ANALYTICAL SOLUTION TO LOCAL FRACTIONAL LANDAU-GINZBURG-HIGGS EQUATION ON FRACTAL MEDIA

by

Shu-Xian DENG[a] ***and Xin-Xin GE***[b*]

[a] School of Science, Henan University of Engineering, Xinzheng, China
[b] School of Management Engineering, Henan University of Engineering, Xinzheng, China

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The main objective of the present article is to introduce a new analytical solution of the local fractional Landau-Ginzburg-Higgs equation on fractal media by means of the local fractional variational iteration transform method, which is coupling of the variational iteration method and Yang-Laplace transform method.

Key words: *Landau-Ginzburg-Higgs equation, fractal media, Yang-Laplace transform, variational iteration method*

Introduction

In the present work, we consider the following local fractional Landau-Ginzburg-Higgs equation on fractal media as follows:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha}u}{\partial x^{2\alpha}} - m^2u + n^2u^3 = 0, \quad 0 < \alpha \le 1 \tag{1}$$

with the initial conditions

$$u(x,0) = f(x^\alpha), \quad \frac{\partial^\alpha}{\partial t^\alpha}u(x,0) = g(x^\alpha) \tag{2}$$

where $\partial^\alpha u/\partial t^\alpha$ denote the local fractional derivative of $u(x,t)$, m and n are parameters, and both $f(x^\alpha)$ and $g(x^\alpha)$ are given functions.

The problem (1)-(2) is widely used to model fractal heat transfer and anomalous heat flow in superconductors [1-8]. The local fractional differential equations have attracted lots of attention among scientists [9-11]. In most cases, the local fractional differential equations were applied to model problems in fractal media. Finding the non-differentiable solutions of the local fractional differential equations is the hot topics [12-15]. However, it is difficult to obtain an exact analytic solution for the non-linear local fractional differential equations.

Recently, some useful techniques have been successfully applied to deal with the local fractional differential equations, such as the local fractional variational iteration method [12, 13], the local fractional Adomian decomposition method [14], the local fractional series expansion method [15], the fractional Laplace transform method [16] and other methods [17-20]. The main objective of the present article is introduce a new analytical solution of the problem

* Corresponding author, e-mail: hngcdsx@163.com

(1)-(2) by means of local fractional variational iteration transform method, which is coupling of variational iteration method and Yang-Laplace transform method.

Preliminaries

In this section, we recall some definitions and properties of local fractional calculus and Yang-Laplace transform. For more details, see [21].

Assume the relation below exists:

$$\left|f(x)-f(x_0)\right|<\varepsilon^{\alpha} \tag{3}$$

with $|x-x_0|<\delta$ for $\varepsilon,\delta>0$. Then $f(x)$ is local fractional continuous at x_0 which is denoted by $\lim_{x\to x_0} f(x)=f(x_0)$. If $f(x)$ is local fractional continuous on the interval (a,b), it is denoted by:

$$f(x)\in C_{\alpha}(a,b)$$

Let $f(x)\in C_{\alpha}(a,b)$. The local fractional derivative of $f(x)$ of fractal order at the point $x=x_0$ is given:

$$\mathrm{D}_x^{\alpha} f(x_0)=\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}f(x)\bigg|_{x=x_0}=f^{(\alpha)}(x_0)=\lim_{x\to x_0}\frac{\Delta^{\alpha}[f(x)-f(x_0)]}{(x-x_0)^{\alpha}} \tag{4}$$

where $\Delta[f(x)-f(x_0)]\cong\Gamma(\alpha+1)[f(x)-f(x_0)]$.

A partition of the interval $[a,b]$ is denoted as (t_j,t_{j+1}), $j=0,1,\cdots,N-1$, $t_0=a$ and $t_N=b$ with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_0,\Delta t_1,\cdots\Delta t_N\}$. The local fractional integral of $f(x)$ in the interval $[a,b]$ is given:

$${}_aI_b^{(\alpha)}f(x)=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(\mathrm{d}x)^{\alpha}=\lim_{\Delta t\to 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^{\alpha} \tag{5}$$

In the fractal space, the Mittag-Leffler-Yang function is given [21]:

$$E_{\alpha}(x^{\alpha})=\sum_{n=0}^{\infty}\frac{x^{(n\alpha)}}{\Gamma(1+n\alpha)},\quad 0<\alpha\le 1 \tag{6}$$

Let

$$\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}|f(x)|(\mathrm{d}x)^{\alpha}<k<\infty,\quad 0<\alpha\le 1$$

The Yang-Laplace transforms of $f(x)$ is given:

$$L_{\alpha}\{f(x)\}=f_s^{L,\alpha}(s)=\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}E_{\alpha}(-s^{\alpha}x^{\alpha})f(x)(\mathrm{d}x)^{\alpha} \tag{7}$$

where the latter integral converges and $s^{\alpha}\in R^{\alpha}$.

