ON (2+1)-DIMENSIONAL EXPANDING INTEGRABLE MODEL OF THE DAVEY-STEWARTSON HIERARCHY

by

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This paper mainly investigates the reductions of an integrable coupling of the Levi hierarchy and an expanding model of the (2+1)-dimensional Davey-Stewartson hierarchy. It is shown that the integrable coupling system of the Levi hierarchy possesses a quasi-Hamiltonian structure under certain constraints. Based on the Lie algebras construct, The type abstraction hierarchy scheme is used to generate the (2+1)-dimensional expanding integrable model of the Davey-Stewartson hierarchy.

Key words: *Levi hierarchy, Lie algebras, Quasi-Hamiltonian structure, type abstraction hierarchy scheme, Davey-Stewartson hierarchy*

Introduction

Searching for the expanding models and the reductions of integrable hierarchies are two important subjects in integrable systems in the different systems [1-4]. Using 2×2 Lie algebras, Tu [5] proposed a scheme for generating integrable Hamiltonian hierarchies, and constructed a united integrable model of the Levi, D-AKNS and TD hierarchies [6]. In [7], we introduced two types of block-matrix Lie algebras to obtain a united integrable model of the Levi and AKNS hierarchies, which provided a new method in search for the standard heat equation and a special Newell-Whitehead equation.

Based on the Hamiltonian structures of the integrable coupling of the Levi hierarchy given in [7], our aim of the paper is to show that its quasi-Hamiltonian structure is deduced under certain constraints and to obtain a (2+1)-D expanding integrable model of the DS hierarchy.

The type abstraction hierarchy scheme

At present, there is less work on the use of the type abstraction hierarchy (TAH) scheme to generate higher-dimensional hierarchies of evolution equations. Here, the TAH scheme is recalled briefly. At first, Tu *et al.* [8] proposed an efficient and direct approach to generate the (2+1)-D equation hierarchies by introducing a residue operator, which is called the TAH scheme. In details, the TAH scheme is as follows.

Let $\mathcal{A}$ be an associative algebra over the field $\mathcal{R}$. The operator $\partial:\mathcal{A}\to\mathcal{A}$ satisfies that:

$$\partial(\alpha f+\beta g)=\alpha\partial f+\beta(\partial g),\quad \partial(fg)=(\partial f)g+f(\partial g)$$

where $\alpha,\beta\in\mathcal{R}; f,g\in\mathcal{A}$.

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Secondly, an associative algebra $\mathcal{A}[\xi]$ σ_t consisting of $\sum_{-\infty}^{N} a_i\xi^i$ is introduced, where the coefficient $a_i \in \mathcal{A}$ and ξ is the operator, given by:

$$\xi f = f\xi + (\partial_y)f, \quad f \in \mathcal{A} \tag{1}$$

It can be verified that [8]:

$$\xi^n f = \sum_{i\geq 0}\binom{n}{i}(\partial^i f)\xi^{n-i}, \quad n \in \mathbf{Z} \tag{2}$$

$$f\xi^n = \sum_{i\geq 0}(-1)^i\binom{n}{i}\xi^{n-i}(\partial^i f), \quad n \in \mathbf{Z} \tag{3}$$

In addition, the residue operator is defined:

$$R = \mathcal{A}[\xi] \to \mathcal{A}, R\left(\sum a_i\xi^i\right) = a_{-1} \tag{4}$$

Finally, the TAH scheme can be stated as follows.

Fixing a matrix operator as:

$$U = U(\lambda, \xi, u) \in \mathcal{A}[\xi]$$

where $u = (u_1, \cdots u_p)^T$.

Solve the matrix-operator equation by:

$$V_x = [U, V] \tag{5}$$

in which $V = \sum V_n\lambda^{-n}$.

From eq. (5), the recursion relation among $g^{(n)} = [g_1^{(n)}, \cdots, g_p^{(n)}]^T$ can be obtained, where $g_i^{(n)}$ comes from the expansion of:

$$< V, \frac{\partial U}{\partial u_i} > = \sum_n g_i^{(n)}\lambda^{-n}$$

where, $< a, b > = trR(ab)$.

