THERMALLY INDUCED VIBRATION SUPPRESSION IN A THERMOELASTIC BEAM STRUCTURE

by

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In this paper, the problem of thermally induced vibration suppression in a thermoelastic beam is studied. Physical equivalent of the present problem is that a thermoelastic beam is suddenly entering into daylight zone and vibrations are induced due to heating on the upper surface of the beam or thermoelastic beam in a spacecraft enters to intensive sunlight area just after leaving a shadow of a planet. Thermally induced vibrations are suppressed by means of minimum using of control forces to be applied to dynamic space actuators. Objective functional of the problem is chosen as a modified quadratical functional of the kinetic energy of the thermoelastic beam. Necessary optimality condition to be satisfied by an optimal control force is derived in the form of maximum principle, which converts the optimal vibration suppression problem to solving a system of distributed parameters system linked by initial-boundary-terminal conditions. Solution of the system is achieved via MATLAB© and simulated results reveal that thermally induced vibration suppression by means of dynamic space actuators are very effective and robust.

Key words: *thermal moment, maximum principle, vibration, thermoelastic*

Introduction

Due to changes in the temperature of a thermoelastic beam, thermal stress is arisen and thermal stress causes the vibrations in the beam. Periodical changes in the temperature of the thermoelastic beams provoke the transverse vibrations. These thermally induced vibrations are need to be suppress for improving the durability, safety and performance of the structure. Especially after starting the space adventure of mankind, the need and demand to structures having more durable and persistence have increased and damping out of thermally induced vibrations has great attention. Especially over the thirty years, number of researchers have considered the vibrations in the thermoelastic structures. Yu [1] studied viscoelastic damping and a viscous fluid damper on the stability with stability criteria of thermally induced vibrations in a space craft booms, and also considered the problem of thermal flutter of a flexible spacecraft boom. Boley [2] derived a general formula for the ratio of the maximum dynamic to static deflection of heated beams and plates. In [3], the effects of damping and axial compressive forces on the structural response are studied. In [4, 5], thermoelastic damping effect is considered for an isotropic and homogeneous Bernoulli-Euler beam undergoing flexural vibration. In [6], a solution to the problem of thermally induced vibration of a uniform, simply supported beam is presented and effect of internal damping is taken into account. Some important studies related

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to thermoelastic beams are shortly listed as follows, but not limited to [7-16]. Particularly, in this paper, thermally induced vibration suppression in a thermoelastic beam is considered. Vibration suppression is managed by application of control force to dynamic point actuators. The existence and uniqueness of solution to the system and controllability properties of the system is discussed. Optimal control problem is introduced and for obtaining the maximum principle which is a necessary condition for the optimal control force function, adjoint system is defined. By means of maximum principle, vibration suppress problem is transformed to the solving a system of PDE including state and adjoint variables, which are subjected to initial-boundary-terminal conditions. Solution of this system is obtained by means of MATLAB and a numerical example is presented for indicating the robustness and efficiency of the introduced vibration suppression actuation. Numerical results reveals that thermally induced vibrations are damped out effectively by means of minimum using of optimal control force applied to dynamic point actuators.

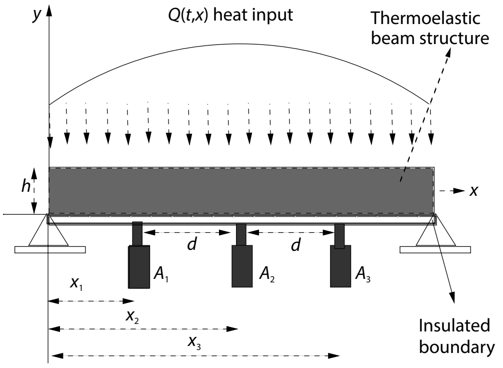

Figure 1. Thermally induced vibration suppression via point actuators A_1, A_3, A_3