The inverse transform of the Yang-Laplace transforms of $f(x)$ is given:

$$L_{\alpha}^{-1}\{f_s^{L,\alpha}(s)\}=f(x)=\frac{1}{(2\pi)^{\alpha}}\int_{\beta-i\varpi}^{\beta+i\varpi}E_{\alpha}(s^{\alpha}x^{\alpha})f_s^{L,\alpha}(S)(\mathrm{d}s)^{\alpha} \tag{8}$$

where $s^{\alpha}=\beta^{\alpha}+i^{\alpha}\varpi^{\alpha}$ fractal imaginary unit i^{α} and $\mathrm{Re}(s)=\beta>0$.

Some useful formulas of local fractional derivative were summarized:

$$\frac{d^{\alpha}(x^{n\alpha})}{dx^{\alpha}}=\frac{\Gamma(1+n\alpha)x^{(n-1)\alpha}}{\Gamma[1+(n-1)\alpha]} \tag{9}$$

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b E_{\alpha}(x^{\alpha})(dx)^{\alpha}=E_{\alpha}(b^{\alpha})-E_{\alpha}(a^{\alpha}) \tag{10}$$

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b x^{n\alpha}(dx)^{\alpha}=\frac{\Gamma(1+n\alpha)[b^{(n+1)\alpha}-a^{(n+1)\alpha}]}{\Gamma[1+(n+1)\alpha]} \tag{11}$$

Now, we recall some basic properties of local fractional Yang-Laplace transform.
Let $L_{\alpha}\{f(x)\}=f_s^{L,\alpha}(s)$ and $L_{\alpha}\{g(x)\}=g_s^{L,\alpha}(s)$, then we have:

$$L_{\alpha}\{af(x)+bg(x)\}=af_s^{L,\alpha}(s)+bg_s^{L,\alpha}(s) \tag{12}$$

$$L_{\alpha}\left\{E_{\alpha}(c^{\alpha}x^{\alpha})f(x)\right\}=f_s^{L,\alpha}(s-c) \tag{13}$$

$$L_{\alpha}\left\{f^{(\alpha)}(x)\right\}=s^{\alpha}f_s^{L,\alpha}(s)-f(0) \tag{14}$$

$$L_{\alpha}\left\{x^{k\alpha}\right\}=\frac{\Gamma(1+k\alpha)}{s^{(k+1)\alpha}} \tag{15}$$

Local fractional variational iteration transform method

Consider the following general non-linear local fractional PDE:

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t)+P_{\alpha}u(x,t)+N_{\alpha}u(x,t)=0,\quad t>0,\quad x\in R,\quad 0<\alpha<1 \tag{16}$$

where P_{α} denotes the linear local fractional differential operator, and N_{α} represents the general non-linear local fractional operator.

Taking local fractional Yang-Laplace transform on eq. (16), we obtain:

$$L_{\alpha}\left[\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t)\right]+L_{\alpha}\left[P_{\alpha}u(x,t)\right]+L_{\alpha}\left[N_{\alpha}u(x,t)\right]=0 \tag{17}$$

By applying eq. (14), we have:

$$s^{2\alpha}L_{\alpha}[u(x,t)]-s^{\alpha}u(x,0)-\frac{\partial^{\alpha}u}{\partial t^{\alpha}}(x,0)=-L_{\alpha}[P_{\alpha}u(x,t)]-L_{\alpha}[N_{\alpha}u(x,t)] \tag{18}$$

or

$$L_{\alpha}[u(x,t)]=\frac{1}{s^{\alpha}}u(x,0)+\frac{1}{s^{2\alpha}}\frac{\partial^{\alpha}u}{\partial t^{\alpha}}(x,0)-\frac{1}{s^{2\alpha}}L_{\alpha}[P_{\alpha}u(x,t)]-\frac{1}{s^{2\alpha}}L_{\alpha}[N_{\alpha}u(x,t)] \tag{19}$$

Operating with the Yang-Laplace inverse on both sides of eq. (19) gives:

$$u(x,t)=u(x,0)+\frac{t^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}u}{\partial t^{\alpha}}(x,0)+L_{\alpha}^{-1}\left\{\frac{1}{s^{2\alpha}}L_{\alpha}\left[-P_{\alpha}u(x,t)-N_{\alpha}(x,t)\right]\right\} \tag{20}$$