Try to find the operator J and form a hierarchy as:

$$u_{t_n} = Jg^{(n)} \tag{6}$$

Using the trace identity:

$$\frac{\delta}{\delta u_i} < V, \frac{\partial U}{\partial \lambda} > = \left(\lambda^{-r}\frac{\partial}{\partial\lambda}\lambda^r\right) < V, \frac{\partial U}{\partial u_i} >, \quad i = 1, 2, \ldots, p \tag{7}$$

we can derive the Hamiltonian structure of the (2+1)-D hierarchy given in eq. (6).

Tu *et al.* [8] have obtained the Kadomtsev-Petviashvili (KP) hierarchy and the Davey-Stewartson (DS) hierarchy by the TAH scheme in the frame of the subalgebra of the Lie algebra A_1. In this paper, we would extend the Lie algebra A_1 to the case of the Lie algebra A_3 presented by block-matrix forms, in which some new integrable hierarchies including the (2+1)-D model of the DS hierarchy are obtained.

The quasi-Hamiltonian structure of the Levi hierarchy

The following Lie algebra G_1 and the corresponding loop algebra can be given by [7]:

$$\tilde{G}_1 : G_1 = \{f_1, \cdots, f_8\}, \quad \tilde{G}_1 = \{f_1(n), \cdots, f_8(n)\}$$

Further, we established the Lie algebra, given as follows:

$$G_2 : G_2 = \Delta_1 \oplus \Delta_2,\ \left[\Delta_i, \Delta_i\right] \subset \Delta_i,\ i = 1, 2, \left[\Delta_1, \Delta_2\right] \subset \Delta_2$$

where the corresponding loop algebra is:

$$\tilde{G}_2 : \tilde{G}_2 = \left\{g_1(n), g_2(n), \cdots, g_7(n)\right\}$$

Using an isospectral problem $\psi_x = U\psi,\ \psi_t = V\psi$, we have:

$$\begin{aligned} &q_t = -V_{2,n+1} - q(V_{3n} - V_{2n} + 2V_{1n}),\ r_t = V_{3,n+1} + r(V_{3n} - V_{2n} + 2V_{1n}) \\ &s_{1,t} = -V_{6,n+1} - s_1(V_{3n} - V_{2n} + 2V_{1n}) = V_{7n,x} + (r-q)V_{6n} + (q+r+2s_1)V_{5n} - s_1 V_{3n} + s_1 V_{2n} \\ &s_{2,t} = -V_{7,n+1} - s_2(V_{3n} - V_{2n} + 2V_{1n}) = V_{6n,x} + (r-q)V_{7n} + (q-r+2s_1)V_{5n} - s_1 V_{3n} + s_1 V_{2n} \end{aligned} \tag{8}$$

When $s_1 = s_2 = 0$, eq. (8) is reduced to the Levi hierarchy. According to the theory of integrable couplings, eq. (8) is an integrable coupling of the Levi hierarchy [7]. Based on approach generating the Hamiltonian structures of the integrable couplings [9, 10], we deduce the Hamiltonian structures of the integrable couplings of the Levi hierarchy under the constraints between potential and parameters.

Set:

$$a = \sum_{i=1}^{8} a_i f_i,\quad b = \sum_{i=1}^{8} b_i f_i$$

Then, we have:

$$\begin{aligned} \left[a, b\right] &= (a_3b_4 - a_4b_3)f_1 + \left(a_4b_3 - a_3b_4\right)f_2 + (a_1b_3 - a_3b_1 + a_3b_2 - a_2b_3)f_3 + (a_2b_4 - a_4b_2 + \\ &+ a_4b_1 - a_1b_4)f_4 + (a_3b_7 - a_7b_3 + 2a_8b_7 - 2a_7b_8 + a_8b_3 - a_3b_8 + a_8b_4 - a_4b_8 + a_7b_4 - \\ &- a_4b_7)f_6 + (a_1b_8 - a_8b_1 + a_4b_6 - a_6b_4 + 2a_6b_8 - 2a_8b_6 + a_8b_2 - a_2b_8 + a_6b_3 - a_3b_6)f_7 + \\ &+ (a_1b_7 - a_7b_1 + a_7b_2 - a_2b_7 + a_6b_3 - a_3b_6 + a_6b_4 - a_4b_6 + 2a_6b_7 - 2a_7b_6)f_8 \equiv \sum_{i=1, i\neq 5}^{8} F_i f_i \end{aligned} \tag{9}$$

