THE Y FUNCTION APPLIED IN THE STUDY OF AN ANOMALOUS DIFFUSION

by

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In this article, we propose a new family of the extended analogues to the Y function for the first time. The relationships among the Y function, Fox H function, Meijer G function, Wright generalized hypergeometric function, and Clausen hypergeometric function are discussed in detail. This result is used to represent the solutions for the anomalous diffusion problems.

Key words: Y function, Fox H function, Meijer G function, anomalous diffusion, generalized hypergeometric function

Introduction

The Pincherle-Mellin-Barnes integrals, which being studied first by Pincherle [1], Barnes [2], and Mellin [3], have played the important role in investigating the structure of the special functions [4] as for example, the Fox H function [5], and Meijer G function [6], Wright generalized hypergeometric function [7, 8], hypergeometric function [9, 10], and others [8, 10].

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$ be the sets of the complex, real number, positive real number, and integer numbers. The Fox H function $\mathbb{H}_{p,q}^{m,n}(x)$, proposed by Fox [5], is defined by the Mellin-Barnes type integral [11]:

$$\mathbb{H}_{p,q}^{m,n}(x)=\mathbb{H}_{p,q}^{m,n}\left[x;\begin{matrix}\{a_j,\alpha_j\}_1^p\\ \{b_j,\beta_j\}_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}x^s\mathrm{d}s \tag{1}$$

where L is the infinite contour in the complex plane, $x \in \mathbb{C}\backslash\{0\}$, $i = (-1)^{1/2}$, $q \geq 1$, $0 \leq n \leq p$, $0 \leq m \leq q$, $\{a_j, b_j\} \in \mathbb{C}$, $\{\alpha_j, \beta_j\} \in \mathbb{R}_+$, and the poles are [11]:

$$k_j^{\lambda}=-\frac{b_j+\lambda}{\beta_j}\left(j=1,\ n;\ \lambda\in\mathbb{N}\cup\{0\}\right) \tag{2}$$

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and

$$k_j^{\hbar} = \frac{1 - a_j + \hbar}{\alpha_j}\left(j = 1,\ n;\ \hbar \in \mathbb{N} \cup \{0\}\right) \tag{3}$$

The Meijer G-function $\mathbb{G}_{p,q}^{m,n}(x)$, introduced by Meijer [6], is defined by the Mellin-Barnes type integral [11]:

$$\mathbb{G}_{p,q}^{m,n}(x) = \mathbb{G}_{p,q}^{m,n}\left[x; \begin{matrix} a_1, a_2, a_3, \cdots, a_p \\ b_1, b_2, b_3, \cdots, b_q \end{matrix}\right] = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + s)\prod_{j=n+1}^{p}\Gamma(a_j - s)} x^s \mathrm{d}s \tag{4}$$

where L is the infinite contour in the complex plane, $x \in \mathbb{C}\backslash\{0\}$, $0 \le n \le p$, $0 \le n \le q$, $\{a_j, b_j\} \in \mathbb{C}$, and the poles are [11]:

$$k_j^{*\lambda} = -\left(b_j + \lambda\right)\left(j = 1, n; \lambda \in \mathbb{N} \cup \{0\}\right) \tag{5}$$

and

$$k_j^{*\hbar} = 1 - a_j + \hbar\left(j = 1, n; \hbar \in \mathbb{N} \cup \{0\}\right) \tag{6}$$

The relationship between the Fox H and Meijer G functions reads [12]:

$$\mathbb{H}_{p,q}^{m,n}\left[x; \begin{matrix} \{a_j, 1\}_1^p \\ \{b_j, 1\}_1^q \end{matrix}\right] = \mathbb{G}_{p,q}^{m,n}\left[x; \begin{matrix} a_1, a_2, a_3, \cdots, a_p \\ b_1, b_2, b_3, \cdots, b_q \end{matrix}\right] \tag{7}$$

