SEMI-DERIVATIVE INTEGRAL METHOD TO TRANSIENT HEAT CONDUCTION
Time-Dependent (Power-Law) Temperature Boundary Conditions

by

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Transient heat conduction in semi-infinite medium with a power-law time-dependent boundary conditions has been solved by an integral-balance integral method applying to a semi-derivative approach. Two versions of the integral-balance method have been applied: Goodman's method with a generalized parabolic profile and Zien's method with exponential (original and modified) profile.

Key words: *transient heat conduction, time-dependent boundary condition, integral-balance solution*

Introduction

The present work addresses approximate solutions of transient heat conduction in a semi-infinite medium with time-dependent boundary conditions by applying a synergism of the semi-derivative version of the diffusion equation and the integral-balance method. Two versions of the integral method are at issue :

– Goodman's approach [1, 2] with a parabolic profile (8) [3] and
– Zien's method [4, 5] with exponential profile (9).

The initial transformation of the governing diffusion (heat conduction) equation into a fractional one with a semi-derivative (with respect to the time), allows only one-step integration procedure to be applied, known as heat balance integral (HBIM) [1, 3] in contrast to the double integration method (DIM) [6-10]. This study applies the semi-derivative integral method (SDIM), proposed in [11], now with time-dependent temperature (power-law) as boundary condition.

In order to be correct in formulation of the background of the problem studied, intensively investigated for years [12], we mention some articles related to it such as [13-15] and the references therein, since all studies on this problem are hard to be encompassed and this is beyond the scope of this work.

Problem formulation

The problem of interest is 1-D transient heat conduction in a semi-infinite medium:

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2}, \;\; \theta(x,t) > 0, \;\; x > 0, \;\; t > 0 \tag{1}$$

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with time-dependent (power-law) temperature at the interface $x = 0$ and zero initial condition. The approximate solutions are developed by the integral-balance method in two versions, briefly outlined next.

Versions of the integral-balance method: single integration

In general, the integral method applies the concept of finite depth and a front of the solution moving with a finite speed, beyond which the medium is undisturbed. In this way the unphysical behaviour of the solution (infinite speed) of the parabolic model (1) is corrected.

Goodman's method

The integral-balance method of Goodman [1, 2] is based on integration of (1) with respect to the spatial co-ordinate x over a fixed distance δ denoting the position (front) of the heat penetrating the medium. Hence, this concept divides the medium into two zones: disturbed zone $0 \le x \le \delta(t)$ with $\theta(x, t) > 0$ and undisturbed zone $\delta(t) \le x < \infty$ with $\theta(x, t) = 0$. This approach replaces the classical boundary condition far away from the interface $x = 0$, *i.e.* $\theta(x \to \infty) = 0$ by $\theta(\delta) = 0$ and $(\partial\theta/\partial x)(\delta) = 0$, which are the well-known Goldman's boundary conditions [1-3]. The integration of eq. (1) with respect to the spatial co-ordinate over a finite penetration depth δ and applying the Leibniz rule, yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\delta}\theta(x,t)\,\mathrm{d}x = -a\frac{\partial\theta}{\partial x}(0,t) \tag{2}$$

The left part of eq. (2) is known as Heat-balance integral [1-3]. Then, replacement θ by an assumed profile θ_a expressed in terms of x/δ results in an ODE about $\delta(t)$. The principle problem emerging in application of the HBI method is the approximation of the gradient of right-side of eq. (2) because it should be defined through the assumed profile, which can be avoided by double integration approaches [6-10].

Zien's method

An alternative approach, consistent with both the integral method and the classical boundary conditions (relevant to a semi-infinite medium) $\theta(\infty) = 0$ and $\partial\theta/\partial x\,(\infty) = 0$ has been developed by Zien [4, 5]. Integrating both sides of eq. (1) with respect to the space co-ordinate from $x = 0$ to infinity we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\infty}\theta(x,t)\,\mathrm{d}x = -a\left(\frac{\partial\theta}{\partial x}\right)_{x=0} \tag{3}$$

Equation (3) is almost the same as eq. (2) of the Goodman's method with the difference in the upper terminal of the HBI. The HBIM defined by eq. (3) has been used by Zien [4, 5], together with a first-moment equation, to develop solutions with assumed exponential profile $T_a = T_s\mathrm{e}^{-x/\delta_Z}$ which goes to zero for $x \to \infty$ (theoretically correct [12], but practically indefinable).