An operation relation in the linear space R^8 is defined:

$$\left[a, b\right] = \left(F_1, F_2, F_3, F_4, 0, F_6, F_7, F_8\right)^T = a^T M_1(b) \tag{10}$$

where

$$M_1(b) = \begin{pmatrix} m_1 & m_2 \\ 0 & m_4 \end{pmatrix}$$

with

$$m_1 = \begin{pmatrix} 0 & 0 & b_3 & -b_4 \\ 0 & 0 & -b_3 & b_4 \\ b_4 & -b_4 & b_2 - b_1 & 0 \\ -b_3 & b_3 & 0 & b_1 - b_2 \end{pmatrix},\quad m_2 = \begin{pmatrix} 0 & 0 & b_8 & b_7 \\ 0 & 0 & -b_8 & -b_7 \\ 0 & b_7 - b_8 & -b_6 & -b_6 \\ 0 & -b_7 - b_8 & b_6 & -b_6 \end{pmatrix}$$

and

$$m_4=\begin{pmatrix}0 & 0 & 0 & b_7\\ 0 & 0 & b_3-b_4+2b_8 & b_3+b_4+2b_7\\ 0 & b_4-b_3-2b_8 & 0 & -b_1+b_2-2b_6\\ 0 & b_3+b_4+2b_7 & -b_1+b_2-2b_6 & 0\end{pmatrix}$$

It can be verified that, equipped with eq. (10), the R^8 becomes a Lie algebra.

Set:

$$\delta: G_1\to R^8,\quad \sum_{i=1,i\neq 5}^{8} a_i f_i \to \left(a_1,a_2,a_3,a_4,0,a_6,a_7,a_8\right)^T$$

Then, we can show that δ is an isomorphism between the Lie algebras G_1 and R^8. Thus, in the Lie algebra R^8, the following Lax matrices:

$$U=f_2\left(1\right)+(r-q)f_2\left(0\right)+qf_3\left(0\right)+rf_4\left(0\right)+u_1f_6\left(0\right)+u_2f_7\left(0\right)+u_3f_8\left(0\right) \tag{11}$$

and

$$V=V_1f_1\left(0\right)+V_2f_3\left(0\right)+V_3f_4\left(0\right)+V_4f_2\left(0\right)+\sum_{i=6}^{8}V_if_i\left(0\right) \tag{12}$$

can be written:

$$U=(0,\lambda+r-q,q,r,0,u_1,u_2,u_3)^T,\quad V=(V_1,V_4,V_2,V_3,0,V_6,V_7,V_8)^T \tag{13}$$

Based on the results in [10, 11], $M_1(b)$ satisfies the following matrix equation:

$$M_1(b)F_1=-\left[M_1(b)F_1\right]^T,\ F_1^T=F_1 \tag{14}$$

where $F_1=(f_{ij})_{8\times 8}$ is a matrix with constant terms. Using the mathematical software MAPLE, we can obtain the solution of eq. (14) as:

$$F_1=\begin{pmatrix}A & B\\ B & C\end{pmatrix},\quad A=\begin{pmatrix}\eta_1 & 0 & 0 & 0\\ 0 & \eta_1 & 0 & 0\\ 0 & 0 & 0 & \eta_1\\ 0 & 0 & \eta_1 & 0\end{pmatrix}$$

$$B=\begin{pmatrix}0 & \eta_2 & 0 & 0\\ 0 & -\eta_2 & 0 & 0\\ 0 & 0 & \eta_2 & -\eta_2\\ 0 & 0 & \eta_2 & \eta_2\end{pmatrix},\quad C=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 2\eta_2 & 0 & 0\\ 0 & 0 & 2\eta_2 & 0\\ 0 & 0 & 0 & -2\eta_2\end{pmatrix}$$

where η_1 and η_2 are different constants.

In terms of F_1, we can construct the linear functional as:

$$\left\{a,b\right\}=a^TF_1b\quad\text{and}\quad a=(a_1,\cdots,a_8)^T,\ b=(b_1,\cdots,b_8)^T \tag{15}$$

Making use of eqs. (13) and (15), we have:

$$\frac{\delta}{\delta u}\int^{x}[(\eta_1 V_4-\eta_2 V_6)]\mathrm{d}x=\left(\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\right)\begin{pmatrix}\eta_1(V_1+V_3)+\eta_2(V_6+V_7-V_8)\\ \eta_1(V_2-V_1)+\eta_2(-V_6+V_7+V_8)\\ \eta_2(2V_1+2V_6)\\ \eta_2(V_2+V_3+2V_7)\\ \eta_2(-V_2+V_3-2V_8)\end{pmatrix}$$