Finding the local fractional derivative in (20), we obtain:

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)-\frac{\partial^\alpha u}{\partial t^\alpha}(x,0)-\frac{\partial^\alpha}{\partial t^\alpha}L_\alpha^{-1}\left\{\frac{1}{s^{2\alpha}}L_\alpha\left[-P_\alpha u(x,t)-N_\alpha(x,t)\right]\right\}=0 \tag{21}$$

By making the correction function, we get:

$$u_{k+1}(x,t)=$$

$$=u_k(x,t)-{}_0I_t^\alpha\left[\frac{\partial^\alpha u_k(x,\tau)}{\partial\tau^\alpha}-\frac{\partial^\alpha u_k(x,0)}{\partial\tau^\alpha}+\frac{\partial^\alpha}{\partial\tau^\alpha}L_\alpha^{-1}\left\{\frac{1}{s^{2\alpha}}L_\alpha\left[P_\alpha u_k(x,\tau)+N_\alpha u_k(x,\tau)\right]\right\}\right]$$

Finally, the solution $u(x,t)$ is given:

$$u(x,t)=\lim_{k\to\infty}u_k(x,t).$$

The solutions of the problem (1)-(2)

In this section, we present the solutions of local fractional Landau-Ginzburg- Higgs equation on fractal media.

Consider the following local fractional Landau-Ginzburg-Higgs equation on fractal media:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}-\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}-m^2u+n^2u^3=0,\quad 0<\alpha\le 1 \tag{22}$$

with the initial conditions:

$$u(x,0)=f(x^\alpha),\qquad \frac{\partial^\alpha}{\partial t^\alpha}u(x,0)=g(x^\alpha) \tag{23}$$

We rewrite eq. (22) as follows:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}=\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}+m^2u-n^2u^3 \tag{24}$$

By taking local fractional Yang-Laplace transform and applying eq. (14), we have:

$$s^{2\alpha}L_\alpha[u(x,t)]-s^\alpha u(x,0)-\frac{\partial^\alpha u}{\partial t^\alpha}(x,0)=L_\alpha\left(\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}+m^2u-n^2u^3\right) \tag{25}$$

or

$$L_\alpha[u(x,t)]=\frac{1}{s^\alpha}f(x)+\frac{1}{s^{2\alpha}}g(x)+\frac{1}{s^{2\alpha}}L_\alpha\left(\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}+m^2u-n^2u^3\right) \tag{26}$$

Then, applying the inverse Yang-Laplace transform to eq. (26), we get:

$$u(x,t)=f(x)+\frac{t^\alpha}{\Gamma(1+\alpha)}g(x)+L_\alpha^{-1}\left\{\frac{1}{s^{2\alpha}}L_\alpha\left[\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u(x,t)+m^2u-n^2u^3\right]\right\} \tag{27}$$

From eq. (27), we have:

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)-g(x)-\frac{\partial^\alpha}{\partial t^\alpha}L_\alpha^{-1}\left[\frac{1}{s^{2\alpha}}L_\alpha\left(\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}+m^2u-n^2u^3\right)\right]=0$$

Making the correction function is given:

$$u_{k+1}(x,t)=$$

$$=u_k(x,t)-{}_0I_t^{\alpha}\left\{\frac{\partial^{\alpha}u_k(x,\tau)}{\partial\tau^{\alpha}}-\frac{\partial^{\alpha}u_k(x,0)}{\partial\tau^{\alpha}}-\frac{\partial^{\alpha}}{\partial\tau^{\alpha}}L_{\alpha}^{-1}\left[\frac{1}{s^{2\alpha}}L_{\alpha}\left(\frac{\partial^{2\alpha}u_k}{\partial x^{2\alpha}}+m^2u_k-n^2u_k\right)\right]\right\}$$

Using the initial condition, we can select:

$$u_0(x,t)=f(x^{\alpha})$$

From this selection, we can get the successive approximations solutions.

Next, two examples are presented in order to demonstrate the efficiency of the aforementioned method.