Remark on differences and equivalence of both integration approaches

Both methods are physically equivalent because the Zien's integration can be presented:

$$\int_0^{\infty}(\bullet)\mathrm{d}x = \int_0^{\delta}(\bullet)\mathrm{d}x + \int_{\infty}^{\delta}(\bullet)\mathrm{d}x$$

The second integral in this sum is always zero in accordance with the concept of finite penetration depth. The Goodman's boundary conditions are more suitable for polynomial or generalized parabolic, see (8), profiles, while the Zien's conditions allow the assumed exponential profile to vanish at the upper bond of the integral. This profile is inapplicable with the Goodman's boundary conditions because at $x = \delta_Z$ we get $T_a/T_s = 1/e \neq 0$ and $\mathrm{d}T_a/\mathrm{d}x(\delta_Z) = -1\ \delta_Z e \neq 0$.

Semi-derivative integral method: The General approach

From integer-order diffusion equation a semi-derivative model

Equation (1) can be represented as [11, 16, 17] a product of two operators by means of a time-fractional Riemann-Liouville (RL) semi-derivative $\partial^{1/2}\theta/\partial t = D_t^{1/2}$, see eq. (5) [16]:

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2} = \left(\frac{\partial^{1/2}\theta}{\partial t} - \sqrt{a}\frac{\partial \theta}{\partial x}\right)\left(\frac{\partial^{1/2}\theta}{\partial t} + \sqrt{a}\frac{\partial \theta}{\partial x}\right) \tag{4}$$

$$\frac{\partial^{1/2}\theta}{\partial t} = \frac{1}{\Gamma(1/2)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^u \frac{\theta(x,t)}{\sqrt{t-u}}\mathrm{d}u - \frac{\theta(x,0)}{\sqrt{\pi t}} \tag{5}$$

In eq. (4) only the second term has a physical meaning [18, 19]. Hence, the time-fractional equivalent of eq. (1), see also the derivation in [18] and applications in [19]:

$$\frac{\partial^{1/2}\theta}{\partial t} = -\sqrt{a}\frac{\partial \theta}{\partial x} \underset{x=0}{\Rightarrow} \frac{\partial^{1/2}\theta}{\partial t} = -\sqrt{a}\frac{\partial \theta(0,t)}{\partial x} \tag{6}$$

where from eq. (6), we mention: the gradient at $x = 0$ is proportional to the semi-derivative of the surface temperature.

Single-integration method applied to semi-derivative equations

Now applying the single-integration from eq. (2) and eq. (6), as well from eq. (3) and eq. (6), we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta \theta(x,t)\mathrm{d}x = \sqrt{a}\frac{D^{1/2}\theta(0,t)}{\partial t} \tag{7}$$

Equation (7) is the principle SDIM single-integration equation relevant to both Goodman's approach and Zien's method. It is noteworthy, to stress the attention on the physically based condition for the front $\delta(t = 0)$ since no heat diffusion takes place for $t < 0$.

Remark on the semi-derivative approach to time-dependent boundary conditions

The semi-derivative approach allows easily to apply different time-dependent boundary conditions $T_s(t)$ and get a semi-derivative ($D_t^{1/2}$) of them. To some extent, this simple approach is equivalent to the DIM [9] as it was commented in *Section 5* of [11]. Its main advantage is that it allows the function (temperature) and its gradient to be related at any point, see the left eq. (6), through a convolution integral, and especially at the boundary $x = 0$ by the second relation in eq. (6).

Aim

The SDIM, conceived in [11], with fixed boundary conditions, will be applied with time-dependent temperature $b_0 t^{m/2}$, $m \geq 0$ as a boundary condition through two examples:

– *Example 1* with the Goodman version of integral method and generalized parabolic profile eq. (8) and
– *Example 2* with Zien's method with exponential profile (9) (original one and with a modified version conceived here).

Generalized parabolic profile satisfying the Goodman's boundary conditions [3]

$$T_a = T_s\left(1-\frac{x}{\delta}\right)^n \tag{8}$$

Zien's exponential profile [4, 5]

$$T_a = T_s \mathrm{e}^{-x/\delta_Z} \tag{9}$$

where $\delta_Z(t)$ plays a slightly modified role of a thermal penetration depth.

Solutions

With a boundary condition $b_0 t^{m/2}$, $m \geq 0$ at $x = 0$ the exact solution, *Section 2* in [12], expressed through the error function $\Phi(x)$:

$$T_e = b_0 \Gamma\left(\frac{m}{2}+1\right)(4t)^{m/2}\, i^m \Phi\left(\frac{x}{2\sqrt{at}}\right),\ \xi = \frac{x}{2\sqrt{at}} \tag{10}$$

The compact description of the last term can be explained:

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-\xi^2}\mathrm{d}\xi,\ \Phi^*(x) = 1-\Phi(x),\ i^m\Phi(x) = \int_x^\infty i^{m-1}\Phi^*(\xi)\mathrm{d}\xi,\ m = 2,3,4... \tag{11}$$

As it mentioned in [12], the solution of eq. (10), can be used with tabulated functions $i^m\Phi(x)$, tab. 1 of *Appendix 2* in [12], which are not always useful to handle in engineering calculations. However, we will use them for comparisons with developed approximate solutions.

