CIRCULATORY INTEGRAL AND ROUTH'S EQUATIONS OF LAGRANGE SYSTEMS WITH RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES

by

Jing-Li FU [a], ***Lijun ZHANG*** [b*], ***Chaudry Masood KHALIQUE*** [c], ***and Ma-Li GUO*** [d]

[a] College of Mechanical and Automotive Engineering,
Zhejiang University of Water Resources and Electric Power, Hangzhou, China
[b] College of Mathematics and Systems Science,
Shandong University of Science and Technology, Qingdao, Shandong, China
[c] International Institute for Symmetry Analysis and Mathematical Modelling,
Department of Mathematical Sciences, North-West University,
Mafikeng Campus, Mmabatho, South Africa
[d] Institute of Mathematical Physics, Zhejiang Sci-Tech University, Hangzhou, China

Original scientific paper
https://doi.org/10.2298/TSCI200520034F

In this paper, the circulatory integral and Routh's equations of Lagrange systems are established with Riemann-Liouville fractional derivatives, and the circulatory integral of Lagrange systems is obtained by making use of the relationship between Riemann-Liouville fractional integrals and fractional derivatives. Thereafter, the Routh's equations of Lagrange systems are given based on the fractional circulatory integral. Two examples are presented to illustrate the application of the results.

Key words: *circulatory integral, routh's equation, Lagrange system, Riemann-Liouville fractional derivatives*

Introduction

Fractional calculus has received considerable attention in recent years. This is largely because it has been demonstrated that in many physical phenomena of nature, for example, in science and engineering, fractional derivatives can be used to develop accurate models of these phenomena. This can be particularly seen in the field of modern engineering, where the fractional calculus has become a powerful tool to modeling the anomalous dissipation phenomenon. It is a matter of fact that with the development of science and technology, the applications of fractional calculus in various fields are becoming increasingly important [1-5].

The fractional calculus first appeared in the letter that the 17th century French mathematician L'Hopital wrote to Leibniz in 1695, and with this, the fractional calculus was born. The first book on fractional calculus was published in 1974. In recent decades, some progress has been made in the study of fractional calculus and as a result its applications have flourished in various fields of applied sciences and engineering. Particularly, the study of conserved quantities of Euler-Lagrange equations with Riemann-Liouville fractional derivatives is a popular subject of current research. Since the definitions of the left and right Riemann-Liouville

* Corresponding author, e-mail: li-jun0608@163.com

fractional integrals and fractional derivatives were proposed, the mathematicians, physicists and dynamics experts were engaged in a serious study about it [6]. In 1996, Riewe [7] applied the fractional calculus to the non-conservative mechanical system and did a preliminary study on the fractional variational problems. Later, Agrawal [8] presented Euler-Lagrange equations with left and right fractional derivatives in the Riemann-Liouville sense for fractional variational problems. Improving on the existing results, Zhou *et al.* [9] established the Lagrange equations of general holonomic systems with fractional derivatives. However, as far as we know, the basal problem of the fractional circulatory integral of Lagrange systems has not been studied yet. It is in this spirit that we study this problem in this paper, we present the circulatory integral of Euler-Lagrange equations with fractional derivatives using the method of direct integration. Thereafter, Routh's equations of Lagrange systems with fractional derivatives are established.

Fractional derivatives and fractional integrals

In this section, we review the basic concepts of left and right Riemann-Liouville fractional integrals and fractional derivatives.