The Wright generalized hypergeometric function, proposed by Wright [7], is defined [8]:

$${}_p\Psi_q\left[\begin{matrix} (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ (b_1, \beta_1), \cdots, (b_q, \beta_q) \end{matrix}; x\right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(a_j + k\alpha_j)}{\prod_{j=1}^{q}\Gamma(b_j + k\beta_j)} \frac{x^k}{k!} \tag{8}$$

which can be connected with the Fox H function by [13, p. 352]:

$${}_p\Psi_q\left[\begin{matrix} (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ (b_1, \beta_1), \cdots, (b_q, \beta_q) \end{matrix}; x\right] = \mathbb{H}_{p,q+1}^{1,p}\left[-x; \begin{matrix} (1 - a_1, \alpha_1), \cdots, (1 - a_p, \alpha_p) \\ (1 - b_1, \beta_1), \cdots, (1 - b_q, \beta_q) \end{matrix}\right] \tag{9}$$

The Clausen hypergeometric function, proposed by Clausen [8], is defined [10]:

$${}_pF_q\left[\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}; x\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!} \tag{10}$$

The special Wright generalized hypergeometric function reads [13, p. 352]:

$${}_p\Psi_q\left[\begin{matrix} (a_1, 1), \cdots, (a_p, 1) \\ (b_1, 1), \cdots, (b_q, 1) \end{matrix}; x\right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(a_j + k)}{\prod_{j=1}^{q}\Gamma(b_j + k)} \frac{x^k}{k!} \tag{11}$$

The relationship between the Wright generalized and Clausen hypergeometric functions can be given by [13, p. 352]:

$$ {}_p\psi_q\left[\begin{matrix}(a_1,1),\cdots,(a_p,1)\\(b_1,1),\cdots,(b_q,1)\end{matrix};x\right]=\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}\,{}_pF_q\left[\begin{matrix}a_1,\cdots,a_p\\b_1,\cdots,b_q\end{matrix};x\right] \tag{12} $$

The main target of this paper is to suggest the Y function, which is an extended version of the Fox H function based on the Pincherle-Mellin-Barnes type integral, to give a new family of the special functions associated with the Y function, and to present the new representations of the solutions for the anomalous diffusion.

The Y function

In this section we present the definition and properties of the Y function, and give a family of the extended analogues to the Y function, and suggest the relationships among Y function, Fox H function, Meijer G function, Wright generalized hypergeometric function, and Clausen hypergeometric function.

Let $y \in \mathbb{C}$, $\alpha \in \mathbb{R}$, $x \in \mathbb{C}$, $q \geq 1$, $0 \leq n \leq p$, $0 \leq m \leq q$, $\{a_j, b_j\} \in \mathbb{C}$, and $\{\alpha_j, \beta_j\} \in \mathbb{R}_+$:

$$ k_j^{\lambda}=-\frac{b_j+\lambda}{\beta_j}\left(j=1,\ n;\ \lambda\in\mathbb{N}\cup\{0\}\right) \tag{13} $$

and

$$ k_j^{\hbar}=\frac{1-a_j+\hbar}{\alpha_j}\left(j=1,\ n;\ \hbar\in\mathbb{N}\cup\{0\}\right) \tag{14} $$

The Y function:

$$ \mathbb{Y}_{p,q}^{m,n}[x;y;\alpha]=\mathbb{Y}_{p,q}^{m,n}\left[x;y;\alpha;\begin{matrix}\{a_j,\alpha_j\}_1^p\\\{b_j,\beta_j\}_1^q\end{matrix}\right] \tag{15} $$

is defined:

$$ \mathbb{Y}_{p,q}^{m,n}[x;y;\alpha]=\frac{1}{2\pi i}\int_L\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}x^{-s}\mathrm{e}^{sy}s^{\alpha}\mathrm{d}s \tag{16} $$

provided the integral exists, where L is the infinite contour in the complex plane:

$$ \mathbf{A}_1(s)=\prod_{j=1}^{m}\Gamma\left(b_j-\beta_j s\right) \tag{17} $$

$$ \mathbf{B}_1(s)=\prod_{j=1}^{n}\Gamma\left(1-a_j+\alpha_j s\right) \tag{18} $$

and

$$ \mathbf{C}_1(s)=\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\beta_j s\right) \tag{19} $$

$$\mathbf{D}_1(s)=\prod_{j=n+1}^{p}\Gamma\left(a_j-\alpha_j s\right) \tag{20}$$

The poles of the Y function reads:

$$k_{ƛ}^{\ell}=-\frac{b_j+ƛ}{\beta_j}\left(j=1,\ n;\ ƛ\in\mathbb{N}\cup\{0\}\right) \tag{21}$$

and

$$k_j^{\hbar}=\frac{1-a_j+\hbar}{\alpha_j}\left(j=1,\ n;\ \hbar\in\mathbb{N}\cup\{0\}\right) \tag{22}$$