Example 1: Goodman approach and a general parabolic profile

With the general parabolic profile (8) and applying the Goodman's boundary conditions we get a profile:

$$T_a(0,t) = T_s = b_0 t^{m/2} \Rightarrow T_a = b_0 t^{m/2}\left(1-\frac{x}{\delta}\right)^n,\ b_0\left[K\mathrm{s}^{-m/2}\right] \tag{12}$$

Now, with the approximate profile defined by eq. (12) and applying the SDIM eq. (7) we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(b_0 t^{m/2}\frac{\delta}{n+1}\right) = \sqrt{a}\left[b_0\frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)}t^{m/2-1/2}\right] \tag{13}$$

The integration of eq. (13) with the initial condition $\delta(t = 0) = 0$ yields:

$$\delta_G = \sqrt{at}\frac{2(n+1)}{(m+1)}G_m \Rightarrow \frac{\delta_G}{\sqrt{at}} = C^n_{m(T)} = \frac{2(n+1)}{(m+1)}G_m,\ G_m = \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)} \tag{14}$$

Therefore, the approximate solution:

$$T_{a(G)} = b_0 t^{m/2} \left\{ 1 - \frac{x}{\sqrt{at}\left[\dfrac{2(n+1)}{(m+1)} G_m\right]} \right\}^n \Rightarrow T_{a(G)} = b_0 t^{m/2} \left[1 - \frac{\eta}{2(n+1)} M_T \right]^n \tag{15}$$

where $\eta = x/(at)^{1/2}$ is the Boltzmann similarity variable and $M_T = (m+1)/G_m$

For $m = 0$ *i.e.* fixed temperature as boundary condition we get

$$G_{m=0} = 1/\Gamma(1/2) = 1/\sqrt{\pi} \text{ and } M_T = \sqrt{\pi}$$

Therefore

$$\delta_{G^n_{m=0}} = \sqrt{at}\,\frac{2(n+1)}{\sqrt{\pi}}$$

i.e. the solution obtained in [11], eq. (12a,b). In thecase the surface flux

$$q_G(x=0), (\lambda \text{ is the heat conductivity})$$

$$\text{is } q_G(x=0) = -\lambda \left(\frac{\partial T_{a(G)}}{\partial x} \right)_{x=0} = \lambda b_0 t^{(m-1)/2} \frac{M_T}{2(n+1) a^{1/2}}$$

Example 2: Zien's exponential profile

With the profile (9) and the principle SDIM eq. (7) we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty b_0 t^{m/2} \mathrm{e}^{-x/\delta_{zT}} \mathrm{d}x = \sqrt{a}\,\frac{D^{1/2}}{\partial t^{1/2}}\left(b_0 t^{m/2}\right) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(b_0 t^{m/2} \delta_{ZT}\right) = \sqrt{a} b_0 \frac{\Gamma\left(\dfrac{m}{2}+1\right)}{\Gamma\left(\dfrac{m}{2}+\dfrac{1}{2}\right)} t^{m/2-1/2} \tag{16}$$

The integration of the second equation in eq. (16) with the initial condition $\delta(t = 0)$ yields:

$$\delta_Z = \sqrt{at}\,\frac{2}{(m+1)} \frac{\Gamma\left(\dfrac{m}{2}+1\right)}{\Gamma\left(\dfrac{m}{2}+\dfrac{1}{2}\right)} \Rightarrow \frac{\delta_Z}{\sqrt{at}} = C_{m(T)} = \frac{2}{(m+1)} G_m,\; G_m = \frac{\Gamma\left(\dfrac{m}{2}+1\right)}{\Gamma\left(\dfrac{m}{2}+\dfrac{1}{2}\right)} \tag{17}$$

the approximate solution:

$$T_{a(Z)} = b_0 t^{m/2}\left(\mathrm{e}^{-\eta M_{ZT}}\right),\; \eta = \frac{x}{\sqrt{at}},\; M_Z = \frac{m+1}{2Gm} \tag{18}$$

In this case the surface flux is:

$$q_z(x=0) = -\lambda \left(\frac{\partial T_{a(z)}}{\partial x} \right)_{x=0} = -\lambda b_0 t^{(m-1)/2} \left(\frac{M_z}{\sqrt{a}} \right)$$

Solutions: Plots and comments

Solution obtained are shown in fig. 1. For the sake of simplicity in the case of the Goodman method a stipulated exponent $n = 2$ was used as in the classical approach [1-3]. In general, with increase in the value of the parameter m, that is, with increases in the rate of the surface heating, the penetration depth decreases which is obvious from the distribution of the temperature profiles. Despite these good and informative results the solutions need optimizations in order to reduce the errors of approximations. The problem with solution optimizations is discussed in the next section.

