MOTION EQUATIONS AND NON-NOETHER SYMMETRIES OF LAGRANGIAN SYSTEMS WITH CONFORMABLE FRACTIONAL DERIVATIVE

by

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In this paper, we present the fractional motion equations and fractional non-Noether symmetries of Lagrangian systems with the conformable fractional derivatives. The exchanging relationship between isochronous variation and fractional derivative, and the fractional Hamilton's principle of the holonomic conservative and non-conservative systems under the conformable fractional derivative are proposed. Then the fractional motion equations of these systems based on the Hamilton's principle are established. The fractional Euler operator, the definition of fractional non-Noether symmetries, non-Noether theorem, and Hojman's conserved quantities for the Lagrangian systems are obtained with conformable fractional derivative. An example is given to illustrate the results.

Key words: non-Noether symmetry, conformable fractional derivative, Lagrangian system

Introduction

The role of symmetry is one of the most prominent attributes in physics, life sciences and engineering technology*, etc.* [1, 2]. Noether symmetry, Lie symmetry, and Mei symmetry have been well applied to study dynamical systems in mathematical physics. In 1992, a theorem concerning the conserved quantities for second-order dynamical systems has been derived by Hojman [3], which showed that the conserved quantity could be constructed in terms of the symmetry transformation vector of the equations of motion only, without using either Lagrangian or Hamiltonian structures. Later in 1994 the geometrical basis of this conservation law was presented by Gonzalez-Gascon in [4].

The symmetries, conservation laws and bifurcation at al of non-linear differential equations in mathematical physics have been paid much attentions [5-20]. Lately, Cai *et al*. [6] has studied the non-Noether conserved quantities for holonomic mechanical system. Fu *et al.*

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[7] and Fu and Chen [8, 9] developed a new non-Noether conserved quantity for non-conservative dynamical systems. Subsequently, the symmetries of fractional derivative have attracted the attention of lots of researchers. Zhou *et al.* [10] presented the symmetry theories of Hamiltonian systems with Riemann-Liouville fractional derivatives. Zhang *et al.* [11] investigated the Hamilton formalism and Noether symmetry for mechanic-electrical systems with fractional derivatives while Sun *et al.* [12] developed a Lie symmetry theorem of fractional non-holonomic systems. Wang and Fu [13] studied fractional cyclic integrals and fractional Lagrange system and non-Noether symmetries of Hamiltonian systems with conformable fractional derivatives. Fu *et al.* [14] derived Lie symmetries and their inverse problems of fractional non-holonomic Hamilton systems. Zhou and Zhang [15] presented Noether theorems for a fractional Birkhoffian system within Riemann-Liouville derivatives. Riemann-Liouville derivative and Caputo derivative play important roles for the dynamical systems with fractional derivatives [16-18]. Agrawal [19], presented the fractional variational problems with left and right Riemann-Liouville derivatives. Zhang [20], further gave the fractional differential equations in terms of combined Riemann-Liouville derivatives. However, the definitions of these fractional order derivatives are mostly in the form of integrals, which bring great difficulties for general researchers and many engineering and technical personnel.

Recently, Abdeljawad [21] and Khalil *et al.* [22] developed the definitions of the conformable fractional derivatives and set the basic concepts in this new simple interesting fractional calculus, which the definitions and properties of the conformable fractional derivative are coincident with the usual derivative. In this paper, we study the motion equations and non-Noether symmetries of fractional Lagrangian systems with conformable fractional derivatives.

Preliminary

We present the definitions and properties of conformable fractional calculus [19, 20] to benefit the readers.

Definition 1 (conformable fractional derivative). The fractional derivative starting from a of a function f: $[a,\infty)$ of order $0 < \alpha \leq 1$ is defined:

$$\left(T_\alpha^a f\right)(t) = \lim_{\varepsilon \to 0} \frac{f\left[t + \varepsilon (t-a)^{1-\alpha}\right] - f(t)}{\varepsilon} \tag{1}$$

When $a = 0$, we write $(T_\alpha f)(t)$. If $(T_\alpha^a f)(t)$ exists on (a, b), then:

$$\left(T_\alpha^a f\right)(t) = \lim_{t \to a^+} \left(T_\alpha^a f\right)(t) \tag{2}$$

Definition 2 (conformable fractional integral). Let $\alpha \in (0, 1]$, then the left fractional integral starting at a if order α is defined:

$$\left(I_\alpha^a f\right)(t) = \int_a^t f(x) \mathrm{d}_\alpha x = \int_a^t \frac{f(x)}{(x-a)^{1-\alpha}} \mathrm{d}x \tag{3}$$