Let f be a function with some smoothness in the interval $[a, b]$. For $\forall t \in [a, b]$, the left Riemann-Liouville fractional derivatives ${}_aD_t^\alpha$ and the right Riemann-Liouville fractional derivatives ${}_tD_b^\alpha$ of order α, are defined [10-14]:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-\theta)^{n-\alpha-1} f(\theta) d\theta \tag{1} $$

$$ {}_tD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_t^b (\theta-t)^{n-\alpha-1} f(\theta) \mathrm{d}\theta \tag{2} $$

The Riemann-Liouville fractional integral ${}_aI_t^\alpha$ and ${}_tI_b^\alpha$ of order α, are defined:

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n+1)} \int_a^t (t-\theta)^{\alpha-n} f(\theta) \mathrm{d}\theta \tag{3} $$

$$ {}_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha-n+1)} \int_t^b (\theta-t)^{\alpha-n} f(\theta) \mathrm{d}\theta \tag{4} $$

where $n \in N$, $n - 1 \leq \alpha < n$, and Γ is the Euler gamma function.

According to the definitions of fractional derivatives and integrals, we know that the following equalities hold for $\lambda, \mu \in R$, $0 < \alpha < 1$ and $f(t) \in C^1(R)$:

$$ {}_aD_t^\alpha [{}_aI_t^\alpha f(t)] = f(t) \tag{5} $$

$$ {}_aI_t^\alpha [{}_aD_t^\alpha f(t)] = f(t) - \frac{[{}_aI_t^{1-\alpha} f(t)]_{t=a}}{\Gamma(\alpha)} (t-a)^{\alpha-1} \tag{6} $$

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-\theta)^{-\alpha} f(\theta) \mathrm{d}\theta = \frac{\mathrm{d}}{\mathrm{d}t}[{}_aI_t^{1-\alpha} f(t)] \tag{7} $$

$$ {}_tD_b^\alpha f(t) = \frac{-1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^b (\theta-t)^{-\alpha} f(\theta) \mathrm{d}\theta = -\frac{\mathrm{d}}{\mathrm{d}t}[{}_tI_b^{1-\alpha} f(t)] \tag{8} $$

$$ {}_aD_t^{\alpha}[\lambda f(t)+\mu g(t)] = \lambda\, {}_aD_t^{\alpha} f(t) + \mu\, {}_aD_t^{\beta} g(t) \tag{9}$$

$$ {}_aD_t^{\alpha} C = \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha}, \quad {}_tD_b^{\alpha} C = \frac{C}{\Gamma(1-\alpha)}(b-t)^{-\alpha} \tag{10}$$

$$ {}_aD_t^{\alpha}(t-a)^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)}(t-a)^{\mu-\alpha}, \quad {}_tD_b^{\alpha}(b-t)^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)}(b-t)^{\mu-\alpha} \tag{11}$$

$$ {}_aI_t^{\alpha}(t-a)^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu+\alpha)}(t-a)^{\mu+\alpha} + C, \quad {}_tI_b^{\alpha}(b-t)^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu+\alpha)}(b-t)^{\mu+\alpha} + C \tag{12}$$

Circulatory integral

Considering a mechanical system of n degrees of freedom, the Lagrangian of this system is given [8, 9]:

$$ \frac{\partial L}{\partial q_s} + {}_tD_b^{\alpha} \frac{\partial L}{\partial\, {}_aD_t^{\alpha} q_s} + {}_aD_t^{\beta} \frac{\partial L}{\partial\, {}_tD_b^{\beta} q_s} = 0, \quad (s = 1, 2, \cdots, n) \tag{13}$$

where $q_1, ..., q_n$ are generalized co-ordinates, $0 < \alpha, \beta < 1$ and $L = L(t, q_s, {}_aD_t^{\alpha} q_s, {}_tD_t^{\beta} q_s)$.

Equation (13) is the fractional Euler-Lagrange equation of holonomic conservative systems with Riemann-Liouville fractional derivatives. The resulting equation is more common than Euler-Lagrange equation containing integral order derivatives. When $\alpha = 1$, fractional Euler-Lagrange equation degenerates into integer order Euler-Lagrange equation. Given the fact that many fractional systems can be modeled more accurately using fractional derivative models.