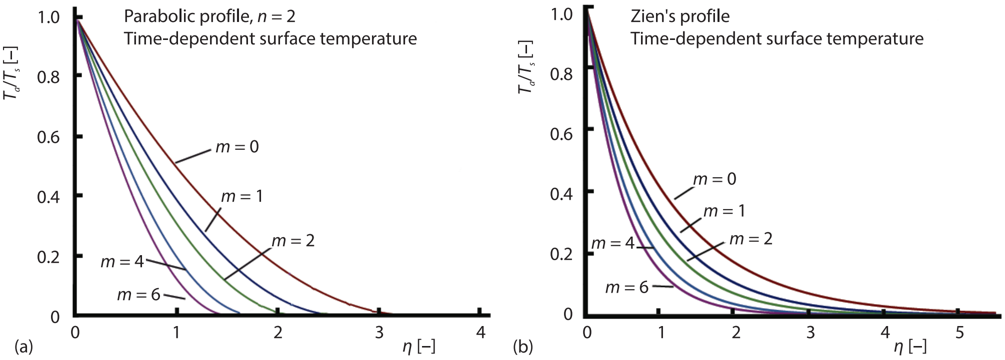

Figure 1. Normalized temperature profiles for various non-linear parameters *m* (different ramping programs of surface heating); (a) Goodman's method and parabolic profile with stipulated exponents *n* = 2 and (b) Zien's method and exponential profile; note: for the sake of simplicity, it is assumed $b_0 = 1$

Optimization of solutions

We have to remember that the obtained approximate solutions satisfy the integral relation (7) but not the initial model (1). Therefore, the residual function (19) should not equal zero, namely:

$$R = \frac{\partial T_a}{\partial t} - a\frac{\partial^2 T_a}{\partial x^2} \neq 0 \tag{19}$$

Since the approximate (assumed) profiles satisfy all boundary conditions, we have to look how to minimize the error of approximation over the entire thermal penetration depth. In this context, we have to mention, that in case of the generalized parabolic profile (8) the profile exponent can be varied in order to minimize the mean squared error of approximation. In case of the Zien's profile such approach is impossible, since there are no additional parameters allowing optimization. Despite this the present article will do a step ahead modifying this profile as it will be demonstrated in section *Zien's method (modified)*.

Goodman's method

Assumed generalized parabolic profile with $T_s = b_0 t^{m/2}$ as boundary condition yields $T_a = b_0 t^{m/2}(1 - x/\delta)^n$. By introduction of the dimensionless variable $0 \leq z = x/\delta \leq 1$ (transforming the moving boundary problem into a fixed boundary one) the residual function:

$$R_G = \frac{m}{2}(1-z)^n t^{-1}\delta^2 + n\left[z\delta\frac{\mathrm{d}\delta}{\mathrm{d}t}(1-z)^{n-1}\right] - a\left[n(n-1)(1-z)^{n-2}\right] \tag{20}$$

The products $t^{-1}\delta^2 = aG_m^2$ and $\delta(\mathrm{d}\delta/\mathrm{d}t) = 1/2(aG_m^2)$ in eq. (20) are time independent (it is easy to check this, so we skip these calculations). Thus, the residual function as a time-independent function of the variable z:

$$R_G = \frac{m}{2}(1-z)^n G_m^2 + nz\frac{1}{2}G_m^2(1-z)^{n-1} - n(n-1)(1-z)^{n-2} \tag{21}$$

Hence, we have the only task to minimize R_G with respect to n, thus finding its optimal value. It is obvious from eq. (21) that $R_G = 0$ for $z = 1$ ($x = \delta$) and the question arising now is: what is the situation at $z = 0$ (at the boundary $x = 0$)? Thus, setting $z = 0$ in eq. (21) we get $R_G(z = 0) = m/2(G^2_m) - n(n - 1)$. Setting, $R_G(z = 0) = 0$ we have an equation about n, namely: $n^2 - n - m/2(G^2_m) = 0$. The solutions of this equation ($n > 0$) are possible to be obtained analytically (or numerically) for all values of m and examples (corresponding to values of m used in the exact solutions of Carslaw and Jaeger [12]) are summarized in tab. 1 (first row). We especially added cases when $0 \le m \le 1$ to the situations studied in the preceding sections. Therefore, we address the mean squared error of approximation over the interval $0 \le x \le \delta$ in the original problem definition, and in terms of eq. (21) over the interval $0 \le z \le 1$. Precisely, we look for minimum of the functional $\int_0^1 (R_T)^2 \mathrm{d}z$ with respect to the variable (exponent) n. We skip the cumbersome expressions, since the method is described elsewhere [8-10]. The results obtained are summarized in tab. 1 (second row). The optimal values of n when $0 \le m \le 1$ are summarized in tab. 2.