When $a = 0$:

$$\left(I_\alpha f\right)(t) = \int_0^t f(x) \mathrm{d}_\alpha x = \int_0^t \frac{f(x)}{x^{1-\alpha}} \mathrm{d}x \tag{4}$$

Notice that if f is $\alpha -$ differentiable and $0 < \alpha \leq 1$, the conclusions are easily made:

$$T_\alpha f(t) = t^{1-\alpha} f'(t) \tag{5}$$

$$T_\alpha(\lambda) = t^{1-\alpha}\lambda' = 0, \text{ for all constant functions } f(t) = \lambda \quad (6)$$

$$T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g), \text{ for all } a, b \in R \quad (7)$$

$$T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f) \quad (8)$$

$$T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2} \quad (9)$$

$$\left(T_\alpha^a h\right)(t) = \left(T_\alpha^a f\right)\left[g(t)\right]\left(T_\alpha^a g\right)(t)\,g(t)^{\alpha-1} = t^{1-\alpha}\frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}t}(t) \quad (10)$$

where $h(t) = f[g(t)]$.

Fractional Hamilton principles and fractional motion equations of holonomicconservative systems

In this section, the exchanging relationship between the isochronous variation and fractional derivative is proved. Furthermore, the fractional Hamilton principles for Lagrangian systems are presented.

Exchanging relationship between the isochronous variation and fractional derivative

The isochronous variation does not depend on t. Assume that q is determined only by the parameter γ. We now study two infinite close orbits $q = q(t, \gamma)$ and $q = q(t, \gamma + d\gamma)$. The difference between them is called the isochronous variation of the variable:

$$\delta q = q(t, \gamma + d\gamma) - q(t, \gamma) \quad (11)$$

Note that $q(t, \gamma + d\gamma)$ can be expanded:

$$q(t, \gamma + \mathrm{d}\gamma) = q(t, \gamma) + \frac{\partial q(t, \gamma)}{\partial \tilde{a}}\mathrm{d}\gamma \quad (12)$$

Substituting eq. (12) into eq. (11):

$$\delta q = \frac{\partial q(t, \gamma)}{\partial \gamma}\mathrm{d}\gamma \quad (13)$$

We know the isochronous variation and fractional derivatives with the fractional gene are interchangeable:

$$\delta\left(t^{1-\alpha}\frac{\mathrm{d}q}{\mathrm{d}t}\right) = t^{1-\alpha}\left(\frac{\mathrm{d}\delta q}{\mathrm{d}t}\right) \quad (14)$$

due to

$$T_\alpha \delta q = T_\alpha\left[\frac{\partial q(t, \gamma)}{\partial \gamma}\mathrm{d}\gamma\right] \quad (15)$$

$$\delta T_\alpha q = T_\alpha q(t, \gamma + \mathrm{d}\gamma) - T_\alpha q(t, \gamma) = T_\alpha\left[q(t, \gamma) + \frac{\partial q(t, \gamma)}{\partial \gamma}\mathrm{d}\gamma\right] - T_\alpha q(t, \gamma) \quad (16)$$

Fractional Hamilton principles and motion equations for holonomic conservative systems

The fractional Hamilton action for holonomic conservative systems is defined:

$$S(\gamma)=\int_{t_1}^{t_2} L(t,q_s,T_\alpha q_s)\,\mathrm{d}t=\int_{t_1}^{t_2} t^{1-\alpha} L(t,q_s,T_\alpha q_s)\,\mathrm{d}_\alpha t \tag{17}$$

where $L(t, q_s, T_\alpha q_s)$ is Lagrangian function of the fractional systems and γ is a curve.

It is well-known that the Hamilton principle says that the real movement, in all possible kinds of motions of the fractional systems, keeps the Hamilton action functional invariant for the same time, starting and ending positions under the same constraint conditions. Hence we know clearly that the fractional Hamilton principle for holonomic conservative systems:

$$\delta S=\delta\int_{t_1}^{t_2} L(t,q_s,T_\alpha q_s)\,\mathrm{d}t=0 \tag{18}$$

with

$$\delta q_s\Big|_{t=t_a}=\delta q_s\Big|_{t=t_b}=0 \quad (s=1,\cdots,n) \tag{19}$$