If a generalized co-ordinate, such as q_1, is independent in the fractional Lagrange function L, we call q_1 the cyclic co-ordinate. From eq. (13):

$$ {}_tD_b^{\alpha} \frac{\partial L}{\partial\, {}_aD_t^{\alpha} q_1} + {}_aD_t^{\beta} \frac{\partial L}{\partial\, {}_tD_b^{\beta} q_1} = 0 \tag{14}$$

By using eqs. (7), (8), and (14):

$$ -\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^b (\theta - t)^{-\alpha} \frac{\partial L}{\partial\, {}_aD_t^{\alpha} q_1} + \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-\theta)^{-\alpha} \frac{\partial L}{\partial\, {}_tD_b^{\beta} q_1} = 0 \tag{15}$$

From eqs. (3) and (4):

$$ \frac{\mathrm{d}}{\mathrm{d}t}\left(-{}_tI_b^{1-\alpha} \frac{\partial L}{\partial\, {}_aD_t^{\alpha} q_1} + {}_aI_t^{1-\beta} \frac{\partial L}{\partial\, {}_tD_b^{\beta} q_1} \right) = 0 \tag{16}$$

Integrating both sides of eq. (16):

$$ -{}_tI_b^{1-\alpha} \frac{\partial L}{\partial\, {}_aD_t^{\alpha} q_1} + {}_aI_t^{1-\beta} \frac{\partial L}{\partial\, {}_tD_b^{\beta} q_1} = C_1 \tag{17}$$

where C_1 is a constant of integration. The integral (17) is known as the fractional circulatory integral of the eq. (13) . Let us now discuss two special cases.

Case 1. Taking into account:

$$ {}_tI_b^{1-\alpha} \frac{\partial L}{\partial\, {}_aD_t^{\alpha} q_1} = C_2 \tag{18}$$

one knows

$$ {}_aI_t^{1-\beta}\frac{\partial L}{\partial\, {}_tD_b^{\beta}q_1}=C_1+C_2 \tag{19}$$

From eqs. (5), (10), (18), and (19), we can get the fractional circulatory integrals:

$$\frac{\partial L}{\partial\, {}_aD_t^{\alpha}q_1}={}_tD_b^{1-\alpha}\left[{}_tI_b^{1-\alpha}\frac{\partial L}{\partial\, {}_aD_t^{\alpha}q_1}\right]={}_tD_b^{1-\alpha}C_2=\frac{C_2}{\Gamma(\alpha)}(b-t)^{\alpha-1} \tag{20}$$

$$\frac{\partial L}{\partial\, {}_tD_b^{\beta}q_1}={}_aD_t^{1-\beta}\left[{}_aI_t^{1-\beta}\frac{\partial L}{\partial\, {}_tD_b^{\beta}q_1}\right]={}_aD_t^{1-\beta}(C_1+C_2)=\frac{C_1+C_2}{\Gamma(\beta)}(t-a)^{\beta-1} \tag{21}$$

Case 2. For the case when:

$$ {}_aI_t^{1-\beta}\frac{\partial L}{\partial\, {}_tD_b^{\beta}q_1}=C_3 \tag{22}$$

we see that

$$ {}_tI_b^{1-\alpha}\frac{\partial L}{\partial\, {}_aD_t^{\alpha}q_1}=C_3-C_1 \tag{23}$$

Similarly, we have the fractional circulatory integrals:

$$\frac{\partial L}{\partial\, {}_tD_b^{\beta}q_1}={}_aD_t^{1-\beta}\left[{}_aI_t^{1-\beta}\frac{\partial L}{\partial\, {}_tD_b^{\beta}q_1}\right]={}_aD_t^{1-\beta}C_3=\frac{C_3}{\Gamma(\beta)}(t-a)^{\beta-1} \tag{24}$$

$$\frac{\partial L}{\partial\, {}_aD_t^{\alpha}q_1}={}_tD_b^{1-\alpha}\left[{}_tI_b^{1-\alpha}\frac{\partial L}{\partial\, {}_aD_t^{\alpha}q_1}\right]={}_tD_b^{1-\alpha}(C_3-C_1)=\frac{C_3-C_1}{\Gamma(\alpha)}(b-t)^{\alpha-1} \tag{25}$$