Definition of the problem

Consider a simply supported thermo-elastic beam with length, ℓ, and thickness, h, in the x- and y-directions, respectively. The beam is exposed to effects of a transient heat input, $Q(t, x)$, along its length and heat input, Q, creates a temperature distribution, $T(t, x, y)$, in the beam structure. Due to existence of $T(t, x, y)$, the thermal moment, M, is defined:

$$M(t) = \int_0^{\ell} \int_{-h/2}^{h/2} \alpha ETy \mathrm{d}y \mathrm{d}x \tag{1}$$

in which E is the Young's modulus and α – the coefficient of thermal expansion. The heat input $Q(t, x)$ is considered as $Q(t)$ along x and applied for $t > 0$ only over the edge $y = h/2$. The edge $y = -h/2$ is isolated:

$$Q(t) = \begin{cases} Q(t), & y = \dfrac{h}{2} \\ 0, & y = \dfrac{-h}{2} \end{cases} \tag{2}$$

The temperature distribution function, $T(t, x, y)$, is independent of x under the step heat input $Q(t)$ and $T(t, y)$ is denoted for a thermoelastic beam structure:

$$T(y,t) = \frac{hQ(t)}{k} \left\{ \frac{\kappa t}{h^2} + \frac{1}{2}\left(\frac{y}{h} + \frac{1}{2}\right)^2 - \frac{1}{6} - \frac{2}{\pi^2} - \sum_{j=1}^{\infty} \frac{(-1)^j \mathrm{e}^{-j^2\pi^2 \frac{\kappa t}{h^2}}}{j^2} cos(j\pi)\left(\frac{y}{h} + \frac{1}{2}\right) \right\} \tag{3}$$

where k and κ are the coefficients of thermal conductivity and thermal diffusivity, respectively. Substituting $T(y, t)$ given by eq. (3), into eq. (1), one obtains thermal moment function:

$$M(t) = \frac{48EIQ(t)\alpha}{\pi^4 k} \left[\frac{\pi^4}{96} - \sum_{j=1,3,...}^{\infty} \frac{\mathrm{e}^{-j^2\pi^2 \frac{\kappa t}{h^2}}}{j^4} \right] \tag{4}$$

For more details about the thermal moment function, please see [17, 18]. Equation of the vibration motion in the thermoelastic beam is given [19]:

$$EI\left[\vartheta_{xxxx} + f\vartheta_{txxxx}\right] + c_d\vartheta_t + \rho\vartheta_{tt} = \sum_{i=1}^{N}\mathcal{F}_i\left(t\right)\delta\left(x - x_i\right), \quad i = 1,..,N \tag{5}$$

where ϑ is transverse beam deflection in:

$$\Omega = \left\{\left(t, x\right), t \in \left(0, t_f\right), x \in \left(0, \ell\right)\right\}$$

where t is time variable, t_f – the predetermined final time of control duration, x – the space variable, ℓ – the length of the beam, EI – the bending rigidity, ρ – the mass per unit length of the beam, f, c_d – the internal and external damping coefficients of the beam material, respectively. The control forces on the beam $\mathcal{F}_i$, $i = 1,..., N$ are exercised by means of N equally spaced actuators at locations $x_i = x_1\ (i – 1)d$, where d is the fixed spacing between actuators and $\delta(x – x_i)$ is Dirac-delta function in which x_i – the location of i^{th} actuator. eq. (5) is subjected to following initial condition:

$$\vartheta\left(t, x\right) = \vartheta_0\left(x\right), \quad \vartheta_t\left(t, x\right) = \vartheta_1\left(x\right) \quad \text{at} \quad t = 0 \tag{6}$$

and boundary thermal moment condition as:

$$\vartheta\left(t, 0\right) = 0, \quad \vartheta\left(t, \ell\right) = 0, \quad EI\vartheta_{xx}\left(t, 0\right) = -M\left(t\right), \quad EI\vartheta_{xx}\left(t, \ell\right) = -M\left(t\right) \tag{7}$$

Let the solution ϑ to the system, given by eqs. (5)-(7), satisfies:

$$\vartheta, \frac{\partial^{i+j}\vartheta}{\partial t^j \partial x^i} \in L^2\left(\Omega\right), \quad j = 0,1,2, \quad i = 0,1,...,4 \tag{8a}$$

$$\vartheta_0\left(x\right) \in H^1\left(0, \ell\right) = \left\{\vartheta_0\left(x\right) \in L^2\left(0, \ell\right) : \frac{\partial\vartheta_0\left(x\right)}{\partial x} \in L^2\left(0, \ell\right)\right\}, \quad \vartheta_1\left(x\right) \in L^2\left(0, \ell\right) \tag{8b}$$

where $L^2\Omega$ means to square-integrable functions space in the manner of Hilbert in the domain Ω in the Lebesque sense with following norm and inner product:

$$\left\|\eta\right\|^2 = <\eta, \eta>, \quad <\eta, \rho>_\Omega = \int_\Omega \rho\eta d\Omega$$

With the assumptions eq. (8) and by means of Cauchy Kov alevskya theorem [20, 21], it is concluded that the system defined eqs. (5)-(7) has a unique solution. It means that system must has unique control function for unique solution. Then system is observable and controllable [20, 22].