Taking $y = 0$ and $\alpha = 0$, the relationship between the Y and Fox H functions reads:

$$\mathbb{Y}_{p,q}^{m,n}\left[x;0;0;\begin{matrix}\{a_j,\alpha_j\}_1^p\\ \{b_j,\beta_j\}_1^q\end{matrix}\right]=\mathbb{H}_{p,q}^{m,n}\left[\frac{1}{x};\begin{matrix}\{a_j,\alpha_j\}_1^p\\ \{b_j,\beta_j\}_1^q\end{matrix}\right] \tag{23}$$

The relationship between the Y and Meijer G functions can be given:

$$\mathbb{Y}_{p,q}^{m,n}\left[x;0;0;\begin{matrix}\{a_j,1\}_1^p\\ \{b_j,1\}_1^q\end{matrix}\right]=\mathbb{G}_{p,q}^{m,n}\left[x;\begin{matrix}a_1,a_2,a_3,\cdots,a_p\\ b_1,b_2,b_3,\cdots,b_q\end{matrix}\right] \tag{24}$$

The relationship between the Y function and Wright generalized hypergeometric function can be expressed:

$$\mathbb{Y}_{p,q+1}^{1,p}\left[-x;0;0;\begin{matrix}(1-a_1,\alpha_1),\cdots,(1-a_p,\alpha_p)\\ (1-b_1,\beta_1),\cdots,(1-b_q,\beta_q)\end{matrix}\right]={}_p\psi_q\left[\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\ (b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix};x\right] \tag{25}$$

The relationship between the Y function and Clausen hypergeometric function can be written:

$$\mathbb{Y}_{p,q+1}^{1,p}\left[-x;0;0;\begin{matrix}(1-a_1,1),\cdots,(1-a_p,1)\\ (1-b_1,1),\cdots,(1-b_q,1)\end{matrix}\right]=\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}\,{}_pF_q\left[\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix};x\right] \tag{26}$$

Let $\Re(s)$ be the real part of a complex variable $s\in\mathbb{C}$.

The Mellin transform of the Y function can be given:

$$M\left\{\mathbb{Y}_{p,q}^{m,n}\left[\kappa x;y;\alpha\right]\right\}=\int_0^\infty\mathbb{Y}_{p,q}^{m,n}\left[\kappa x;y;\alpha\right]x^{s-1}\mathrm{d}x=\kappa^{-s}\mathrm{e}^{sy}s^{\alpha}\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)} \tag{27}$$

where

$$M\{g(x)\}=\int_0^\infty g(x)x^{s-1}\mathrm{d}x$$

is the Mellin transform of g(x) [8], $\kappa\in\mathbb{C}$, $s\in\mathbb{C}$, $y>0$:

$$-\min_{1\le j\le m}\Re\left[\Re(b_j)/\beta_j\right]<\Re(s)<\max_{1\le i\le n}\Re\left[1-\left(1-\Re(a_i)\right)/\alpha_i\right]$$

and $|\arg\kappa|<\pi y/2$ for $y>0$.

The Laplace transform of the Y function can be given:

$$\int_0^\infty \mathbb{Y}_{p,q}^{m,n}\left[x;\kappa y;\alpha\right]\mathrm{e}^{-sy}\mathrm{d}y = \frac{s^\alpha}{\kappa^{\alpha+1}}\frac{\mathbf{A}_1(s/\kappa)\mathbf{B}_1(s/\kappa)}{\mathbf{C}_1(s/\kappa)\mathbf{D}_1(s/\kappa)} \tag{28}$$

where

$$L\{f(y)\} = \int_0^\infty f(y)\mathrm{e}^{-sy}\mathrm{d}y$$

is the Laplace transform of $f(y)$ [10], $\kappa \in \mathbb{C}$, $s \in \mathbb{C}$, $y > 0$:

$$-\min_{1\le j\le m}\Re\left[\Re(b_j)/\beta_j\right] < \Re(s) < \max_{1\le i\le n}\Re\left[1-\left(1-\Re(a_i)\right)/\alpha_i\right]$$

and $|\arg\kappa| < \pi y/2$ for $y > 0$.