Routh's method

Considering the holonomic conservative systems of n free degrees, the fractional Lagrange equations of the system are written by eq. (13), assuming q_1, q_2,... q_k are k cyclic co-ordinates and using eqs. (20), (21), (24) and (25):

$$\frac{\partial L}{\partial\, {}_aD_t^{\alpha}q_j}=\frac{C_{2_j}}{\Gamma(\alpha)}(b-t)^{\alpha-1}\triangleq g_{j_1}(t),\ \frac{\partial L}{\partial\, {}_aD_t^{\alpha}q_j}=\frac{C_{1_j}+C_{2_j}}{\Gamma(\alpha)}(b-t)^{\alpha-1}\triangleq g_{j_2}(t),\ (j=1,2,...,k) \tag{26}$$

or

$$\frac{\partial L}{\partial\, {}_tD_b^{\beta}q_j}=\frac{C_{3_j}}{\Gamma(\beta)}(t-a)^{\beta-1}\triangleq g_{j_3}(t),\quad \frac{\partial L}{\partial\, {}_aD_t^{\alpha}q_j}=\frac{C_{3_j}-C_{1_j}}{\Gamma(\alpha)}(b-t)^{\alpha-1}\triangleq g_{j_4}(t),\ (j=1,2,...,k) \tag{27}$$

Since q_1,..., q_k are independent in the Lagrange function L, then L is in the form:

$$L=L\left(q_{k+1},q_{k+2},\cdots,q_n;{}_aD_t^{\alpha}q_1,{}_tD_b^{\beta}q_1,\cdots,{}_aD_t^{\alpha}q_n,{}_tD_b^{\beta}q_n,t\right) \tag{28}$$

Now we get the results:

$${}_aD_t^{\alpha}q_j=f_{j_1}\left[q_{k+1},\cdots,q_n,{}_aD_t^{\alpha}q_{k+1},{}_tD_b^{\beta}q_{k+1},\cdots,{}_aD_t^{\alpha}q_n,{}_tD_b^{\beta}q_n,g_{1_1}(t),\cdots,g_{k_2}(t),t\right] \tag{29}$$

$$ {}_tD_b^\beta q_j = f_{j_2}\left[q_{k+1},\cdots,q_{n,a},D_t^\alpha q_a D_t^\alpha D_b^\beta q_{k+1},\cdots,{}_aD_t^\alpha q_n,{}_tD_b^\beta q_n,g_{1_1}(t),\cdots,g_{k_2}(t),t\right] \tag{30}$$

or

$$ {}_tD_b^\beta q_j = f_{j_3}\left[q_{k+1},\cdots,q_{n,\,a}D_t^\alpha q_{k+1},{}_{t}^{\beta}{}_{b}q_{k+1},\cdots,{}_aD_t^\alpha q_n,{}_tD_b^\beta q_n,g_{1_3}(t),g_{1_4}(t),\cdots,g_{k_3}(t),g_{k_4}(t),t\right] \tag{31}$$

$$ \begin{aligned} {}_aD_t^\alpha q_j = f_{j_4}\Big[&q_{k+1},q_{k+2},\cdots,q_n,{}_aD_t^\alpha q_{k+1},{}_tD_b^\beta q_{k+1},\cdots,\\ &{}_aD_t^\alpha q_n,{}_tD_b^\beta q_n,g_{1_3}(t),g_{1_4}(t),\cdots,g_{k_3}(t),g_{k_4}(t),t\Big] \end{aligned} \tag{32}$$