Optimal control problem

In this study, thermally induced vibration suppression in a thermoelastic beam structure is aimed by means of minimum using of control forces exercised Via dynamics space actuators. Before stating the objective functional of the system, introduce the admissible control force function set:

$$\mathcal{F}_{ad} = \left\{\mathcal{F}_i\left(t\right) | \mathcal{F}_i \in L^2\left(\Omega\right), \quad i = 1,..., N, \quad \left|\mathcal{F}_i\left(t\right)\right| \le \mathcal{F}_0 < \infty, \quad \mathcal{F}_0 \quad \text{is a constant}\right\} \tag{9}$$

Then, the objective functional of the vibration suppression problem is stated:

$$\mathcal{J}\left[\mathcal{F}(t)\right]=\int_0^\ell\left[\mu_1\vartheta^2\left(t_f,x\right)+\mu_2\vartheta_t^2\left(t_f,x\right)\right]\mathrm{d}x+\mu_3\int_0^{t_f}\sum_{i=1}^N\mathcal{F}_i^2(t)\mathrm{d}t \tag{10}$$

in which $\mu_3 > 0$, $\mu_1 + \mu_2 \neq 0$ and μ_1, $\mu_2 \geq 0$, are weighted constants. In the eq. (10), first integral at the L.H.S. is modified kinetic energy of the beam structure introduced by a quadratical functional of deflection and velocity of a point in the thermo-elastic beam. Second integral on the L.H.S. in eq. (10) is the measure of the total applied control forces on the $(0, t_f)$. Then, optimal vibration suppression problem is stated:

$$\mathcal{J}\left[\mathcal{F}^\circ(t)\right]=\min_{\mathcal{F}_i\in\mathcal{F}_{ad}}\mathcal{J}\left(\mathcal{F}\right) \tag{11}$$

subject to the eqs. (5)-(7). In order to achieve the maximum principle for obtaining control forces optimally, let us introduce an adjoint variable $\upsilon \in L^*$, L^* is the dual to $L^2(\Omega)$ and has the same norm and inner product like in $L^2(\Omega)$. Adjoint system corresponding to eqs. (5)-(7) is expressed:

$$EI\left[\upsilon_{xxxx}-f\upsilon_{txxxx}\right]-c_d\upsilon_t+\rho\upsilon_{tt}=0 \tag{12}$$

and eq. (12) is subjected to following boundary and terminal conditions, respectively:

$$\upsilon(t,0)=0,\quad \upsilon(t,\ell)=0,\quad EI\upsilon_{xx}(t,0)=0,\quad EI\upsilon_{xx}(t,\ell)=0 \tag{13}$$

$$-2\mu_1\vartheta(t,x)=\rho\upsilon_t(t,x)-c_d\upsilon(t,x)-EIf\upsilon_{xxxx}(t,x),\quad 2\mu_2\vartheta_t(t,x)=\rho\upsilon(t,x),\quad t=t_f \tag{14}$$

The existence and uniqueness of the solutions to adjoint system defined by eqs. (12)-(14) is shown similar to eqs. (5)-(7). Maximum principle is derived a necessary condition for the optimal control function in terms of Hamiltonian functional. In case of some convexity assumptions, satisfied by eq. (10), maximum principle is also sufficient condition for optimal control function. Let us derive the maximum principle as follows;

