# REPRODUCING KERNEL FUNCTIONS AND HOMOGENIZING TRANSFORMS

by

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*A lot of problems of the physical world can be modeled by non-linear ODE with their initial and boundary conditions. Especially higher order differential equations play a vital role in this process. The method for solution and its effectiveness are as important as the modelling. In this paper, on the basis of reproducing kernel theory, the reproducing kernel functions have been obtained for solving some non-linear higher order differential equations. Additionally, for each problem the homogenizing transforms have been obtained.*

Key words: *reproducing kernel functions, PDE, bounded linear operator, hilbert spaces*

## Introduction

In the universe, there are countless phonemes waiting to be understood. In order to understand these phonemes, it is necessary to model them first. At this point, non-linear ODE (NODE) arise as an indispensable tool for modelling. It appears in two form: initial value problem (IVP) and boundary value problem (BVP). In a big variety of fields such as engineering, biology, astronomy, fluid dynamics, economics, physics, electric circuits, control theory and so on, higher order BVP and IVP have an important role. So, this study will be focused on some kind of higher order boundary and initial value problems.

Many approaches have been used and there have been lots of efforts for solving non-linear higher order ODE in researches. For instance, Abbasbandy [1] used homotopy perturbation method to investigate quadratic Riccati differential equations. Adomian [2] implemented Adomian decomposition method to stochastic operator equations. Dascioglu and Yaslan [3] derived Chebyshev collocation method for the solution of high-order NODE by Chebyshev series. Ozturk and Gulsu [4] reported improved collocation method. Also Wazwaz [5] used Adomian decomposition method for solving initial value problems in second-order ordinary differential equations. Lu *et al*. [6] used a method based on least squares support vector machines. Furthermore Runge-Kutta method [7-9], predictor-corrector method [10, 11], decomposition method [12], direct block method [13], linearization method [14] have been used for solving IVP and BVP. For a further reading see [10, 15-17].

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The reproducing kernel method (RKM) have been used as an efficient way to solve different types of differential equations by many researcher for years. The reproducing kernel functions are the basis of this method to approach the solution. The initial establish of the general theory of RKM was begin with the research of Aronszajn and Bergman [18, 19]. Since the method is very effective, many researcher applied the method to the several kind of problems. For instance Cui *et al*. [20] published a book about numerical analysis in the reproducing kernel space which is a comprehensive study. Syam *et al*. [21] used the method to solve a class of fractional Sturm-Liouville eigenvalue problems. Jiang and Tian [22] reported the Volterra integro-differential equations of fractional order by the RKM. Li *et al*. [23] applied the method for numerical solutions of fractional Riccati differential equations. For more details see [24-28].

In this article, we aim to get appropriate reproducing kernel functions for some higher order non-linear differential equations:

$$\frac{\mathrm{d}^M y}{\mathrm{d}x^M} = u(x,y),\ x \in [a,c],\quad M \geq 2$$

## Preliminaries

We consider:

$$\frac{\mathrm{d}^M u}{\mathrm{d}t^M} = h(t,u),\quad t \in [a,c],\quad M \geq 2 \tag{1}$$

### *Second order NODE for initial value problems*

We consider:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = h(t,u),\ t \in [a,c] \tag{2}$$

with initial conditions

$$u(a) = p_0,\ u'(a) = p_1$$

### *Second order NODE for boundary value problems*

We consider:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = u(x,y),\ x \in [a,c] \tag{3}$$

with boundary conditions

$$u(a) = p_0,\ u(c) = q_0$$

### *The $M^{th}$ order NODE for IVP*

Consider the following $M^{\mathrm{th}}$ order NODE for IVP:

$$\frac{\mathrm{d}^M u}{\mathrm{d}t^M} = h(t,u),\ t \in [a,c] \tag{4}$$

with boundary conditions

$$u^{(j)}(a) = p_j,\ j = 0,1,\ldots,M-1$$

## Construction of reproducing kernel functions

In order to solve the problem (1) using RKM, we first construct reproducing kernel Hilbert spaces. In this section we present some essentials and results in the reproducing kernel theory. Let we start with some basic definitions which play a very important role in the study of proposed method.