Define

$$ R = L - \sum_{j=1}^{k}\left({}_aD_t^\alpha q_j\frac{\partial L}{\partial\,{}_aD_t^\alpha q_j} + {}_tD_b^\beta q_j\frac{\partial L}{\partial\,{}_tD_b^\beta q_j}\right) \tag{33}$$

which we name as the Routh function. The R can be re-written in the following form: Moreover, the function R will be represented in the form:

$$ R = R\left[q_{k+1},\cdots,q_n,{}_aD_t^\alpha q_{k+1},{}_tD_b^\beta q_{k+1},\cdots,{}_aD_t^\alpha q_n,{}_tD_b^\beta q_n,g_{1_1}(t),g_{1_2}(t),\cdots,g_{k_1}(t),g_{k_2}(t),t\right] \tag{34}$$

Considering the variation of eq. (33):

$$ \begin{aligned} \delta R = \delta\left[L - \sum_{j=1}^{k}\left({}_aD_t^\alpha q_j\frac{\partial L}{\partial\,{}_aD_t^\alpha q_j} + {}_tD_b^\beta q_j\frac{\partial L}{\partial\,{}_tD_b^\beta q_j}\right)\right] = \delta L -\\ -\sum_{j=1}^{k}\left[\frac{\partial L}{\partial\,{}_aD_t^\alpha q_j}\delta\,{}_aD_t^\alpha q_j + {}_aD_t^\alpha q_j\delta\left(\frac{\partial L}{\partial\,{}_aD_t^\alpha q_j}\right) + \frac{\partial L}{\partial\,{}_tD_b^\beta q_j}\delta\,{}_tD_b^\beta q_j + {}_tD_b^\beta q_j\delta\left(\frac{\partial L}{\partial\,{}_tD_b^\beta q_j}\right)\right] \end{aligned} \tag{35}$$

Taking into account of the variations of eqs. (26), (28), and (34), we get:

$$ \delta g_{j_1}(t) = \delta\left(\frac{\partial L}{\partial\,{}_aD_t^\alpha q_j}\right),\ \delta g_{j_2}(t) = \delta\left(\frac{\partial L}{\partial\,{}_tD_b^\beta q_j}\right) \tag{36}$$

$$ \begin{aligned} \delta L = \sum_{r=k+1}^{n}\frac{\partial L}{\partial q_r}\delta q_r + \sum_{j=1}^{k}\left(\frac{\partial L}{\partial\,{}_aD_t^\alpha q_j}\delta\,{}_aD_t^\alpha q_j + \frac{\partial L}{\partial\,{}_tD_b^\beta q_j}\delta\,{}_tD_b^\beta q_j\right) +\\ + \sum_{r=k+1}^{n}\left(\frac{\partial L}{\partial\,{}_aD_t^\alpha q_r}\delta\,{}_aD_t^\alpha q_r + \frac{\partial L}{\partial\,{}_tD_b^\beta q_r}\delta\,{}_tD_b^\beta q_r\right) \end{aligned} \tag{37}$$

$$ \begin{aligned} \delta R = \sum_{r=k+1}^{n}\frac{\partial R}{\partial q_r}\delta q_r + \sum_{r=k+1}^{n}\left(\frac{\partial L}{\partial\,{}_aD_t^\alpha q_r}\delta\,{}_aD_t^\alpha q_r + \frac{\partial L}{\partial\,{}_tD_b^\beta q_r}\delta\,{}_tD_b^\beta q_r\right) +\\ + \sum_{j=1}^{k}\frac{\partial R}{\partial g_{j_1}(t)}\delta g_{j_1}(t) + \sum_{j=1}^{k}\frac{\partial R}{\partial g_{j_2}(t)}\delta g_{j_2}(t) \end{aligned} \tag{38}$$