Theorem 1. (maximum principle) The maximization problem is presented for each control forces$\mathcal{F}_i$, $i = 1,..., N$:

$$\text{If}\quad \mathcal{H}\left[t;\vartheta^\circ,\upsilon^\circ,\mathcal{F}_i^\circ(t)\right]=\max_{\mathcal{F}_i(t)\in\mathcal{F}_{ad}}\mathcal{H}\left[t;\vartheta,\upsilon,\mathcal{F}_i\right] \tag{15}$$

in which $\mathcal{F}_i^\circ$ is the optimal control force corresponding to optimal deflection ϑ° and Hamiltonian function is presented:

$$\mathcal{H}\left[t;\vartheta,\upsilon,\mathcal{F}_i\right]=-\mathcal{F}_i\upsilon(t,x_i)-\mu_3\mathcal{F}_i^2(t),\quad i=1,...,N \tag{16}$$

then

$$\mathcal{J}\left[\mathcal{F}_i^\circ(t)\right]\leq\mathcal{J}\left[\mathcal{F}_i(t)\right],\quad \forall\mathcal{F}_i\in\mathcal{F}_{ad} \tag{17}$$

Proof 1. Before starting to proof, let us define following operator and its adjoint operator, respectively:

$$\varphi(\Delta\vartheta)=EI\left[\Delta\vartheta_{xxxx}+f\Delta\vartheta_{txxxx}\right]+c_d\Delta\vartheta_t+\rho\Delta\vartheta_{tt} \tag{18}$$

$$\varphi^\star(\upsilon)=EI\left[\upsilon_{xxxx}-f\upsilon_{txxxx}\right]-c_d\upsilon_t+\rho\upsilon_{tt} \tag{19}$$

The deflections in the state variable and its derivatives with respect to the time variable are defined:

$$\Delta\vartheta = \vartheta - \vartheta^{\circ}, \quad \Delta\vartheta_t = \vartheta_t - \vartheta_t^{\circ}$$

The operator

$$\varphi(\Delta\vartheta) = \sum_{i=1}^{N} \Delta\mathcal{F}_i(t)\delta(x - x_i)$$

satisfies homogeneous boundary and initial conditions, respectively:

$$\Delta\vartheta(t,x) = \Delta EI\vartheta_{xx}(t,x) = 0 \quad \text{at} \quad x = 0, \ell \tag{20}$$

$$\Delta\vartheta(t,x) = \Delta\vartheta_t(t,x) = 0 \quad \text{at} \quad t = 0 \tag{21}$$

Now, consider the following functional:

$$\iint_{\Omega} \left\{ \upsilon\varphi(\Delta\vartheta) - \Delta\vartheta\varphi^*(\upsilon) \right\} \mathrm{d}\Omega = \iint_{\Omega} \upsilon \sum_{i=1}^{N} \Delta\mathcal{F}_i(t)\delta(x - x_i)\mathrm{d}\Omega \tag{22}$$

Employing the integration by parts to each term in eq. (22) and using eqs. (13) and (14), eq. (22) becomes:

$$\iint_{\Omega} \left\{ \upsilon\varphi(\Delta\vartheta) - \Delta\vartheta\varphi^*(\upsilon) \right\} \mathrm{d}\Omega = 2\int_0^{\ell} \left\{ \mu_1\vartheta(t_f,x)\Delta\vartheta(t_f,x) + \mu_2\vartheta_t(t_f,x)\Delta\vartheta_t(t_f,x) \right\} \mathrm{d}x =$$

$$= \iint_{\Omega} \upsilon \sum_{i=1}^{N} \Delta\mathcal{F}_i(t)\delta(x - x_i)\mathrm{d}\Omega \tag{23}$$

Evaluating the Dirac-delta function inside the above integral, one observes an equality:

$$\iint \upsilon \sum_{i=1}^{N} \Delta\mathcal{F}_i(t)\delta(x - x_i)\mathrm{d}\Omega = \int \sum_{i=1}^{N} \Delta\mathcal{F}_i(t)\upsilon(t,x_i)\mathrm{d}t \tag{24}$$

The difference of the objective functional is introduced:

$$\Delta\mathcal{J}\left[\mathcal{F}(t)\right] = \mathcal{J}\left[\mathcal{F}(t)\right] - \mathcal{J}\left[\mathcal{F}^{\circ}(t)\right] =$$

$$= \int_0^{\ell} \left\{ \mu_1\left[\vartheta^2(t_f,x) - \vartheta^{\circ^2}(t_f,x)\right] + \mu_2\left[\vartheta_t^2(t_f,x) - \vartheta_t^{\circ^2}(t_f,x)\right] \right\} \mathrm{d}x + \mu_3\int_0^{t_f} \sum_{i=1}^{N} \left[\mathcal{F}_i^2(t) - \mathcal{F}_i^{\circ^2}(t)\right] \mathrm{d}t \tag{25}$$