By using the properties of isochronous variation and fractional derivative:

$$\delta S=\delta\int_{t_1}^{t_2} L(t,q_s,T_\alpha q_s)\,\mathrm{d}t=\int_{t_1}^{t_2}\delta L(t,q_s,T_\alpha q_s)\,\mathrm{d}t=\int_{t_1}^{t_2}\sum_{s=1}^{n}\left(\frac{\partial L}{\partial q_s}\delta q_s+\frac{\partial L}{\partial T_\alpha q_s}\delta T_\alpha q_s\right)\mathrm{d}t \tag{20}$$

$$\frac{\partial L}{\partial T_\alpha q_s}T_\alpha\,\delta q_s=T_\alpha\left(\frac{\partial L}{\partial T_\alpha q_s}\delta q_s\right)-T_\alpha\left(\frac{\partial L}{\partial T_\alpha q_s}\right)\delta q_s \tag{21}$$

Substituting eq. (21) into eq. (22):

$$\begin{aligned}\delta S&=\int_{t_1}^{t_2}\left\{\sum_{s=1}^{n}\left[\frac{\partial L}{\partial q_s}-T_\alpha\left(\frac{\partial L}{\partial T_\alpha q_s}\right)\right]\delta q_s\,\mathrm{d}t+\int_{t_1}^{t_2}\sum_{s=1}^{n}T_\alpha\left(\frac{\partial L}{\partial T_\alpha q_s}\delta q_s\right)\right\}\mathrm{d}t=\\&=\int_{t_1}^{t_2}\left\{\sum_{s=1}^{n}\left[\frac{\partial L}{\partial q_s}-T_\alpha\left(\frac{\partial L}{\partial T_\alpha q_s}\right)\right]\delta q_s\,\mathrm{d}t+\left(\sum_{s=1}^{n}\frac{\partial L}{\partial T_\alpha q_s}\delta q_s\right)\Bigg|_{t_1}^{t_2}\right\}\end{aligned} \tag{22}$$

Since:

$$\left(\sum_{s=1}^{n}\frac{\partial L}{\partial T_\alpha q_s}\delta q_s\right)\Bigg|_{t}^{t_2}=\left(\sum_{s=1}^{n}\frac{\partial L}{\partial T_\alpha q_s}\delta q_s\right)\Bigg|_{t=t_2}-\left(\sum_{s=1}^{n}\frac{\partial L}{\partial T_\alpha q_s}\delta q_s\right)\Bigg|_{t=t_1}=0 \tag{23}$$

Here $\delta q_1,\ldots,\delta q_\mathrm{n}$ are independent, respectively:

$$T_\alpha\left(\frac{\partial L}{\partial T_\alpha q_s}\right)-\frac{\partial L}{\partial q_s}=0 \tag{24}$$

So the motion equations of fractional holonomic conservative systems are given:

$$t^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial T_\alpha q_s}\right)-\frac{\partial L}{\partial q_s}=0,\quad (s=1,2,\cdots n) \tag{25}$$

Fractional motion equations of the non-conservative systems

We know that if the Lagrange function for the fractional holonomic non-conservative systems is $L = L(t, q_s, T_\alpha q_s)$ $(s = 1, 2,..., n)$, then the fractional Hamilton principles for holonomic non-conservative systems is defined:

$$\int_{t_1}^{t_2} \left(\delta L + \delta' A\right) \mathrm{d}t = \int_{t_1}^{t_2} t^{1-\alpha} \left(\delta L + \delta' A\right) \mathrm{d}_\alpha t = 0 \tag{26}$$

which satisfies $\delta q_s|_{t=t_\alpha} = \delta q_s|_{t=t_\chi} = 0$ and $\delta' A = \sum_{s=1}^{n} Q_s \delta q_s$ where $Q_s = Q_s(t, q_s, T_\alpha q_s)$ are non-conservative forces. According to the variation of Lagrangian:

$$\delta L = \sum_{s=1}^{n} \left(\frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial T_\alpha q_s} \delta T_\alpha q_s \right) = \sum_{s=1}^{n} \left(\frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial T_\alpha q_s} T_\alpha \delta q_s \right) \tag{27}$$

Substituting eq. (27) into eq. (26):

$$\begin{aligned} &\int_{t_1}^{t_2} \left(\sum_{s=1}^{n} \frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial T_\alpha q_s} T_\alpha \delta q_s + Q_s \delta q_s \right) \mathrm{d}t = \\ &= \int_{t_1}^{t_2} \sum_{s=1}^{n} \left[\frac{\partial L}{\partial q_s} \delta q_s + T_\alpha \left(\frac{\partial L}{\partial T_\alpha q_s} \delta q_s \right) - T_\alpha \left(\frac{\partial L}{\partial T_\alpha q_s} \right) \delta q_s + Q_s \delta q_s \right] \mathrm{d}t = \\ &= \int_{t_1}^{t_2} \sum_{s=1}^{n} \left[\frac{\partial L}{\partial q_s} - T_\alpha \left(\frac{\partial L}{\partial T_\alpha q_s} \right) + Q_s \delta q_s \right] \mathrm{d}t + \int_{t_1}^{t_2} \sum_{s=1}^{n} T_\alpha \left(\frac{\partial L}{\partial T_\alpha q_s} \delta q_s \right) \mathrm{d}t \end{aligned} \tag{28}$$