*Definition 1*. (Reproducing kernel). Let $E$ be a non-empty set. A function $R: E \times E \to \mathbb{C}$ is called a reproducing kernel of the Hilbert space $H$ if and only if:

$$(a)\ R(\cdot, x) \in H,\ \forall x \in E$$

$$(b)\ \langle \psi, R(\cdot, x)\rangle = \psi(x)$$

The item (b) is called *reproducing property* of kernel $R$. The value of the function $\psi$ at the point $x$ is reproduced by the inner product of $\psi$ with $R(\cdot, x)$.

*Lemma 1*. If a Hilbert space has a reproducing kernel, it is called a reproducing kernel Hilbert space (RKHS).

*Definition 2*. [20] The space $V_2^m[a, b]$ consist of the functions $u$:$[a, b] \to \mathbb{R}$ and define:

$$V_2^m[a,b] = \left\{u(x) \mid u^{(m-1)}(x) \in AC[a,b],\ \ u^{(m)}(x) \in L^2[a,b],\ \ x \in [a,b]\right\}$$

$V_2^m[a, b]$ equipped with the inner product:

$$< u, v >_{V\frac{m}{2}} = \sum_{i=0}^{m-1} u^{(i)}(a) v^{(i)}(a) + \int_a^b u^{(m)}(x) v^{(m)}(x) \mathrm{d}x$$

Here we denote the vector space of absolutely continuous (real-valued) functions with AC[a, b] and the quadratically integrable functions on the interval $[a, b]$ with $L^2[a, b]$.

*Lemma 2* [20]. The $V_2^m[a, b]$ function space is a reproducing kernel space.

We now give some special reproducing kernel spaces which will be using in the solution of the implements in section *Illustrative examples*.

### *The* $V_2^3[1, 2]$ *reproducing kernel space*

Let we define a function space

$$V_2^3[1,2] = \left\{u(x) \mid u''(x) \in AC[1,2],\ \ u'''(x) \in L^2[1,2],\ \ x \in [1,2]\right\}$$

with the inner product

$$\langle u, R_y \rangle_{V_2^3[1,2]} = u_1 R_y(1) + u'(1) R_y'(1) + u''(1) R_y''(1) + \int_1^2 u^{(3)}(x) R_y^{(3)}(x) \mathrm{d}x$$

We use integration by parts and obtain:

$$< u, R_y >_{V_2^3[1,2]} = u_1 R_y(1) + u'(1) R_y'(1) + u''(1) R_y''(1) + u''(2) R_y^{(3)}(2) - u''(1) R_y^3(1) -$$

$$-u'(2) R^{(4)}(2) + u'(1) R^{(4)}(1) + u(2) R^{(5)}(2) - u(1) R^{(5)}(2) - \int_1^2 u(x) R^{(6)}(x) \mathrm{d}x$$

We have $R_y(1) = 0 = R_y(2)$ by the conditions. Therefore, we get:

$$< u, R_y >_{V_2^3[1,2]} = u'(1)R_y'(1) + u''(1)R_y''(1) + u''(2)R_y^{(3)}(2) - u''(1)R_y^3 - u'(2)R_y^4(2) + u'(1)R_y^4(1) - \int_1^2 u(x)R_y^{(6)}(x)\mathrm{d}x$$

If we have the following equations:

$$\begin{array}{ll} 1.\ R'_y(1)+R_y^{(4)}(1)=0 & 3.\ R_y^{(3)}(2)=0 \\ 2.\ R''_y(1)-R''''_y(1)=0 & 4.\ R_y^{(4)}(2)=0 \end{array} \tag{5}$$

we will get:

$$< u, R_y >_{V_2^3[1,2]} = -\int_1^2 u(x) R_y^{(6)}(x)\,\mathrm{d}x$$

Note that property of the reproducing kernel is:

$$< u, R_y >_{V_2^3[1,2]} = u(y)$$

thus we reach

$$-\int_1^2 u(x) R_y^{(6)}(x)\,\mathrm{d}x = u(y)$$

This gives us the Dirac-Delta function:

$$-R_y^{(6)}(x)\,\mathrm{d}x = \delta(x-y)$$

When $x \neq y$, we get:

$$R_y^{(6)}(x) = 0$$

therefore we obtain the reproducing kernel function $R_y$:

$$R_y(x) = \begin{cases} \displaystyle\sum_{k=1}^{6} c_k x^{k-1}, & x \le y \\ \displaystyle\sum_{k=1}^{6} \mathrm{d}_k x^{k-1}, & x > y \end{cases}$$