By comparing the coefficients of λ^{-n-1}, it follows that:

$$\frac{\delta}{\delta u}\int^{x}[(\eta_1 V_{4,n+1}-\eta_2 V_{6,n+1})]\mathrm{d}x=(-n+\gamma)\begin{pmatrix}\eta_1(V_{1n}+V_{3n})+\eta_2(V_{6n}+V_{7n}-V_{8n})\\ \eta_1(V_{2n}-V_{1n})+\eta_2(-V_{6n}+V_{7n}+V_{8n})\\ 2\eta_2(V_{1n}+V_{6n})\\ \eta_2(V_{2n}+V_{3n}+2V_{7n})\\ \eta_2(-V_{2n}+V_{3n}-2V_{8n})\end{pmatrix}\equiv(-n+\gamma)P_n$$

Thus, the system, expressed by:

$$\begin{aligned}
&q_t=2\alpha q_{xx}-2\alpha(q^2-2qr)_x,\, r_t=-2\alpha r_{xx}-2\alpha(2qr-r^2)_x\\
&u_{1,t}=-2\alpha(qu_3-ru_3-ru_2-qu_2-u_2^2)_x\\
&u_{2,t}=2\alpha u_{3,xx}-2\alpha(ru_2-qu_2+2u_1u_2-ru_1+qu_1)_x-2\alpha(q-r+2u_1)(u_{2,x}-qu_3+ru_3-\\
&\quad -2u_1u_2-ru_1-qu_1)_x+2\alpha(u_1+u_3)(r_x+qr-r^2)-2\alpha(u_1-u_3)(q_x-q^2+qr)_x\\
&u_{3,t}=2\alpha u_{2,xx}-2\alpha(u_3q-u_3r+2u_1u_3-ru_1-qu_1)_x+2\alpha(q-r-2u_1)(u_{3,x}-u_2r+u_2q+\\
&\quad +2u_1u_2+ru_1-qu_1)-2\alpha(u_1-u_2)(r_x+qr-r^2)-2\alpha(u_2-u_1)(q_x-q^2+qr)
\end{aligned}\tag{16}$$

can be written:

$$u_t=\begin{pmatrix}q & r & u_1 & u_2 & u_3\end{pmatrix}_t^T=\tilde{J}P_n=\tilde{J}\frac{\delta H}{\delta u}\tag{17}$$

where

$$N_1=\frac{1}{\eta_1}+\frac{2}{2\eta_2-2\eta_1}+\frac{1-\eta_2}{\eta_1(2\eta_2-2\eta_1)}-\frac{(1+\eta_2)\eta_2}{\eta_1(2\eta_2-2\eta_1)^2},\quad N_2=N_1-\frac{2}{2\eta_2-2\eta_1}$$

$$N_3=\frac{1}{2\eta_2-2\eta_1}+\frac{1+\eta_2}{(2\eta_2-2\eta_1)^2},\quad M_1=\frac{1}{\eta_2-\eta_1}\left(\frac{\partial}{2}+2u_1+u_3+q-r\right),\quad M_2=\frac{\partial}{2\eta_2}-M_1$$

$$Q_1=\frac{\eta_1^2}{2\eta_2^2(\eta_1-1)}(r-q+2u_1),\quad Q_2=\frac{\eta_1}{2\eta_2(\eta_1-1)}(q-r-2u_1)$$

$$H_n=\int^{x}\left(\frac{-\eta_1V_{4,n+1}+\eta_2V_{6,n+1}}{n}\right)\mathrm{d}x$$

Obviously, $\tilde{J}$ is not the Hamiltonian. It is remarkable that some constrained relations exist between η_i and the potential functions q, r, u_i. Using the Lie algebra G_2, we have:

$$\left[a,b\right]=a^T M_2(b)\tag{18}$$

where

$$a^T=(a_1,\cdots,a_7)^T,\quad M_2(b)=\begin{pmatrix} n_1 & n_2\\ 0 & n_3\end{pmatrix}$$

with

$$n_1=\begin{pmatrix} 0 & 0 & b_3 & -b_4\\ 0 & 0 & -b_3 & b_4\\ b_4 & -b_4 & b_2-b_1 & 0\\ -b_3 & b_3 & 0 & b_1-b_2\end{pmatrix},\quad n_2=\begin{pmatrix} 0 & b_7 & b_6\\ 0 & -b_7 & b_6\\ -b_7+b_6 & -b_5 & -b_5\\ -b_6+b_7 & b_5 & -b_5\end{pmatrix}$$

$$n_3=\begin{pmatrix} 0 & b_3-b_4+2b7 & b_3+b_4+2b_6\\ b_4-b_3-2b_7 & 0 & b_2-b_1-2b_5\\ b_3+b_4+2b_6 & b_2-b_1-2b_5 & 0\end{pmatrix}$$