Family A: The J function

The J function

$$\mathbb{J}_{p,q}^{m,n}(\alpha) = \mathbb{J}_{p,q}^{m,n}\left[\alpha;\begin{matrix}\{a_j,\alpha_j\}_1^p\\\{b_j,\beta_j\}_1^q\end{matrix}\right] \tag{29}$$

is defined

$$\mathbb{J}_{p,q}^{m,n}(\alpha) = \frac{1}{2\pi i}\int_L \frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}s^\alpha\mathrm{d}s \tag{30}$$

provided the integral exists, where L is the infinite contour in the complex plane, $\alpha \in \mathbb{R}$, and $\mathbf{A}_1(s)$, $\mathbf{B}_1(s)$, $\mathbf{C}_1(s)$, and $\mathbf{D}_1(s)$ are defined by eqs. (17)-(20).

Here

$$\mathbb{J}_{p,q}^{m,n}(\alpha) = \frac{1}{2\pi i}\int_L \frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}\mathrm{e}^{\alpha\log s}\mathrm{d}s \tag{31}$$

and

$$\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)} = \int_0^\infty \mathbb{J}_{p,q}^{m,n}(\alpha)\mathrm{e}^{-\alpha\log s}\mathrm{d}\alpha \tag{32}$$

Here, eqs. (31) and (32) can be derived from the special integral transforms of the book in [8, 10].

By eqs. (16) and (30), the relationship between the Y and J functions can be written:

$$\mathbb{J}_{p,q}^{m,n}(\alpha) = \mathbb{Y}_{p,q}^{m,n}(1;0;\alpha) \tag{33}$$

Family B: The L function

The L function

$$\mathbb{L}_{p,q}^{m,n}(y) = \mathbb{L}_{p,q}^{m,n}\left[y;\begin{matrix}\{a_j,\alpha_j\}_1^p\\\{b_j,\beta_j\}_1^q\end{matrix}\right] \tag{34}$$

is defined:

$$\mathbb{L}_{p,q}^{m,n}(y)=\frac{1}{2\pi i}\int_L \frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}\mathrm{e}^{sy}\mathrm{d}s \tag{35}$$

provided the integral exists, where L is the infinite contour in the complex plane, $y\in\mathbb{C}$, and $\mathbf{A}_1(s)$, $\mathbf{B}_1(s)$, $\mathbf{C}_1(s)$, and $\mathbf{D}_1(s)$ are defined by eqs. (17)-(20).

There exists:

$$\mathbb{J}_{p,q}^{m,n}(0)=\mathbb{Y}_{p,q}^{m,n}(1;0;0)=\mathbb{L}_{p,q}^{m,n}(0) \tag{36}$$

and

$$\mathbb{L}_{p,q}^{m,n}(y)=\mathbb{Y}_{p,q}^{m,n}(1;y;0) \tag{37}$$

Hence, there exist:

$$\mathbb{L}_{p,q}^{m,n}(y)=\frac{1}{2\pi i}\int_L \frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}\mathrm{e}^{sy}\mathrm{d}s \tag{38}$$

and

$$\mathbb{L}\left\{\mathbb{L}_{p,q}^{m,n}(y)\right\}=\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)} \tag{39}$$

Moreover,

$$\mathbb{L}_{p,q}^{m,n}(y)=\sum_{\ell=0}^{\infty}\frac{\mathbb{J}_{p,q}^{m,n}(\ell)}{\Gamma(1+\ell)}y^{\ell} \tag{40}$$

Family C: The D function

The D function

$$\mathbb{D}_{p,q}^{m,n}[x;y]=\mathbb{D}_{p,q}^{m,n}\left[x;y;\begin{matrix}\{a_j,\alpha_j\}_1^p\\ \{b_j,\beta_j\}_1^q\end{matrix}\right] \tag{41}$$

is defined:

$$\mathbb{D}_{p,q}^{m,n}[x;y]=\frac{1}{2\pi i}\int_L \frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}x^{-s}\mathrm{e}^{sy}\mathrm{d}s \tag{42}$$

provided the integral exists, where L is the infinite contour in the complex plane, $x\in\mathbb{C}$, $y\in\mathbb{C}$, $\mathbf{A}_1(s)$, $\mathbf{B}_1(s)$, $\mathbf{C}_1(s)$, and $\mathbf{D}_1(s)$ are defined by eqs. (17)-(20).