Considering $[t_1, t_2]$ of arbitrariness, the independence of δq_s and the endpoint conditions, we give the fractional motion equations of the non-conservative systems:

$$\frac{\partial L}{\partial q_s} - T_\alpha \frac{\partial L}{\partial T_\alpha q_s} + Q_s = 0 \tag{29}$$

i. e.

$$t^{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial T_\alpha q_s} \right) - \frac{\partial L}{\partial q_s} = Q_s, \quad (s = 1, 2, \cdots n) \tag{30}$$

The non-Noether symmetry of the fractional Lagrangian systems

Now we introduce a fractional Euler operator:

$$E_s^\alpha = \frac{\mathrm{d}}{\mathrm{d}_\alpha t} \frac{\partial}{\partial T_\alpha q_s} - \frac{\partial}{\partial q_s} = t^{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial T_\alpha q_s} - \frac{\partial}{\partial q_s} \tag{31}$$

the motion eq. (30) of the fractional Lagrangian system turns into:

$$E_s^\alpha (L) = Q_s \tag{32}$$

Assuming the fractional Lagrangian system is non-singular, which satisfies:

$$\det \left(\frac{\partial^2 L}{\partial T_\alpha q_s \partial T_\alpha q_k} \right) \neq 0 \tag{33}$$

we can work out all the generalized acceleration based on the motion eq. (29):

$$T_\alpha \left(T_\alpha q_s \right) = \alpha_s \left(t, q_s, T_\alpha q_s \right) \tag{34}$$

Introducing the infinitesimal transformations on time and generalized co-ordinates:

$$t^* = t,\ q_s^*\left(t^*\right) = q_s\left(t\right) + \Delta q_s\left(t\right) \tag{35}$$

and their expansion formulae:

$$t^* = t,\ q_s^*\left(t^*\right) = q_s\left(t\right) + \varepsilon\xi_s\left(t, q_s, T_\alpha q_s\right) \tag{36}$$

where ε is the infinitesimal parameter and ξ_s – the generating function of the infinitesimal transformations.

Suppose that an infinitesimal transformation vector of generator:

$$X^{(0)} = \xi_s \frac{\partial}{\partial q_s} \tag{37}$$

and its extensions are:

$$X^{(1)} = \xi_s \frac{\partial}{\partial q_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\xi_s \frac{\partial}{\partial T_\alpha q_s},\ X^{(2)} = X^{(1)} + \frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\xi_s \frac{\partial}{\partial T_\alpha\left(T_\alpha q_s\right)} \tag{38}$$

Under the infinitesimal transformations (36), eq. (34) lead to the determining equations of non-Noether symmetry:

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\xi_s = \frac{\partial \alpha_s}{\partial q_k}\xi_k + \frac{\partial \alpha_s}{\partial T_\alpha q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\xi_k \tag{39}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t} = \frac{\partial}{\partial_\alpha t} + T_\alpha q_s \frac{\partial}{\partial q_s} + \alpha_s \frac{\partial}{\partial T_\alpha q_s} = \frac{\partial}{\partial_\alpha t} + t^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}q_s \frac{\partial}{\partial q_s} + \alpha_s \frac{\partial}{\partial T_\alpha q_s}$$

In terms of the theory of invariance of the differential equations under the infinitesimal transformations, if eq. (34) is invariant under the infinitesimal transformations:

$$X^{(2)}\left[T_\alpha\left(T_\alpha q_s\right) - \alpha_s\left(t, q_s, T_\alpha q_s\right)\right] = T_\alpha\left(T_\alpha \xi_s\right) - X^{(1)}\left(\alpha_s\right) = 0 \tag{40}$$

thus, eq. (39) is proved. By using fractional Euler operator, eq. (40) is written:

$$X^{(2)}\left\{E_s^\alpha\left(L\right)\right\} = X^{(1)}\left(\alpha_s\right) \tag{41}$$