Substituting eqs. (35) and eq. (37) into eq. (38) gives:

$$\sum_{r=k+1}^{n}\left(\frac{\partial R}{\partial q_r}-\frac{\partial L}{\partial q_r}\right)\delta q_r+\sum_{r=k+1}^{n}\left(\frac{\partial R}{\partial\, {}_aD_t^\alpha q_r}-\frac{\partial L}{\partial\, {}_aD_t^\alpha q_r}\right)\delta\, {}_aD_t^\alpha q_r+\sum_{r=k+1}^{n}\left(\frac{\partial R}{\partial\, {}_tD_b^\beta q_r}-\frac{\partial L}{\partial\, {}_tD_b^\beta q_r}\right)\cdot$$

$$\cdot\delta\, {}_tD_b^\beta q_r+\sum_{j=1}^{k}\left(\frac{\partial R}{\partial g_{j_1}(t)}+{}_aD_t^\alpha q_j\right)\delta g_{j_1}(t)+\sum_{j=1}^{k}\left(\frac{\partial R}{\partial g_{j_2}(t)}+{}_tD_b^\beta q_j\right)\delta g_{j_2}(t)=0 \tag{39}$$

The coefficients of $\delta q_{k+1},\ldots,\delta q_n$, $\delta\, {}_aD_t^\alpha q_{k+1},\ldots,\delta_a D_t^\alpha q_n$, $\delta\, {}_tD_{b\,k+1}^\beta$, $\delta\, {}_tD_b^\beta q_n$, $\delta g_{l1}(t),\ldots,\delta g_{k1}(t)$ are equal to zero because they are independent of each other. Therefore, we have:

$$\frac{\partial R}{\partial q_r}-\frac{\partial L}{\partial q_r}=0,\quad \frac{\partial R}{\partial\, {}_aD_t^\alpha q_r}-\frac{\partial L}{\partial\, {}_aD_t^\alpha q_r}=0,\quad \frac{\partial R}{\partial\, {}_tD_b^\beta q_r}-\frac{\partial L}{\partial\, {}_tD_b^\beta q_r}=0 \tag{40}$$

$$\frac{\partial R}{\partial g_{j_1}(t)}+{}_aD_t^\alpha q_j=0,\quad \frac{\partial R}{\partial g_{j_2}(t)}+{}_tD_b^\beta q_j=0\quad (r=k+1,\ \ k+2,\cdots,n;\ j=1,2,\cdots,k) \tag{41}$$

We know from eqs. (40) and (13):

$$\frac{\partial R}{\partial q_r}+{}_tD_b^\alpha\frac{\partial R}{\partial\, {}_aD_t^\alpha q_r}+{}_aD_t^\beta\frac{\partial R}{\partial\, {}_tD_b^\beta q_r}=0,\ \ (r=k+1,k+2,\cdots,n) \tag{42}$$

which we name as fractional Routh's equations. From eq. (42), we find that eqs. (13) and (42) are in the same form, while the order of eq. (42) reduces to n – k. Furthermore eq. (41) can be written:

$${}_aD_t^\alpha q_j=-\frac{\partial R}{\partial g_{j_1}(t)},\quad {}_tD_b^\beta q_j=-\frac{\partial R}{\partial g_{j_2}(t)} \tag{43}$$

Integrating both sides of eq. (43), we have:

$${}_aI_t^\alpha\left({}_aD_t^\alpha q_j\right)=-{}_aI_t^\alpha\frac{\partial R}{\partial g_{j_1}(t)},\quad {}_tI_b^\beta\left({}_tD_b^\beta q_j\right)=-{}_tI_b^\beta\frac{\partial R}{\partial g_{j_2}(t)} \tag{44}$$

To summarize our results, the fractional circulatory integral is the first integral of Euler-Lagrange equation of holonomic conservative systems. Using the fractional circulatory integral, we can reduce the order of Euler-Lagrange equation, which becomes Routh's equation as eq. (42).