From eqs. (16) and (42), it follows:

$$\mathbb{D}_{p,q}^{m,n}[x;y]=\mathbb{Y}_{p,q}^{m,n}(x;y;0) \tag{43}$$

and

$$\mathbb{D}_{p,q}^{m,n}[1;y]=\mathbb{L}_{p,q}^{m,n}(y)=\mathbb{Y}_{p,q}^{m,n}(1;y;0) \tag{44}$$

In view of eq. (42), there exist:

$$\mathbb{D}_{p,q}^{m,n}[x;y]=\frac{1}{2\pi i}\int_L \frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}x^{-s}\mathrm{e}^{sy}\mathrm{d}s \tag{45}$$

and

$$L\left\{\mathbb{D}_{p,q}^{m,n}\left[x;y\right]\right\}=\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}x^{-s} \tag{46}$$

Similarly, we have:

$$\mathbb{D}_{p,q}^{m,n}\left[x;y\right]=\frac{1}{2\pi i}\int_L\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}x^{-s}\mathrm{e}^{sy}\mathrm{d}s \tag{47}$$

and

$$M\left\{\mathbb{D}_{p,q}^{m,n}\left[x;y\right]\right\}=\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}\mathrm{e}^{sy} \tag{48}$$

From eq. (41), we obtain:

$$\mathbb{D}_{p,q}^{m,n}\left[x;y\right]=\sum_{\ell=0}^{\infty}\frac{\mathbb{J}_{p,q}^{m,n}(\ell)(y-\log x)^{\ell}}{\Gamma(1+\ell)} \tag{49}$$

Taking $y = 0$ in eq. (49), we get:

$$\mathbb{D}_{p,q}^{m,n}\left[x;0\right]=\mathbb{Y}_{p,q}^{m,n}(x;0;0)=\mathbb{H}_{p,q}^{m,n}\left(\frac{1}{x}\right)=\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\,\mathbb{J}_{p,q}^{m,n}(\ell)(\log x)^{\ell}}{\Gamma(1+\ell)} \tag{50}$$

From eq. (50), we arrive:

$$\mathbb{D}_{p,q}^{m,n}\left[\mathrm{e}^{x};0\right]=\mathbb{Y}_{p,q}^{m,n}\left(\mathrm{e}^{x};0;0\right)=\mathbb{H}_{p,q}^{m,n}\left(\mathrm{e}^{-x}\right)=\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\,\mathbb{J}_{p,q}^{m,n}(\ell)x^{\ell}}{\Gamma(1+\ell)} \tag{51}$$

where $x \in \mathbb{C}$.

Taking $x = 0$ in eq. (49), we show:

$$\mathbb{D}_{p,q}^{m,n}\left[1;y\right]=\mathbb{L}_{p,q}^{m,n}(y)=\sum_{\ell=0}^{\infty}\frac{\mathbb{J}_{p,q}^{m,n}(\ell)y^{\ell}}{\Gamma(1+\ell)} \tag{52}$$

Family D: The T function

The T function

$$\mathbb{T}_{p,q}^{m,n}\left[x;\alpha\right]=\mathbb{T}_{p,q}^{m,n}\left[x;\alpha;\begin{matrix}\left\{a_j,\alpha_j\right\}_1^p\\ \left\{b_j,\beta_j\right\}_1^q\end{matrix}\right] \tag{53}$$

is defined:

$$\mathbb{T}_{p,q}^{m,n}\left[x;\alpha\right]=\frac{1}{2\pi i}\int_L\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}x^{-s}s^{\alpha}\mathrm{d}s \tag{54}$$

provided the integral exists, where L is the infinite contour in the complex plane, $\alpha \in \mathbb{R}$, $x \in \mathbb{C}$, $\mathbf{A}_1(s)$, $\mathbf{B}_1(s)$, $\mathbf{C}_1(s)$, and $\mathbf{D}_1(s)$ are defined by eqs. (17)-(20).

