process with the Mandelbrots scaling-law phenomena in the future.

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Nomenclature

c – specific heat capacity, [Jkg^{-1}K^{-1}]
K – heat conductivity, [Wm^{-1}K^{-1}]
T – temperature, [K]
t – time, [s]
x, y, z – space co-ordinates, [m]

Greek symbol

ρ – density, [kgm^{-1}]

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