Two illustrative examples

We consider the motion of two particles of linear damped oscillator with the left Riemann-Liouville fractional derivative firstly. The Lagrangian of the system:

$$L=\frac{1}{2}\left[\left({}_aD_t^\alpha q_1\right)^2+\left({}_aD_t^\alpha q_2\right)^2\right]\exp(\gamma t),\ \ (\gamma=\text{const}) \tag{45}$$

let us study motion of the system. Note that q_1, q_2 are independent of the Lagrange function, L. We know from eq. (17) that the fractional circulatory integral:

$$-{}_tI_b^{1-\alpha}\frac{\partial L}{\partial\, {}_aD_t^\alpha q_s}=C_s$$

which implies that:

$$\frac{\partial L}{\partial\, {}_aD_t^\alpha q_s} = \frac{-C_s}{\Gamma(\alpha)}(b-t)^{\alpha-1}, \quad s=1,2 \tag{46}$$

In view of eq. (45):

$$\frac{\partial L}{\partial\, {}_aD_t^\alpha q_s} = \exp(\gamma t)\, {}_aD_t^\alpha q_s \tag{47}$$

Substituting eq. (47) into (46) gives:

$${}_aD_t^\alpha q_s = \frac{-C_s}{\Gamma(\alpha)} \frac{(b-t)^{\alpha-1}}{\exp(\gamma t)} \tag{48}$$

Thus from eqs. (6), (12), and (48), we obtain the motion of this system. When $\alpha = 1$, eq. (45) becomes:

$$L = \frac{1}{2}\left[(q_1')^2 + (q_2')^2\right]\exp(\gamma t), \quad (\gamma = \text{const}) \tag{49}$$

Considering the definition of the usual circulatory integral:

$$\frac{\partial L}{\partial q_s'} = \beta_s, \quad (s=1,2) \tag{50}$$

We see from eq. (49):

$$\frac{\partial L}{\partial q's} = \exp(\gamma t) q_s' \tag{51}$$

Consequently:

$$q_s' = \frac{\beta_s}{\exp(\gamma t)} \tag{52}$$

It is easy to see that eqs. (48) and (52) are in the same form. So when $\alpha = 1$, the fractional circulatory integral is similar to the usual circulatory integral.

Now we consider a mechanical system of two degrees of freedom as the second example. The Lagrangian of this system is given:

$$L = \frac{1}{2}\left[\left({}_aD_t^\alpha q_1\right)^2 + \left({}_tD_b^\beta q_2\right)^2\right] - q_2 \tag{53}$$

We now study motion of the system. Since q_1 is independent in the Lagrange function L, from eqs. (10) and (17), we obtain the fractional circulatory integration:

$$-{}_tI_b^{1-\alpha} \frac{\partial L}{\partial\, {}_aD_t^\alpha q_1} = C_1 \tag{54}$$

Using eq. (11), we have:

$$\frac{\partial L}{\partial\, {}_aD_t^\alpha} = \frac{-C_1}{\Gamma(\alpha)}(b-t)^{\alpha-1} \tag{55}$$

Considering the differentiation of eq. (53), we have:

$$\frac{\partial L}{\partial\, {}_aD_t^\alpha q_1} = {}_aD_t^\alpha q_1 \tag{56}$$

and hence:

$$ {}_aD_t^\alpha q_1 = \frac{-C_2}{\Gamma(\alpha)}(b-t)^{\alpha-1} \tag{57}$$

According to eqs. (33), (53), and (57), we obtain the fractional Routh function of system:

$$ R = L - {}_aD_t^\alpha q_1 \frac{\partial L}{\partial\, {}_aD_t^\alpha q_1} = L - \left({}_aD_t^\alpha q_1\right)^2 = -\frac{1}{2}\left({}_aD_t^\alpha q_1\right)^2 + \frac{1}{2}\left({}_aD_t^\alpha q_2\right)^2 - q_2 \tag{58}$$

From eq. (42), the fractional Routh's equation of system can be written:

$$ \frac{\partial R}{\partial q_2} + {}_tD_b^\alpha \frac{\partial R}{\partial\, {}_aD_t^\alpha q_2} = 0 \tag{59}$$