There is:

$$M\left\{\mathbb{T}_{p,q}^{m,n}\left[x;\alpha\right]\right\}=\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}s^{\alpha} \tag{55}$$

From eq. (54), we may see:

$$\mathbb{T}_{p,q}^{m,n}[x;\alpha]=\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}(\log x)^{\ell}\,\mathbb{J}_{p,q}^{m,n}(\ell+\alpha)}{\Gamma(1+\ell)} \tag{56}$$

Here, we also find:

$$\mathbb{T}_{p,q}^{m,n}[1;\alpha]=\mathbb{J}_{p,q}^{m,n}(\alpha)=\mathbb{Y}_{p,q}^{m,n}(1;0;\alpha) \tag{57}$$

and

$$\mathbb{T}_{p,q}^{m,n}[x;0]=\mathbb{Y}_{p,q}^{m,n}(x;0;0)=\mathbb{D}_{p,q}^{m,n}[x;0]=\mathbb{H}_{p,q}^{m,n}\left(\frac{1}{x}\right) \tag{58}$$

Family E: The S function

The S function

$$\mathbb{S}_{p,q}^{m,n}[y;\alpha]=\mathbb{S}_{p,q}^{m,n}\left[y;\alpha;\begin{matrix}\{a_j,\alpha_j\}_1^p\\ \{b_j,\beta_j\}_1^q\end{matrix}\right] \tag{59}$$

is defined:

$$\mathbb{S}_{p,q}^{m,n}[y;\alpha]=\frac{1}{2\pi i}\int_L\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}\mathrm{e}^{sy}s^{\alpha}\mathrm{d}s \tag{60}$$

provided the integral exists, where L is the infinite contour in the complex plane, $y\in\mathbb{C}$, $\alpha\in\mathbb{R}$, $\mathbf{A}_1(s)$, $\mathbf{B}_1(s)$, $\mathbf{C}_1(s)$, and $\mathbf{D}_1(s)$ are defined by eqs. (17)-(20).

By using, it follows:

$$L\left\{\mathbb{S}_{p,q}^{m,n}[y;\alpha]\right\}=\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}s^{\alpha} \tag{61}$$

From eq. (60), we give:

$$\mathbb{S}_{p,q}^{m,n}[y;\alpha]=\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\,\mathbb{J}_{p,q}^{m,n}(\ell+\alpha)}{\Gamma(1+\ell)}y^{\ell} \tag{62}$$

With eqs. (56) and (62), we may suggest:

$$\mathbb{S}_{p,q}^{m,n}[y;\alpha]=\mathbb{T}_{p,q}^{m,n}\left[\mathrm{e}^{y};\alpha\right] \tag{63}$$

and

$$\mathbb{D}_{p,q}^{m,n}[1;y]=\mathbb{L}_{p,q}^{m,n}(y)=\mathbb{S}_{p,q}^{m,n}[y;0] \tag{64}$$

Family F: The Fox H function

From eq. (6), we have:

$$\mathbb{H}_{p,q}^{m,n}(x)=\mathbb{T}_{p,q}^{m,n}[1/x;0]=\mathbb{Y}_{p,q}^{m,n}(1/x;0;0)=\mathbb{D}_{p,q}^{m,n}[1/x;0]=\sum_{\ell=0}^{\infty}\frac{(\log x)^{\ell}\,\mathbb{J}_{p,q}^{m,n}(\ell)}{\Gamma(1+\ell)} \tag{65}$$

where $x\in\mathbb{C}$.

In view of eqs. (7) and (65), we obtain:

$$\mathbb{G}_{p,q}^{m,n}\left[x;\begin{matrix}a_1,a_2,a_3,\cdots,a_p\\ b_1,b_2,b_3,\cdots,b_q\end{matrix}\right]=\sum_{\ell=0}^{\infty}\frac{(\log x)^{\ell}}{\Gamma(1+\ell)}\mathbb{J}_{p,q}^{m,n}\left[\ell;\begin{matrix}\{a_j,1\}_1^p\\ \{b_j,1\}_1^q\end{matrix}\right] \tag{66}$$

Applying (9) and (65), it is seen that:

$$ {}_p\psi_q\left[\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix};x\right]=\sum_{\ell=0}^{\infty}\frac{\left[\log(-x)\right]^{\ell}}{\Gamma(1+\ell)}\mathbb{J}_{p,q+1}^{1,p}\left[\ell;\begin{matrix}(1-a_1,\alpha_1),\cdots,(1-a_p,\alpha_p)\\(1-b_1,\beta_1),\cdots,(1-b_q,\beta_q)\end{matrix}\right] \tag{67} $$