Solving the following matrix equation:

$$M_2(b)F_2=-\left[M_2(b)F_2\right]^T,\ F_2^{\ T}=F_2$$

yields

$$F_2=\begin{pmatrix} D_1 & D_2\\ D_3 & D_4\end{pmatrix},\quad D_1=\begin{pmatrix} \eta_1 & 0 & 0 & 0\\ 0 & \eta_1 & 0 & 0\\ 0 & 0 & 0 & \eta_1\\ 0 & 0 & \eta_1 & 0\end{pmatrix},\quad D_2=\begin{pmatrix} \eta_2 & 0 & 0\\ -\eta_2 & 0 & 0\\ 0 & \eta_2 & -\eta_2\\ 0 & \eta_2 & \eta_2\end{pmatrix}$$

$$D_3=\begin{pmatrix} \eta_2 & -\eta_2 & 0 & 0\\ 0 & 0 & \eta_2 & \eta_2\\ 0 & 0 & -\eta_2 & \eta_2\end{pmatrix},\quad D_4=\begin{pmatrix} 2\eta_2 & 0 & 0\\ 0 & -2\eta_2 & 0\\ 0 & 0 & -2\eta_2\end{pmatrix}$$

Let us consider the linear functional:

$$\{a,b\}=a^TF_2b \tag{19}$$

In the Lie algebra R^7:

$$U=g_2(1)+(r-q)g_2(0)+qg_3(0)+rg_4(0)+s_1g_6(0)+s_2g_7(0) \tag{20}$$

and

$$V=V_1g_1(0)+V_2g_3(0)+V_3g_4(0)+V_4g_2(0)+V_5g_5(0)+V_6g_6(0)+V_7g_7(0) \tag{21}$$

can be written:

$$U=(0,\lambda+r-q,q,r,0,s_1,s_2)^T,\ V=(V_1,V_4,V_2,V_3,V_5,V_6,V_7)^T$$

Similarly, we can also derive a quasi-Hamiltonian structure of eq. (8), which would not be detailed here.

An expanding model of the DS hierarchy

In this section, we only make use of the loop algebra $\tilde{G}_1$ to discuss the expanding model of the DS hierarchy. Set:

$$U=\begin{bmatrix}(\lambda+\xi)h_4+qh_2+rh_3 & u_2e_1+u_3e_2\\ 0 & (\lambda+\xi)h_4+qh_2+rh_3+u_2e_1+u_3e_2\end{bmatrix} \tag{22}$$

and

$$\bar{V}=\begin{pmatrix}\sum_{i=1}^{4}V_ih_i & V_6h+V_7e_1+V_8e_2\\ 0 & \sum_{i=1}^{4}V_ih_i+V_6h+V_7e_1+V_8e_2\end{pmatrix}=\begin{pmatrix}V_1 & V_2 & V_6 & V_7+V_8\\ V_3 & V_4 & V_7-V_8 & -V_6\\ 0 & 0 & V_1+V_6 & V_2+V_7+V_8\\ 0 & 0 & V_3+V_7-V_8 & V_4-V_6\end{pmatrix} \tag{23}$$

Since the potential functions are not interchangeable each other, eq. (23) is changed to the following form:

$$V=\begin{pmatrix}V_1 & V_2 & V_6 & \tilde{V}_7\\ V_3 & V_4 & \tilde{V}_8 & \bar{V}_6\\ 0 & 0 & V_1+V_6 & V_2+\tilde{V}_7\\ 0 & 0 & V_3+\tilde{V}_8 & V_4+\bar{V}_6\end{pmatrix} \tag{24}$$

where

$$V_i=\sum_{m\ge 0}V_{im}\lambda^{-m},\ \ i=1,2,3,4,\ \ \bar{V}_6=\sum_{m\ge 0}\bar{V}_{6m}\lambda^{-m},\ \ \tilde{V}_j=\sum_{m\ge 0}V_j\lambda^{-m},\ \ j=7,8$$