After expanding $\vartheta^2(t_f, x)$ and $\vartheta_t^2(t_f, x)$ in Taylor series around $\vartheta^{\circ 2}(t_f, x)$ and $\vartheta_t^{\circ 2}(t_f, x)$ and ignoring the positive small terms in the series:

$$\Delta\mathcal{J}\left[\mathcal{F}(t)\right] \geq \int_0^{t_f} \sum_{i=1}^{N} \Delta\mathcal{F}_i(t)\upsilon(t,x_i)\mathrm{d}t + \mu_3\int_0^{t_f} \sum_{i=1}^{N} \left[\mathcal{F}_i^2(t) - \mathcal{F}_i^{\circ^2(t)}\right] \mathrm{d}t \geq 0 \tag{26}$$

which leads to

$$\mathcal{F}_i(t)\upsilon(t,x_i) + \mu_3\mathcal{F}_i^2(t) \geq \mathcal{F}_i^{\circ}(t)\upsilon^{\circ}(t,x_i) + \mu_3\mathcal{F}_i^{\circ^2(t)} \tag{27}$$

that is

$$\mathcal{H}\left[t;\vartheta^{\circ},\upsilon^{\circ},\mathcal{F}_i^{\circ}\right] \geq \mathcal{H}\left[t;\vartheta,\upsilon,\mathcal{F}_i\right]$$

Hereby, we obtain:

$$\mathcal{J}\left[\mathcal{F}_i\right] \geq \mathcal{J}\left[\mathcal{F}_i^{\circ}\right], \quad \forall \mathcal{F}_i \in \mathcal{F}_{ad}$$

Therefore, the optimal control force function is given:

$$\mathcal{F}_i(t) = \frac{-\upsilon(t, x_i)}{2\mu_3} \tag{28}$$

Simulation results and discussions

In this section, in order to show the effectiveness and robustness of the introduced vibration suppression algorithm in a thermoelastic beam, obtained results are simulated by obtaining solution of following system of PDE via MATLAB:

$$EI\left[\vartheta_{xxxx} + f\vartheta_{txxxx}\right] + c_d\vartheta_t + \rho\vartheta_{tt} = \sum_{i=1}^{N} \mathcal{F}_i(t)\delta(x - x_i), \; \mathcal{F}_i(t) = \frac{-\upsilon(t, x_i)}{2\mu_3} \tag{29a}$$

$$\vartheta(t,0) = 0, \quad \vartheta(t,\ell) = 0, \quad EI\vartheta_{xx}(t,0) = -M(t) \quad EI\vartheta_{xx}(t,\ell) = -M(t) \tag{29b}$$

$$\vartheta(0,x) = \vartheta_0(x), \quad \vartheta_t(0,x) = \vartheta_1(x) \tag{29c}$$

$$EI\left[\upsilon_{xxxx} - f\upsilon_{txxxx}\right] - c_d\upsilon_t + \rho\upsilon_{tt} = 0 \tag{30a}$$

$$\upsilon(t,0) = 0, \quad \upsilon(t,\ell) = 0, \quad EI\upsilon_{xx}(t,0) = 0, \quad EI\upsilon_{xx}(t,\ell) = 0 \tag{30b}$$

$$-2\mu_1\vartheta(t_f, x) = \rho\upsilon_t(t_f, x) - c_d\upsilon(t_f, x) - EIf\upsilon_{xxxx}(t_f, x) \tag{30c}$$

$$2\mu_2\vartheta_t(t_f, x) = \rho\upsilon(t_f, x) \tag{30d}$$

Before discussing the numerical results in the graphics, take into account the optimal control force function defined by eq. (28), in which, it is easy to see that as the value of μ_3 is decreasing, the value of the control voltage function is increasing. As a conclusion of this, modified kinetic energy functional of the thermoelastic beam is minimized by means of minimum using of control forces to be applied the dynamic space actuators. In the numerical computations:

$$EI = 667\,\text{kgm}^2, \quad \rho = 3.12\,\text{kg/m}, \;\; \alpha = 7.2\cdot 10^{-6}\,\text{m}^2/s, \quad k = 20, \quad \kappa = 1$$