By using eqs. (51) and (65), it follows:

$$ \mathbb{D}_{p,q}^{m,n}\left[e^{-x};0\right]=\mathbb{Y}_{p,q}^{m,n}\left(e^{-x};0;0\right)=\mathbb{H}_{p,q}^{m,n}\left(e^{x}\right)=\sum_{\ell=0}^{\infty}\frac{\mathbb{J}_{p,q}^{m,n}(\ell)}{\Gamma(1+\ell)}x^{\ell} \tag{68} $$

The series repression for the Y function

Write

$$ x^{-s}e^{sy}=e^{s(y-\log x)}=\sum_{\ell=0}^{\infty}\frac{\left[s(y-\log x)\right]^{\ell}}{\Gamma(1+\ell)} \tag{69} $$

where $s\in\mathbb{C}$, $y\in\mathbb{C}$, and $x\in\mathbb{C}$.

On putting eq. (69) into eq. (16), it follows:

$$ \mathbb{Y}_{p,q}^{m,n}\left[x;y;\alpha\right]=\frac{1}{2\pi i}\int_L\frac{\mathbf{A}_1(s)\mathbf{B}_1(s)}{\mathbf{C}_1(s)\mathbf{D}_1(s)}\left\{\sum_{\ell=0}^{\infty}\frac{\left[s(y-\log x)\right]^{\ell}}{\Gamma(1+\ell)}\right\}s^{\alpha}ds \tag{70} $$

which yields that:

$$ \mathbb{Y}_{p,q}^{m,n}\left[x;y;\alpha\right]=\sum_{\ell=0}^{\infty}\frac{(y-\log x)^{\ell}}{\Gamma(1+\ell)}\mathbb{J}_{p,q}^{m,n}(\ell+\alpha) \tag{71} $$

This implies that:

$$ \mathbb{Y}_{p,q}^{m,n}\left[1/x;y;\alpha\right]=\sum_{\ell=0}^{\infty}\frac{(y+\log x)^{\ell}}{\Gamma(1+\ell)}\mathbb{J}_{p,q}^{m,n}(\ell+\alpha) \tag{72} $$

We now show that:

$$ \mathbb{Y}_{p,q}^{m,n}\left[e^{x};y;\alpha\right]=\sum_{\ell=0}^{\infty}\frac{(y-x)^{\ell}}{\Gamma(1+\ell)}\mathbb{J}_{p,q}^{m,n}(\ell+\alpha) \tag{73} $$

and

$$ \mathbb{Y}_{p,q}^{m,n}\left[e^{-x};y;\alpha\right]=\sum_{\ell=0}^{\infty}\frac{(y+x)^{\ell}}{\Gamma(1+\ell)}\mathbb{J}_{p,q}^{m,n}(\ell+\alpha) \tag{74} $$

Here, we also show that:

$$ \mathbb{Y}_{p,q}^{m,n}\left[x;\log y;\alpha\right]=\sum_{\ell=0}^{\infty}\frac{\mathbb{J}_{p,q}^{m,n}(\ell+\alpha)}{\Gamma(1+\ell)}\left(\log\frac{y}{x}\right)^{\ell} \tag{75} $$

and

$$ \mathbb{Y}_{p,q}^{m,n}\left[1/x;\log y;\alpha\right]=\sum_{\ell=0}^{\infty}\frac{\mathbb{J}_{p,q}^{m,n}(\ell+\alpha)}{\Gamma(1+\ell)}\left[\log(xy)\right]^{\ell} \tag{76} $$

If:

$$\mathfrak{M}_{p,q}^{m,n}[x;\alpha]=\sum_{\ell=0}^{\infty}\frac{x^{\ell}}{\Gamma(1+\ell)}\mathbb{J}_{p,q}^{m,n}(\ell+\alpha) \tag{77}$$

we obtain:

$$\mathbb{Y}_{p,q}^{m,n}[x;\log y;\alpha]:=\mathfrak{M}_{p,q}^{m,n}[\log(y/x);\alpha]=\sum_{\ell=0}^{\infty}\frac{\mathbb{J}_{p,q}^{m,n}(\ell+\alpha)}{\Gamma(1+\ell)}\left(\log\frac{y}{x}\right)^{\ell} \tag{78}$$

and

$$\mathbb{Y}_{p,q}^{m,n}[1/x;\log y;\alpha]:=\mathfrak{M}_{p,q}^{m,n}[\log(xy);\alpha]=\sum_{\ell=0}^{\infty}\frac{[\log(xy)]^{\ell}}{\Gamma(1+\ell)}\mathbb{J}_{p,q}^{m,n}(\ell+\alpha) \tag{79}$$