According to the TAH scheme, solving the matrix-operator equation:

$$V_x=[U,V]$$

yields that

$$\begin{gathered}
V_{1m,x}=qV_{3m}-V_{2m}r, V_{2,m+1}=-V_{2m,x}+qV_{4m}-V_{1m}q-V_{2m}\xi\\
V_{3,m+1}=V_{3m,x}-rV_{1m}-V_{3m}\xi-V_{3m,y}+V_{4m}r\\
V_{4m,x}=rV_{2m}+\xi V_{4m}-V_{3m}q-V_{4m}\xi=rV_{2m}-V_{3m}q+V_{4m,y}\\
V_{6m,x}=(q+u_2+u_3)\tilde{V}_{8m}+\tilde{V}_{7m}(r+u_2-u_3)+(u_2+u_3)V_{3m}-V_{2m}(u_2-u_3)\\
\bar{V}_{6m,x}=-\tilde{V}_{8m}(q+u_2+u_3)+(r+u_2-u_3)\tilde{V}_{7m}+V_{6m,y}-V_{3m}(u_2+u_3)+(u_2-u_3)V_{2m}\\
\tilde{V}_{7,m+1}=-\tilde{V}_{7m,x}+(q+u_2+u_3)\bar{V}_{6m}-V_{6m}(q+u_2+u_3)-\\
-\tilde{V}_{7m}\xi+(u_2+u_3)V_{4m}-V_{1m}(u_2+u_3)\\
\tilde{V}_{8,m+1}=\tilde{V}_{8m,x}-(r+u_2)V_{6m}-\bar{V}_{6m}(r+u_2-u_3)-\tilde{V}_{8m}\xi-\tilde{V}_{8m,y}+V_{4m}(u_2-u_3)-u_2V_{1m}
\end{gathered} \tag{25}$$

Let $V_{1,0}=-\xi^{-1},\ V_{4,0}=\xi^{-1},\ V_{2,0}=V_{3,0}=V_{4,0}=V_{6,0}=\bar{V}_{6,0}=\tilde{V}_{7,0}=\tilde{V}_{8,0}=0.$

Then, we have:

$$\begin{gathered}
V_{1,1}=\bar{V}_{1,1}\xi^{-2}+o(\xi^{-3}),\quad V_{2,1}=2q\xi^{-1}-q_y\xi^{-2}+q_{yy}\xi^{-3}+o(\xi^{-4})\\
V_{3,1}=2r\xi^{-1}-r_y\xi^{-2}+r_{yy}\xi^{-3}+o(\xi^{-4}),\quad V_{4,1}=\bar{V}_{4,1}\xi^{-2}+o\left(\xi^{-3}\right)\\
V_{6,1}=\tilde{V}_{6,1}\xi^{-1}+\tilde{(\tilde{V}_{6,1})}\xi^{-2}+o(\xi^{-3})\\
\tilde{V}_{7,1}=(2u_2+2u_3)\xi^{-1}-(u_2+u_3)_y\xi^{-2}+(u_{2,yy}+u_{3,yy})\xi^{-3}+o(\xi^{-4})\\
\tilde{V}_{8,1}=(2u_2-u_3)\xi^{-1}-(u_2-u_3)_y\xi^{-2}+(u_{2,yy}+u_{3,yy})\xi^{-3}+o(\xi^{-4})
\end{gathered}$$

where

$$\partial\bar{V}_{1,1}=(qr)_y,\partial_-\bar{V}_{4,1}=(rq)_y,\quad \partial_-=\partial_x-\partial_y,\quad \partial\tilde{V}_{6,1}=4u_2^2+4u_3u_2-4u_2u_3-qu_3+2u_2r-3u_3^2$$

$$\partial(\tilde{V}_{6,1})=-qu_{3,y}+3u_2u_{3,y}+qu_{2,y}-3u_2u_{2,y}-3u_2u_{2,y}-3u_2r_y-3u_3r_y-u_{2,y}r-$$