Table 1. Values of the exponent *n* defined from the condition $R_G = 0$ (first row) and optimal values (second row) for $m \ge 1$

$m \ge 1$	1	2	3	4	5	6
$n(z = 0)$	1.336	1.618	1.822	1.919	2.158	2.302
$n_{(\text{optimal})}$	2.827	2.928	2.997	3.008	3.108	3.210

Table 2. Values of the exponent *n* defined from the condition $R_G = 0$ (first row) and optimal values (second row) for $0 \le m \le 1$

$m \le 1$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$n(z = 0)$	1.047	1.091	1.132	1.170	1.207	1.241	1.247	1.306	1.336
$n_{(\text{optimal})}$	1.784	1.784	1.784	1.784	1.784	1.784	1.784	1.784	1.784

Zien's method (modified)

With the Zien's exponential profile we make a modification by introducing a dimensionless factor β such that $T = T_s\exp[-\beta(x/\delta_z)]$, $\beta > 0$; for $\beta = 1$ we get the original Zien's profile. In this case the penetration depth is $\delta_z/(at)^{1/2} = C_{m(T)}\beta = [2/(m + 1)]G_m\beta$ and therefore, the residual function can be expressed:

$$R_{\beta Z} = \left(b_0 t^{m/2} \mathrm{e}^{-\beta \frac{x}{\delta_z}} \right) B_{\beta z}, \quad B_{\beta z} = a\left(\frac{\beta^2}{\delta_{\beta z}^2}\right)\frac{4}{(m+1)^2}\left[\frac{m}{2} - \beta\frac{z}{2} + \frac{(m+1)^2}{4G_m^2}\right] \tag{22}$$

The first (exponential term of eq. (22) goes to zero as $x \to \infty$ (recall the Zien's boundary condition) and to finite value when $x = \delta_z$ ($x/\delta_z = 1$). Therefore, from (22) it follows that the optimization should address the second term $B_{\beta z}$ which depends on $\delta_{\beta z}$ and the parameter β, taking into account:

$$\frac{\mathrm{d}\delta_{\beta z}}{\mathrm{d}t} = \beta\frac{a}{2\sqrt{t}}\left[\frac{2Gm}{m+1}\right], \quad \delta_{\beta z}^2 t^{-1} = aG_m^2\left[\frac{4}{(m+1)^2}\right], \quad \delta_{\beta z}\left(\frac{\mathrm{d}\delta_{\beta z}}{\mathrm{d}t}\right) = 2a\beta^2\left[\frac{G_m^2}{(m+1)^2}\right] \tag{23}$$

the last two terms in eq. (23) are time-independent and that $\beta^2/\delta_z^2 \equiv 1/t$.

The introduction of the dimensionless variable $z = x/\delta_{\beta z}$ as it was explained in the preceding section and bearing in mind that the mean squared error $R^2_{\beta z} \equiv 1/\delta_z^4$ will decay in time with a rate proportional to $1/t^2$, focuses on minimization of $B_{\beta z}$ with respect to β over the range $0 \le x/\delta_z \le 1$. The Zien's method imposes integration from zero to infinity but it is easy to check that such integration of squared term of (22) does not yields adequate results. First of all, let us see what happens at $z = 0$. At the boundary $z = 0$ we get $\beta(z = 0) = (m/2)^{1/2}[G_m/(m + 1)]$. This condition defines the lower boundary of values of the parameter β, see tab. 3 first row, since the general condition is $R_{\beta z} \ge 0$. Further, following the already demonstrated techniques the optimal values of β for $m < 1$ are summarized in tab. 4.