Proposition. Let ξ_s be the generators of infinitesimal transformations which satisfy eq. (39). If there exists a function $\mu = \mu(t, q_s, T_\alpha q_s)$ such that:

$$\frac{\partial \alpha_s}{\partial T_\alpha q_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\left(\ln \mu\right) = 0 \tag{42}$$

then system (34) possesses a fractional Hojman's conserved quantity

$$I_H^\alpha = \frac{1}{\mu}\frac{\partial}{\partial q_s}\left(\mu\xi_s\right) + \frac{1}{\mu}\frac{\partial}{\partial T_\alpha q_s}\left(\mu \frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\xi_s\right) = \text{const.} \tag{43}$$

Proof. Consider:

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_H^\alpha = t^{1-\alpha}\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}I_H^\alpha = t^{1-\alpha}\left\{\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\left[\frac{1}{\mu}\frac{\partial}{\partial q_s}\left(\mu\xi_s\right)\right] + \frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\left[\frac{1}{\mu}\frac{\partial}{\partial T_\alpha q_s}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\xi_s\right)\right]\right\} \tag{44}$$

It is straightforward to show:

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\frac{\partial}{\partial q_s}\mu-\frac{\partial}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\mu=-\frac{\partial\alpha_k}{\partial q_s}\frac{\partial\mu}{\partial T_\alpha q_k} \tag{45}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\frac{\partial}{\partial T_\alpha q_s}\mu-\frac{\partial}{\partial T_\alpha q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\mu=-\frac{\partial\mu}{\partial q_s}-\frac{\partial\alpha_k}{\partial T_\alpha q_s}\frac{\partial\mu}{\partial T_\alpha q_k} \tag{46}$$

From eqs. (39), (42), (45), and (46), we know that then we have $(\bar{\mathrm{d}}/\mathrm{d}_\alpha t)I_H^\alpha = 0$, $(\bar{\mathrm{d}}/\mathrm{d}t)I_H^\alpha = t^{1-\alpha}(\bar{\mathrm{d}}/\mathrm{d}_\alpha t)I_H^\alpha = 0$. Now we have proved that $(\bar{\mathrm{d}}/\mathrm{d}_\alpha t)I_H^\alpha = 0$ and $(\bar{\mathrm{d}}/\mathrm{d}t)I_H^\alpha = 0$ are equivalent based on the property (6) of conformable fractional derivative.

Remark. It is worth pointing out that eqs. (38)-(43) are all new results, which are the fractional differential equations with conformable fractional derivative.

Example

Let us consider the following fractional non-conservative system, the Lagrangian functions and non-conservative forces:

$$L=\frac{1}{2}\left[\left(T_\alpha q_1\right)^2+\left(T_\alpha q_2\right)^2\right],\ Q_1=-t\left(1+t^2\right)^{-1}T_\alpha q_1,\ Q_2=\left(1+t^2\right)^{-1}T_\alpha q_1$$

According to the fractional motion equations of non-conservative system:

$$T_\alpha\left(T_\alpha q_1\right)=-t\left(1+t^2\right)^{-1}T_\alpha q_1,\ T_\alpha\left(T_\alpha q_2\right)=\left(1+t^2\right)^{-1}T_\alpha q_1$$

From the determining of Lie symmetry:

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\xi_1=-\frac{t}{1+t^2}\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\xi_1,\ \frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\xi_2=\frac{1}{1+t^2}\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\xi_1$$

Solving previous system yields:

$$\xi_1=\xi_2=1,\ \xi_1=0,\ \xi_2=T_\alpha q_1\left(1+t^2\right)^{1/2}\left(T_\alpha q_1+tT_\alpha q_2-q_2\right),\ \xi_1=0,\ \xi_2=\left(T_\alpha q_1+tT_\alpha q_2-q_2\right)^2$$

hence

$$-\frac{t}{1+t^2}=\frac{\bar{\mathrm{d}}}{\mathrm{d}_\alpha t}\left(\ln\mu\right)$$

It is easy to see that:

$$\mu=\left(1+t^2\right)^{1/2},\ \mu=\left(1+t^2\right)^{1/2}\left(T_\alpha q_1+tT_\alpha q_2-q_2\right)$$

are solutions of aforementioned equation. Consequently, the fractional Hojman's conserved quantities:

$$I_H^\alpha=0,\ I_H^\alpha=-\left(T_\alpha q_1+tT_\alpha q_2-q_2\right)^{-1}=\text{const.}$$

$$I_H^\alpha=-T_\alpha q_1(1+t^2)^{1/2}=\text{const},\ I_H^\alpha=-2T_\alpha q_1\left(1+t^2\right)^{1/2}=\text{const.}$$

$$I_H^\alpha=-2\left(T_\alpha q_1+tT_\alpha q_2-q_2\right)=\text{const},\ I_H^\alpha=-3\left(T_\alpha q_1+tT_\alpha q_2-q_2\right)=\text{const.}$$

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