Hence, we have:

$$\mathfrak{M}_{p,q}^{m,n}[\log x;\alpha]=\sum_{\ell=0}^{\infty}\frac{(\log x)^{\ell}}{\Gamma(1+\ell)}\mathbb{J}_{p,q}^{m,n}(\ell+\alpha) \tag{80}$$

such that

$$\mathbb{H}_{p,q}^{m,n}(x)=\mathbb{T}_{p,q}^{m,n}[1/x;0]=\mathbb{Y}_{p,q}^{m,n}(1/x;0;0)=\mathbb{D}_{p,q}^{m,n}[1/x;0]=\mathfrak{M}_{p,q}^{m,n}[\log x;0] \tag{81}$$

Form eqs. (12) and (67), we derive:

$$\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}\,{}_pF_q\left[\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix};x\right]=\sum_{\ell=0}^{\infty}\frac{[\log(-x)]^{\ell}}{\Gamma(1+\ell)}\mathbb{J}_{p,q+1}^{1,p}\left[\ell;\begin{matrix}(1-a_1,1),\cdots,(1-a_p,1)\\ (1-b_1,1),\cdots,(1-b_q,1)\end{matrix}\right] \tag{82}$$

A typical application in anomalous diffusion

We now consider the multidimensional anomalous diffusion, given by [14]:

$$\partial_t^{\varepsilon}u(\mathbf{x},t)=(-\Delta)^{\frac{\varphi}{2}}u(\mathbf{x},t)\quad\left(\mathbf{x}\in\mathbb{R}^2,t>0,0<\varepsilon\le 1\right) \tag{83}$$

subject to the conditions:

$$u(\mathbf{x},0)=\phi(\mathbf{x})\left(\mathbf{x}\in\mathbb{R}^2\right) \tag{84}$$

and

$$\partial_x u(\mathbf{x},t)=0\left(\mathbf{x}\in\mathbb{R}^2\right) \tag{85}$$

where $(-\Delta)^{\varphi/2}$ is the fractional Laplacian with $\varphi\in(1,2]$ [8, 14] and ∂_t^{ε} is the Caputo time-fractional derivative of the order $\varepsilon\in(1,2]$ [8,14].

Taking:

$$\phi(\mathbf{x})=\delta(\mathbf{x})\left(\mathbf{x}\in\mathbb{R}^2\right) \tag{86}$$

and $\varphi=2\varepsilon$, we arrive [14]

$$u(\mathbf{x},t)=\frac{1}{4\pi t}\frac{1}{2\pi i}\int_{\Re(s)=\psi}\frac{\Gamma(\varphi/2-s)\Gamma(1-\varphi/2+s)}{\Gamma(1-\varphi s/2)}\left(\frac{|\mathbf{x}|}{2\sqrt{t}}\right)^{-\varphi s}\mathrm{d}s \tag{87}$$

where $2/\varphi-1<\psi<2/\varphi$.

Taking

$$x = \left[|\mathbf{x}| / \left(2\sqrt{t} \right) \right]^{\varphi}$$

$y = 0$ and $\alpha = 0$ and using eq. (16), we can obtain:

$$u(\mathbf{x},t) = \frac{1}{4\pi t} \mathbb{Y}_{1,0}^{1,1} \left[\left(\frac{|\mathbf{x}|}{2\sqrt{t}} \right)^{\varphi}; 0; 0; \begin{matrix} (\varphi/2,1),(1-\varphi/2,1) \\ (1,\varphi/2) \end{matrix} \right] \tag{88}$$

Conclusion

In the present work, we suggested the Y function and a new family related to the Y function. We presented that the Fox H and Meijer G functions are the special cases of the Y function. The relationships among the Y, Fox H, and Meijer G functions, and Wright and Clausen hypergeometric function were considered by the series representations of the J function. We also obtained a new application for the representation theory for the anomalous diffusion process.

Nomenclature

t – space co-ordinate, [s]
$u(\mathbf{x}, t)$ – temperature function, [K]
$\mathbf{x}$ – space co-ordinate, [m]

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