Substituting the differentiation of eq. (59) into (58), we get:

$$ {}_tD_b^\alpha \left({}_aD_t^\alpha q_2\right) = 1 \tag{60}$$

From eqs. (8) and (60):

$$ {}_tI_b^{1-\alpha} \left({}_aD_t^\alpha q_2\right) = -t + C_2 \tag{61}$$

Substituting eqs. (9) and (11) into (61):

$$ {}_aD_t^\alpha q_2 = \frac{2}{\Gamma(1+\alpha)}(b-t)^\alpha + \frac{C_2 - b}{\Gamma(\alpha)}(b-t)^{\alpha-1} \tag{62}$$

Thus from eqs. (6), (12), (57), and (62), we obtain the motion of this system. When $\alpha = 1$, eq. (53) becomes:

$$ L = \frac{1}{2}\left[(q_1')^2 + (q_2')^2\right] - q_2 \tag{63}$$

Considering the usual circulatory integral and Routh's equation of the system:

$$ q_1' = \beta_1, \quad q_2' = -t + \beta_2 \tag{64}$$

We observe from eqs. (57), (62), and (64), that the fractional circulatory integral can be used to obtain the usual circulatory integral. Thus, through the aforementioned two examples, one can conclude that the usual circulatory integral and Routh's equation are special cases of the fractional circulatory integral and Routh's equation.

Acknowledgment

This work is supported by the National Natural Science Foundations of China (Grant No. 11872335, 11672270 11472247).

References

[1] Hilfer, R., *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, Singapore, 2000
[2] Zaslavsky, G. M., *Hamiltonian Chaos and Fractional Dynamics*, Oxford University Press, Oxford, UK, 2005
[3] Naber, M., Time Fractional Schrodingere Equation, *Journale Math. Phys.*, *45* (2004), 8, pp. 3339-3352
[4] Surguladze, T. A., On Certain Applications of Fraction Calculus to Viscoelasticity, *Journal Math. Sciences*, *112* (2002), 112, pp. 4517-4557
[5] Zhang, L., *et al.*, Bifurcations and Exact Traveling Wave Solutions of the Zakharov-Rubenchik Equation, *Disc. Cont. Dyna. Syst.*, *13* (2020), 10, pp. 2927-2939

[6] Podlubny, I., *Fractional Differential Equations*, Academic Press, New York, USA, 1999
[7] Riewe, F., Nonconservative Lagrangian and Hamiltonian Mechanics, *Phys. Review E*, *53* (1996), pp. 1890-1899
[8] Agrawal, O. P., Formulation of Euler-Lagrange Equations for Variational Problems, *Journal Math. Anal. Appl*., *272* (2002), 1, pp. 368-379
[9] Zhou, S., *et al*., Langrage Equations of Non-Holonomic Systems with Fractional Derivatives, *Chinese Phys. B*, *19* (2010), 120310
[10] Lacroix, S. F., Traite du calcul diffrentiel et du calcul integral, Courcier Paris (in Franch), France, 1819
[11] Kilbas, A. A., *et al*., *Theory and Applications of Fractional Differential Equations*, Elsevier, New York, USA, 2006
[12] Ahn, H. S., *et al*., Robust Stability Test of a Class of Linear Time Invariant Interval Fractional-Order System Using Lyapunov Inequality, *Appl. Math. Comp*., *187* (2007), 1, pp. 27-34
[13] Shahin, A. M., *et al*., On Fractional Order Quantum Mechanics, *Int. J. Non-linear Sci*., *4* (2009), 4, pp. 469-472
[14] Podlubny, I., *Fractional Differential Equations*, Academic Press, New York, USA, 1999

Paper submitted: May 20, 2020
Paper revised: June 20, 2020
Paper accepted: June 20, 2020