Table 3. Values of the factor β in the modified Zien's exponential profile: defined from the condition $R_{\beta z} = 0$ (lower limits) (first row) and optimal values (second row) for $m \ge 1$

$m \ge 1$	1	2	3	4	5	6
$\beta(z = 0)$	0.313	0.376	0.407	0.425	0.437	0.446
$\beta_{(\text{optimal})}$	0.495	0.505	0.515	0.525	0.530	0.535

Table 4. Values of the factor β in the modified Zien's exponential profile: defined from the condition $R_{\beta z} = 0$ (lower limits) (first row) and optimal values (second row) for m ≤ 1

$m \le 1$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\beta(z = 0)$	0.122	0.168	0.200	0.225	0.246	0.263	0.278	0.291	0.303
$\beta_{(\text{optimal})}$	0.392	0.482	0.537	0.582	0.618	0.648	0.668	0.693	0.713

Numerical experiments with optimized profiles and comparison exact solutions

Comparisons with the Carlaw and Jaeger tabulated solutions [12]

The points correspond to tabulated data of (4) $^{m/2}i^m\Phi[x/2(at)^{1/2}]$ in tab. 1 of *Appendix 2* in [12] multiplied by $\Gamma(m/2 + 1)$. Recall, the similarity variable in the approximate solution is $\eta = x/(at)^{1/2}$, so the factor 1/2 is taken into account when exact and approximate solutions are plotted together. It is noteworthy to mention that in order the tabulated values to be compared to the approximate solutions, the function of T_a was multiplied by a factor $1/K$, where $K = (1/k)(4^{m/2})\Gamma(m/2 + 1)$. The value of k depends on m in accordance with tab. 1 of *Appendix 2* in [12], namely: $k_{m=1} = 2$, $k_{m=2} = 4$, $k_{m=3} = 6$, $k_{m=4} = 8$, $k_{m=5} = 10$, $k_{m=6} = 12$. The alternative way is to multiply all tabulated data by K and here the first approach was chosen.

Goodman's method

The plots in figs. 2(a) and 2(b) are optimized solutions based on the generalized parabolic profile (8) and exact solutions, tabulated data from eq. (10), for various values of m. In general the optimized solutions work well for small values of η, but the differences between them and the exact solutions become significant close to the edge of the penetration depth which is inherent problem of the method [3, 15]. We have to take into account that the boundary conditions when $\eta \gg 1$ for the exact solution and the Goodman's method are different.

Zien's method (modified)

The plots in figs. 3(a) and 3(b) allow to compare of optimized solutions with the modified ($\beta \ne 1$) (and the original ($\beta = 1$) Zien's profiles (9), and exact solutions (10) for various values of m. The optimized solutions demonstrate very good approximations, almost matching

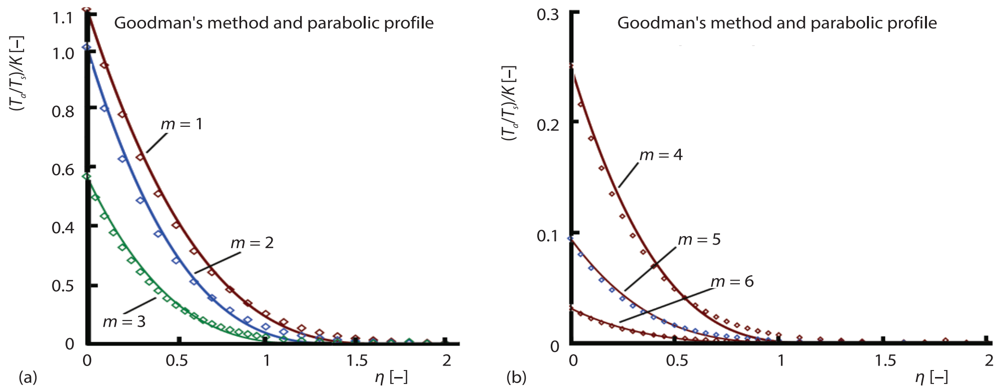

Figure 2. Normalized temperature profiles (Goodman's method and general parabolic profile with optimal exponents) for various parameters *m* (different ramping programs of surface heating) compared with tabulated data in [12] (the points); note: for the sake of simplicity, it is assumed $b_0 = 1$; the vertical scale depends on the data tabulated in [12]

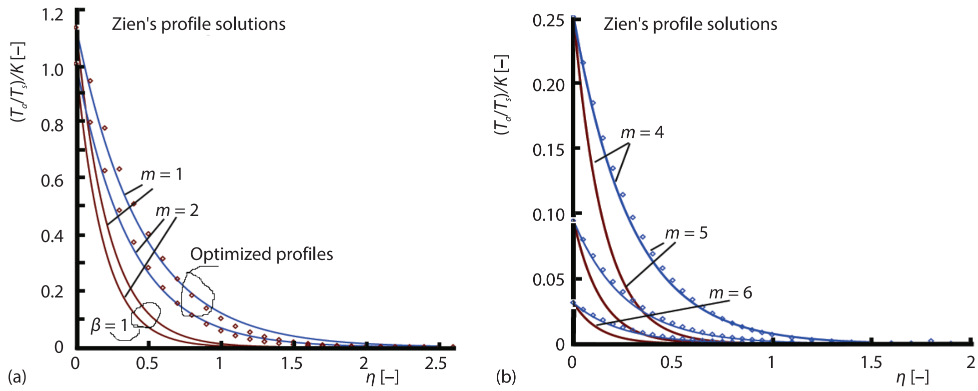

Figure 3. Normalized temperature profiles (Modified Zien's profile) for various parameters *m* (different ramping programs of surface heating) compared with tabulated data in [12] (the points); note: for the sake of simplicity, it is assumed $b_0 = 1$; the vertical scale depends on the data tabulated in [12]