Example 1: We consider the following linear local fractional Landau-Ginzburg-Higgs equation on fractal media:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}-\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}=u,\quad 0<\alpha\le 1$$

with the initial condition:

$$u(x,0)=2\sqrt{2}E_{\alpha}(-x^{\alpha}),\quad \frac{\partial^{\alpha}u(x,0)}{\partial t^{\alpha}}=-4E_{\alpha}(-x^{\alpha})$$

Firstly, from the initial condition, we select:

$$u_0(x,t)=2\sqrt{2}E_{\alpha}(-x^{\alpha})$$

Then, applying previous algorithm, we get the following approximations:

$$u_1(x,t)=u_0(x,t)-{}_0I_t^{\alpha}\left\{\frac{\partial^{\alpha}u_0(x,\tau)}{\partial\tau^{\alpha}}-\frac{\partial^{\alpha}u_0(x,0)}{\partial\tau^{\alpha}}-\frac{\partial^{\alpha}}{\partial\tau^{\alpha}}L_{\alpha}^{-1}\left[\frac{1}{s^{2\alpha}}L_{\alpha}\left(\frac{\partial^{2\alpha}u_0}{\partial x^{2\alpha}}+u_0\right)\right]\right\}$$

$$u_2(x,t)=u_1(x,t)-{}_0I_t^{\alpha}\left\{\frac{\partial^{\alpha}u_1(x,\tau)}{\partial\tau^{\alpha}}-\frac{\partial^{\alpha}u_1(x,0)}{\partial\tau^{\alpha}}-\frac{\partial^{\alpha}}{\partial\tau^{\alpha}}L_{\alpha}^{-1}\left[\frac{1}{s^{2\alpha}}L_{\alpha}\left(\frac{\partial^{2\alpha}u_1}{\partial x^{2\alpha}}+u_1\right)\right]\right\}$$

$$u_3(x,t)=u_2(x,t)-8E_{\alpha}(-x^{\alpha})\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$$

Finally, the solution is:

$$u(x,t)=2\sqrt{2}E_{\alpha}(-x^{\alpha})\left[1-\frac{\sqrt{2}t^{\alpha}}{\Gamma(1+\alpha)}+\frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}+\cdots\right]$$

Example 2: We consider the following non-linear local fractional Landau-Ginzburg--Higgs equation on fractal media:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}-\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}-m^2u+n^2u=0,\quad 0<\alpha\le 1$$

with the initial conditions:

$$u(x,0)=\sqrt{2}\,\mathrm{sech}_\alpha(x^\alpha), \quad \text{and} \quad \frac{\partial^\alpha u(x,0)}{\partial t^\alpha}=-\tanh_\alpha(x^\alpha)\,\mathrm{sech}_\alpha(x^\alpha)$$

By the initial conditions, we select:

$$u_0(x,t)=2\sqrt{2}\,\Omega E_\alpha(x^\alpha), \quad \text{and} \quad \Omega=\frac{1}{1+E_\alpha(2x^\alpha)}$$

Then, we obtain:

$$u_1(x,t)=u_0(x,t)-{}_0I_t^\alpha\left\{\frac{\partial^\alpha u_0(x,\tau)}{\partial\tau^\alpha}-\frac{\partial^\alpha u_0(x,0)}{\partial\tau^\alpha}-\frac{\partial^\alpha}{\partial\tau^\alpha}L_\alpha^{-1}\left[\frac{1}{s^{2\alpha}}L_\alpha\left(\frac{\partial^{2\alpha}u_0}{\partial x^{2\alpha}}+u_0-u_0^3\right)\right]\right\}=$$

$$=u_0(x,t)-4E_\alpha(x^\alpha)[E_\alpha(2x^\alpha)-1]\Omega^2\frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$u_2(x,t)=u_1(x,t)-{}_0I_t^\alpha\left\{\frac{\partial^\alpha u_1(x,\tau)}{\partial\tau^\alpha}-\frac{\partial^\alpha u_1(x,0)}{\partial\tau^\alpha}-\frac{\partial^\alpha}{\partial\tau^\alpha}L_\alpha^{-1}\left[\frac{1}{s^{2\alpha}}L_\alpha\left(\frac{\partial^{2\alpha}u_1}{\partial x^{2\alpha}}+u_1-u_1^3\right)\right]\right\}=$$

$$=u_1(x,t)-2\sqrt{2}E_\alpha(x^\alpha)[6E_\alpha(2x^\alpha)-E_\alpha(4x^\alpha)-1]\,\Omega^3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

and so on.

Conclusion

In the present work, the analytical solution for the local fractional Landau-Ginzburg-Higgs equation is obtained by the local fractional variational iteration transform method. The results obtained show that the solution is convergent very rapidly. The present method is very efficient for finding the analytical solutions for a wide class of non-linear local fractional differential equations.

Nomenclature

t – time, [s]
x – space co-ordinates, [m]

Greek symbol

α – fractal dimension, [–]

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