There are twelve unknown coefficients. So we need twelve equations to find these unknown coefficients. By Dirac-Delta function:

$$\begin{array}{ll} 5.\ R_{y^+}(y)=R_{y^-}(y) & 8.\ R'''_{y^+}(y)=R'''_{y^-}(y) \\ 6.\ R'_{y^+}(y)=R'_{y^-}(y) & 9.\ R_{y^+}^{(4)}(y)=R_{y^-}^{(4)}(y) \\ 7.\ R''_{y^+}(y)=R''_{y^-}(y) & 10.\ R_{y^+}^{(5)}(y)\quad R_{y^-}^{(5)}(y) \end{array} \tag{6}$$

We have the following equations:

$$\begin{array}{l} 11.\ R_y(1)=0 \\ 12.\ R_y(2)=0 \end{array} \tag{7}$$

So we have twelve unknown coefficients and twelve equations. If we solve these equations, we get the reproducing kernel function for $x \le y$:

$$R_y(x) = -\frac{1355}{1872}x - \frac{1355}{1872}y - \frac{7}{117}y^2 + \frac{53}{234}y^3 - \frac{53}{936}y^4 + \frac{53}{9360}y^5 + \frac{115}{1872}xy^4 - \frac{1}{234}xy^2 +$$

$$+\frac{3595}{3744}xy - \frac{23}{3744}xy^5 - \frac{115}{468}xy^3 + \frac{1}{468}x^2y^5 - \frac{5}{234}x^2y^4 + \frac{10}{117}x^2y^3 - \frac{10}{117}x^2y^2 -$$

$$-\frac{1}{234}x^2y+\frac{79}{468}x^3y^2-\frac{1}{468}x^3y^5+\frac{5}{234}x^3y^4-\frac{10}{117}x^3y^3-\frac{115}{468}x^3y+\frac{37}{1872}x^4y+$$

$$+\frac{1}{1872}x^4y^5-\frac{5}{936}x^4y^4+\frac{5}{234}x^4y^3-\frac{5}{234}x^4y^2-\frac{1}{18720}x^5y^5+\frac{1}{1872}x^5y^4-\frac{1}{468}x^5y^3+$$

$$+\frac{1}{468}x^5y^2-\frac{23}{3744}x^5y+\frac{2807}{4680}-\frac{7}{117}x^2+\frac{53}{234}x^3-\frac{53}{936}x^4+\frac{131}{9360}x^5.$$

*The* $V_2^4[1, 0.5]$ *reproducing kernel space*

Let we define a function space:

$$V_2^4[0,0.5]=\left\{u(x)\,|\,u''(x)\in AC[0,0.5],\quad u'''(x)\in L^2[0,0.5],\quad x\in[0,0.5]\right\}$$

with the inner product

$$<u,R_y>_{V_2^4[0,0.5]}=u(0)R_y(0)+u'(0)R_y'(0)+u''(0)R_y''(0)-u^{(3)}(0)R_y^{(3)}(0)+\int_0^{0.5}u+(4)(x)R_y^{(4)}(x)\mathrm{d}x$$

Integrating this equation by parts for four times:

$$<u,R_y>_{V_2^4[0,0.5]}=u(0)R_y(0)+u'(0)R_y'(0)+u''(0)R_y''(0)-u^{(3)}(0)R_y^{(3)}(0)+$$

$$+u^{(3)}(0.5)R_y^{(4)}(0.5)-u^{(3)}(0)R_y^{(4)}(0)-u''(0.5)R_y^{(5)}(0.5)+u''(0)R_y^{(5)}(0)+$$

$$+u'(0.5)R_y^{(6)}(0.5)\quad u'(0)R_y^{(6)}(0)-u(0.5)R_y^{(7)}(0.5)+u(0)R_y^{(7)}(0)+\int^{0.5}u(x)R_y^{(8)}(x)\mathrm{d}x$$

We have the following equations:

$$1.\ R_y(0)=0$$
$$2.\ R_y'(0)=0 \tag{8}$$
$$3.\ R_y''(0)=0$$

Therefore, we obtain:

$$<u,R_y>_{V_2^4[0,0.5]}=u^{(3)}(0)R_y^{(3)}(0)+u^{(3)}(0.5)R_y^{(4)}(0.5)-u^{(3)}(0)R_y^{(4)}(0)-$$

$$-u''(0.5)R_y^{(5)}(0.5)+u'(0.5)R_y^{(6)}(0.5)-u(0.5)R_y^{(7)}(0.5)+\int_0^{0.5}u(x)R_y^{(8)}(x)\mathrm{d}x$$