the exact solutions. Detailed numerical experiments in different sections of the approximate solutions (*i.e.* different ranges of variations of η) indicate that the profile works very well at the edge of the solution, while for low η the approximation decreases, a behaviour which is just opposite to that exhibited by the generalized parabolic profile and the Goodman's solution. A numerical experiment with a value of the exponent β higher than the optimal one for a given m shows changes in the approximation behaviour: for low η (the accuracy decreases), while the solution approaches the exact one close to the front (see further the inset in the right panel of fig. 4(b) and the relevant comments).

Comparisons with the analytical solution of Sahin [14]

The Sahin solutions [14] are based on similarity transforms and expressed through the Kummer function [20]. Two cases of theses solutions can be compared for $m = 1$ and $m = 2$, namely:

$$T_{\mathrm{sah},m=1} = \exp\left(-\frac{\eta^2}{4}\right)\left[1+\sqrt{\pi}\left(\frac{\eta}{2}\right)\exp\left(\frac{\eta^2}{4}\right)\mathrm{erfc}\left(\frac{\eta}{2}\right)-\left(\frac{\eta}{2}\right)\sqrt{\pi}\exp\left(\frac{\eta^2}{4}\right)\right] \tag{24}$$

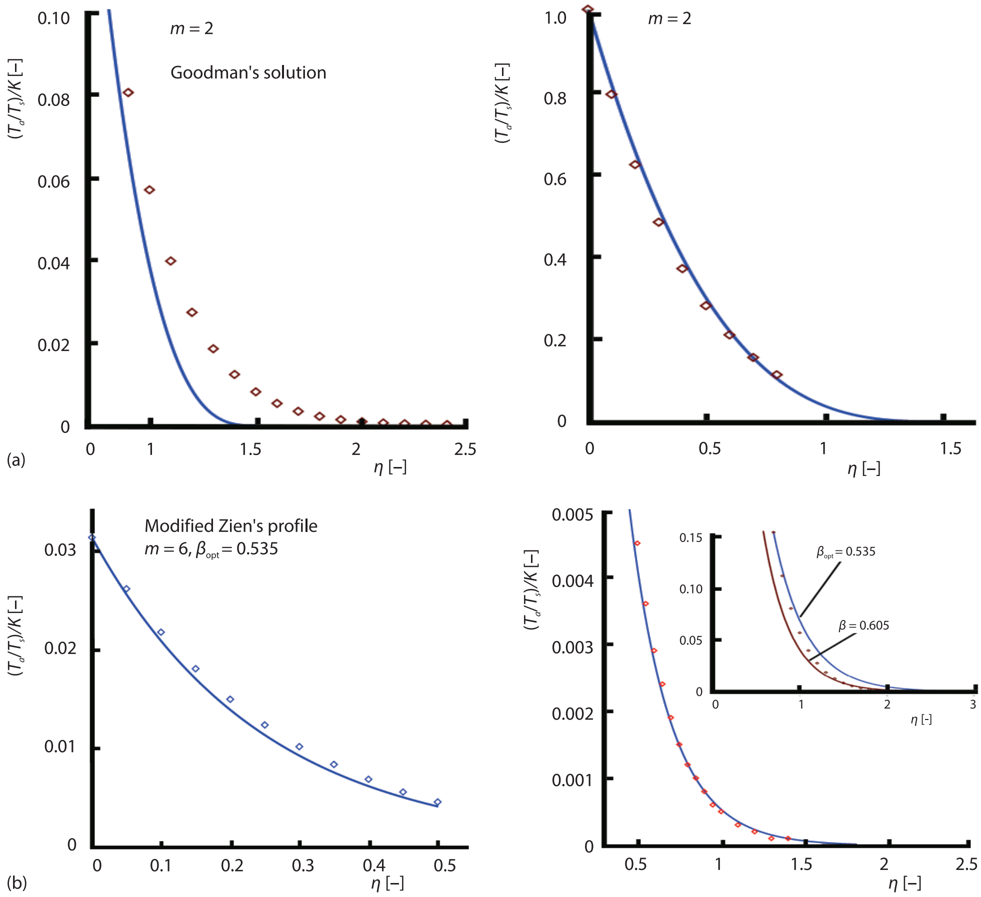

Figure 4. Normalized temperature profiles (as in figs. 2 and 3): sections corresponding to different values of the dimensionless variable *η* and demonstrating the approximation accuracy of the assumed profiles; (a) Goodman's method and generalized parabolic profile: a section close to the solution front (left panel); a section close to the interface *x* = 0 (*η* = 0) (right panel) and (b) modified Zien's method:a section close to the interface *x* = 0 (*η* = 0) (left panel) and a section for large values of *η* (where the Goodman's method does not work); inset: profiles with optimal parameter *β* (blue) and with a value of *β* higher than the optimal (the red line closer to the points)

$$T_{\text{sah},m=2} = \left[1+2\left(\frac{\eta^2}{4}\right)\right]\text{erfc}\left(\frac{\eta}{2}\right) - \frac{2}{\sqrt{\pi}}\left(\frac{\eta}{2}\right)\exp\left(-\frac{\eta^2}{4}\right), \quad \eta = \frac{x}{\sqrt{at}} \tag{25}$$