The heat input function is evaluated as $Q(t) = \exp(t)(1 + t)$ and number of point actuator is $N = 1$. Predetermined control terminal time is $t_f = 1$. The length of the beam is considered as $\ell = 1$ m. All selected parameters in this section are compatible with the [6]. Also, thermal moment function is taken into account for $j = 1$. The weighted coefficients $\mu_{1,2}$ is considered as 1 for the controlled and uncontrolled situations. Also, the weighted coefficient μ_3 in the control force function is 10^6 and 10^{-6} for the uncontrolled and controlled cases, respectively. All simulated results are obtained at $x = 0.5$ which is middle point of the thermoelastic beam structure. Also, in order the see the capacity of the control actuation, initial effects are considered different than zero:

$$\vartheta_0(x) = \sqrt{(2)}\sin(\pi x), \quad \vartheta_1(x) = \sqrt{(2)}\cos(\pi x)$$

Coefficients of internal and external damping effects, f_d, c_d are considered as 10^{-4}. In the figs. 2 and 3, Un/controlled displacements with/out internal and external damping effects are plotted and thermally induced vibrations are suppressed effectively. The kinetic energy functional of the thermoelastic beam is defined:

$$\mathcal{J}\left[\vartheta(t)\right]=\int_0^{\ell}\left[\vartheta^2\left(t_f,x\right)+\vartheta_t^2\left(t_f,x\right)\right]\mathrm{d}x \tag{31}$$

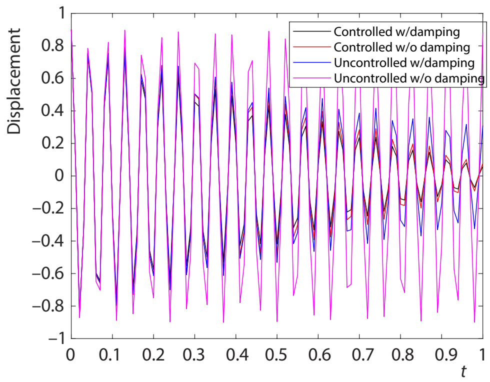

Figure 2. Un/controlled displacements with/out internal and external damping

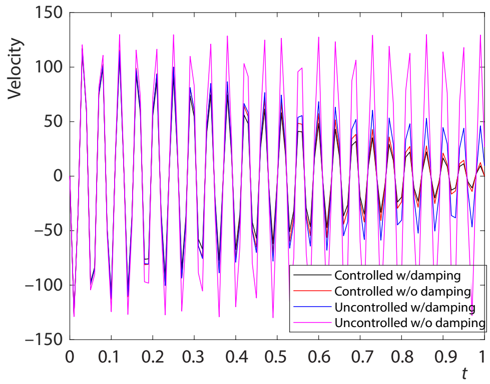

Figure 3. Un/controlled velocities with/out internal and external damping

The value of kinetic energy functional of the thermoelastic beam defined eq. (31) is computed 713.35 in the case of absent both control force and damping effects. After throwing the internal/external damping effects to the system and in case of absent the control force, the value of the kinetic energy functional of the structure is observed as 92.11. This decreasing (713.35-92.11) in the kinetic energy functional clearly shows the effects of the internal/external damping in the system. Also, it is obviously observed from figs. 2 and 3 that suppression in the displacements and velocities of the thermoelastic beam is very rapid in the presence of active damping effects. In the step third, internal and external damping effects are excluded from the system and control force to be applied the actuators are activated, then the value of the kinetic energy functional is measured as 0.006 and finally the kinetic energy functional of the system is calculated as 0.003 in the presence of both active control force and internal/external damping effects in the system. In the presence of internal/external damping effects, value of kinetic energy functional given by eq. (31) is decreased from 92.11-0.003 as a conclusion of successful control actuation. Obtained observations from figs. 2 and 3 reveal that introduced control algorithm is very effective and it has the capacity for applying the other kind of thermoelastic beams and plates.

Conclusion

In this study, thermally induced vibration suppression in a thermoelastic beam structure problem is solved by means of maximum principle, which is a necessary condition for the optimality and converts the vibration suppression problem to solving a system of PDE linked by initial-boundary-terminal conditions. By means of MATLAB, obtained theoretical results are simulated in the form of graphics and it reveals that introduced suppression actuation is very effective and has the application capacity for other kind of vibration suppression problems in the beams, plates and shells.

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