If we have:

$$4.\ R_y^{(3)}(0)-R_y^{(4)}(0)=0 \qquad 7.\ R_y^{(6)}(0.5)=0$$
$$5.\ R_y^{(4)}(0.5)=0 \qquad 8.\ R_y^{(7)}(0.5)=0 \tag{9}$$
$$6.\ R_y^{(5)}(0.5)=0$$

then we will get

$$< u, R_y > V_2^4[0.0.5] = \int_0^{0.5} u(x) R_y^{(8)}(x)\,\mathrm{d}x$$

We have:

$$< u, R_y >_{V_2^4[0,0.5]} = u(y)$$

by the reproducing property. Thus, we reach:

$$\int_0^{0.5} u(x) R_y^{(8)}(x)\,\mathrm{d}x = u(y)$$

The aforementioned equation will give us the Dirac-Delta function. Then, we have:

$$R_y^{(8)}(x) = \delta(x-y)$$

When $x \neq y$, we obtain the reproducing kernel function $R_y$:

$$R_y(x) = \begin{cases} \sum_{k=1}^{8} c_k x^{k-1}, & x \le y, \\ \sum_{k=1}^{8} \mathrm{d}_k x^{k-1}, & x > y. \end{cases}$$

Since there are sixteen unknown coefficients, the same number equations is needed to find them. By the properties of Dirac-Delta function, we have:

$$\begin{array}{ll} 9.\ R_{y^+}(y) = R_{y^-}(y) & 13.\ R_{y^+}^{(4)}(y) = R_{y^-}^{(4)}(y) \\ 10.\ R'_{y^+}(y) = R'_{y^-}(y) & 14.\ R_{y^+}^{(5)}(y) = R_{y^-}^{(5)}(y) \\ 11.\ R''_{y^+}(y) = R''_{y^-}(y) & 15.\ R_{y^+}^{(6)}(y) = R_{y^-}^{(6)}(y) \\ 12.\ R'''_{y^+}(y) = R'''_{y^-}(y) & 16.\ R_{y^+}^{(7)}(y) - R_{y^-}^{(7)}(y) = 1 \end{array} \tag{10}$$

If we solve the aforementioned equations then we get the reproducing kernel function for $x \le y$:

$$R_y(x) = 0.02777777778 y^3 x^3 + 0.006944444444 y^3 x^4 4 - 0.004166666667 y^2 x^5 + \\ + 0.001388888889 y x^6 - 0.0001984126984 x^7$$

## Illustrative examples

We consider the following problems:

*Example 1.* We take into consideration [6]:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{y^3 - 2y^2}{2x^2}, \quad x \in [1,2]$$

with boundary conditions

$$y(1) = 1, \ y(2) = \frac{4}{3}$$

In order to apply the RKM we need to homogenize the boundary conditions. Therefore, we use:

$$u(x)=y(x)-\frac{1}{3}(x+2)$$

Then we obtain:

$$y(x)=u(x)+\frac{1}{3}(x+2)$$

$$y'(x)=u'(x)+\frac{1}{3}$$

$$y''(x)=u''(x)$$

We use the previous equations and get:

$$u''(x)=\frac{\left[u(x)+\frac{1}{3}(x+2)\right]^3-2\left[u(x)+\frac{1}{3}(x+2)\right]^2}{2x^2}=$$

$$=\frac{1}{2x^2}\left[u^3(x)+u^2(x)(x+2)+\frac{u(x)(x+2)^2}{3}+\frac{(x+2)^3}{27}-2u^2(x)-\frac{4}{3}u(x)(x+2)-\frac{2}{9}(x+2)^2\right]$$