These solutions are compared with the approximate solutions developed here and the tabulated data of Carslaw and Jaeger [12], see fig. 5. Unfortunately, they are far away from either the exact (tabulated) results and the approximate solutions. It is important to mention that in [14] either comparisons with the solution of Carslaw and Jaeger [12] or graphical presentations are completely missing. The plot corresponding to the case m = 1 (square-root surface heating) is convex, that is a common curve for non-linear heat conduction equation when the diffusivity a is temperature dependent [8-10]; for constant thermal diffusivities the profiles are concave in shape [3, 15].

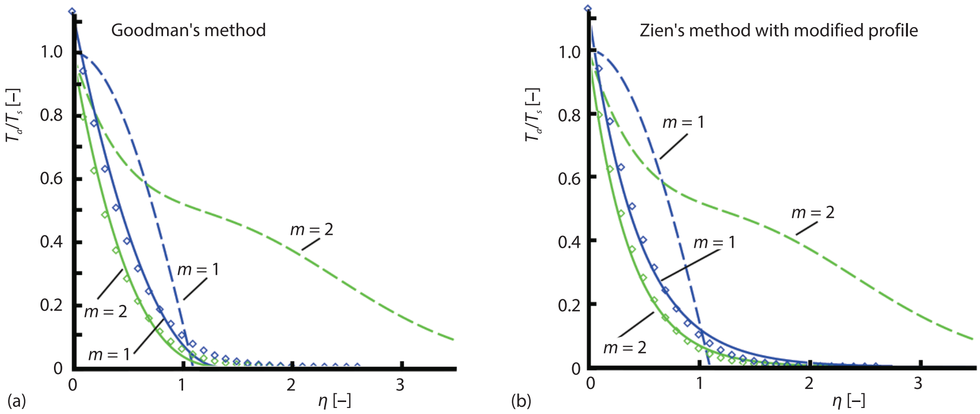

Figure 5. Comparison of the approximate solutions (solid lines) with the tabulated data from [12] (the points) and the analytical solutions of [14] (dashed lines)

Final comments and main results outline

The semi-derivative based approach applied is a further extension of the method developed in [11]. The solutions techniques allow reveal some main points, among them.

- The use of the semi-derivative model allows the function (temperature) and its gradient to be related at any point of the of the medium trough a half-time convolution integral. This is especially important at the interface $x = 0$. In this case only one step integration in the integral-balance method is needed.
- The Goodman's approach with a generalized parabolic profile as well as with the Zien's method with exponential profile provides similar results even though both assumed profiles require different boundary conditions far away from the interface $x = 0$. The optimizations of the approximate solutions allow comparing with tabulated solutions of Carslaw and Jaeger [12] and the results demonstrate the adequacy of the used approaches. In this context, the modification of the Zien's exponential profile (never done before) allows this profile to be more flexible in the solution optimizations.
- It is well-known that the Goodman's method provides good approximation for low values of the space variable x, or low values of the similarity variable $\eta = x/(at)^{1/2}$, while the biggest error occur close to the edge of the solution (the front of penetration depth). In contrast, the solutions with modified Zien's profile provide very good approximations for large $\eta = x/(at)^{1/2}$. These differences could be attributed to different boundary conditions in both methods.

The semi-derivative method explored in this work is applicable not only to problems with ramped power-law temperature boundary condition, but with a heat flux ramping in time as well as with other time of functional relations of the boundary condition (not oscillating).

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