$$u_{2,y}u_2+u_{2,y}u_3-u_{3,y}r-u_{3,y}u_2+u_{3,y}u_3+q_yu_2-q_yu_3,\quad \bar{V}_{6,1}=\tilde{(}\tilde{V}_{6,1})\xi^{-1}+(\bar{V}_{6,1})\xi^{-2}+o(\xi^{-3})$$

$$\partial_-\tilde{(}\tilde{V}_{6,1})=-u_3q-u_3u_2-u_3^2$$

$$\partial_-(\bar{V}_{6,1})=u_2q_y+u_2u_{2,y}-u_3u_{2,y}+u_{2,y}q+u_{2,y}u_2+u_{2,y}u_3-u_{3,y}q-u_{3,y}u_2-u_{3,y}u_3+$$

$$+ru_{2,y}+ru_{3,y}+u_3u_{2,y}+r_yu_2+r_yu_3,\quad V_{2,2}=-2q+(q_y-2q_x)\xi^{-1}+$$

$$+(q_{xy}-q_{yy}+q\bar{V}_{4,1}-\bar{V}_{1,1}q)\xi^{-2}+o(\xi^{-3})$$

$$V_{3,2}=-2r+(2r_x-r_y)\xi^{-1}-(r_{xy}+2r_{yy}+r\bar{V}_{1,1}-\bar{V}_{4,1}r)\xi^{-2}+o(\xi^{-3})$$

$$\tilde{V}_{7,2}=-2u_2-2u_3+[2u_{2,x}+2u_{3,x}+u_{2,y}+u_{3,y}+(q+u_2+u_3)\tilde{(}\bar{V}_{6,1})-$$

$$-\tilde{V}_{6,1}(q+u_2+u_3)]\xi^{-1}+[-u_{2,yy}-u_{3,yy}+(q+u_2+u_3)(\bar{V}_{6,1})-\tilde{(}\tilde{V}_{6,1})(q+u_2+u_3)+$$

$$+(u_2+u_3)\bar{V}_{4,1}-\bar{V}_{1,1}(u_2+u_3)]\xi^{-2}+o(\xi^{-3})$$

$$\tilde{V}_{8,2}=u_3-2u_2+[2u_{2,x}-u_{3,x}-2u_{2,x}+2u_{3,x}-(r+u_2)\tilde{V}_{6,1}-\tilde{(}\bar{V}_{6,1})(r+u_2-u_3)]\xi^{-1}$$

$$+[u_{3,xy}-u_{2,xy}-(r+u_2)\tilde{(}\bar{V}_{6,1})+\tilde{(}\bar{V}_{6,1})(r_y+u_{2,y}-u_{3,y})-(\bar{V}_{6,1})(r+u_2-u_3)+$$

$$+\bar{V}_{4,1}(u_2-u_3)-u_2\bar{V}_{1,1}]\xi^{-2}+o(\xi^{-3})$$

According to the TAH scheme and the integrable hierarchy, we obtain the (2+1)-D hierarchy of the evolution equations as follows:

$$u_t=\begin{pmatrix}q\\r\\u_1\\u_2\end{pmatrix}_t=\begin{pmatrix}-R(V_{2,n+1})\\R(V_{3,n+1})\\-R(\tilde{V}_{8,n+1})\\-R(\tilde{V}_{7,n+1})\end{pmatrix}=J\begin{pmatrix}R(V_{3,n+1})\\R(V_{2,n+1})\\R(\tilde{V}_{7,n+1})\\-R(\tilde{V}_{8,n+1})\end{pmatrix}. \tag{26}$$

When $u_2=u_3=0$, eq. (26) reduces to the DS hierarchy. According to the theory of integrable couplings, eq. (26) can be regarded as an expanding model of the DS hierarchy.

The Hamiltonian structure of the DS hierarchy obtained by the trace identity in eq. (7) has been presented in [8]. However, eq. (7) cannot be used to derive the Hamiltonian structure of the (2+1)-dimensional expanding model shown in eq. (27), which would be the focus in our next research.

Conclusion

In this work, we mainly discussed the reduction of the integrable couplings of the Levi hierarchy and an expanding model of the (2+1)-D DS hierarchy. Under some constraints, the quasi-Hamiltonian structure of the integrable couplings of Levi hierarchy was obtained by deducing their Hamiltonian structures firstly. Under the frame of the TAH scheme, a (2+1)-D expanding integrable model of the DS hierarchy was generated by the proper Lie algebras constructed by ourselves. The result may be used in the nondifferentiable system [1-4].

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