Then we obtain:

$$u''(x)=\frac{u^3(x)}{2x^2}+\frac{x+2}{2x^2}u^2(x)+\frac{(x+2)^2}{6x^2}u(x)+\frac{(x+2)^3}{54x^2}-\frac{u^2(x)}{x^2}-\frac{2(x+2)}{3x^2}u(x)-$$

$$-\frac{1}{9x^2}(x+2)^2u''(x)+\left[\frac{2(x+2)}{3x^2}-\frac{(x+2)^2}{6x^2}\right]u(x)=\frac{u^3(x)}{2x^2}+\frac{x+2}{2x^2}u^2(x)-\frac{1}{x^2}u^2(x)+\frac{(x+2)^3}{54x^2}-$$

$$-\frac{(x+2)^2}{9x^2}\Rightarrow u''(x)-\frac{x^2-4}{6x^2}u(x)=\frac{1}{2x^2}u^3(x)+\frac{1}{2x}u^2(x)+\frac{x^3+4x-16}{54x^2} \quad (11)$$

*Example 2*. We take into consideration [6]

$$\frac{d^2y}{dx^2}=2y^3, \quad x\in[0,0.5]$$

with initial conditions $y(0)=1$, $y'(0)=-1$. We use the following transformation homogenize the initial conditions:

$$v(x)=y(x)+x-1$$

Then we get:

$$y(x)=v(x)-x+1$$

$$y'(x)=v'(x)-1$$

$$y''(x)=v''(x)$$

If we use the aforementioned equations, we will obtain:

$$\begin{aligned} v''(x) &= 2\left[v(x) - x + 1\right]^3 \\ &= 2\left[v(x) + (1-x)\right]^3 \\ &= 2\left[v^3(x) + 3v^2(1-x) + 3v(x)(1-x)^2 + (1-x)^3\right] \end{aligned} \tag{12}$$

By that we obtain:

$$v''(x) = 2v^3(x) + 6v^2(1-x) + 6(1-x)^2 v(x) + 2(1-x)^3$$

$$v''(x) - 6(1-x)^2 v(x) = 2v^3(x) + 6v^2(1-x) + 2(1-x)^3$$

Therefore, the problem coverts to homogeneous form:

$$v''(x) - 6(1-x)^2 v(x) = 2v^3 + 6v^2(1-x) + 2(1-x)^3$$

$$v(0) = 0 = v'(0)$$

*Example 3.* We consider [6]:

$$\frac{d^3 y}{dx^3} = -y^2 - \cos(x) + \sin^2(x), \quad x \in [0, 0.5] \tag{13}$$

with the initial conditions $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$.

Similar to before examples, we will be using a transformation homogenize the initial conditions. For this problem we use the transformation:

$$v(x) = y(x) - x$$

By calculating the derivatives we get:

$$y(x) = v(x) + x$$

$$y'''(x) \quad v'''(x)$$

If we put the aforementioned equations into the problem we arrive to:

$$v'''(x) = -\left[v(x) + x\right]^2 - \cos x + \sin' x$$

$$v'''(x) = -v^2(x) - 2xv(x) - x^2 - \cos x + \sin^2 x$$

Thus, our problem turns into the form which is homogeneous:

$$v'''(x) + 2xv(x) = -v^2(x) - x^2 - \cos x + \sin^2 x$$

$$v(0) = v'(0) = v''(0) = 0$$

*Example 4*. Let us consider the problem [6]:

$$\frac{d^2 y}{dx^2} = y + y^3, \quad x \in [0, 1] \tag{14}$$

subject to boundary conditions $y(0) = 0$, $y(1) = 1$.

In order to transform the non-homogeneous differential equation the homogeneous one, we use the transformation:

$$v(x) = y(x) - x$$

By using that, we get:

$$y(x) = v(x) + x$$
$$y''(x) = v''(x)$$

If we use the aforementioned equations, we obtain:

$$v''(x) = v(x) + x + (v(x) + x)^3 =$$
$$= v(x) + x + v^3(x) + 3xv^3 2(x) + 3v(x)x^2 + x^3 =$$
$$= (3x^2 + 1)v(x) + v^3(x) + 3xv^2(x) + x^3$$

Thus, we reach:

$$v''(x) - (3x2 + 1)v(x) = v^3(x) + 3xv^2(x) + x^3$$
$$v(0) = 0 = v(1)$$

## Conclusion

In this article we obtained the reproducing kernel functions for solving some non-linear higher order differential equations. We obtained homogenizing transforms for the non-linear ordinary differential equations. These transforms are interesting and essential to apply the reproducing kernel Hilbert space method. Therefore, it will be very useful for researchers.

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