PERIODIC SOLUTION OF FRACTAL PHI-4 EQUATION

by

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This paper focuses on a fractal Phi-4 equation with time-space fractal derivatives, though its solitary solutions have been deeply studied, its periodic solution was rarely revealed due to its strong non-linearity. Now the condition is completely changed, He's frequency formulation provides with a universal tool to having a deep insight into the periodic property of the fractal Phi-4 equation. The two-scale transform is used to convert approximately the fractal Phi-4 equation a differential model, and a criterion is suggested for the existence of a periodic solution of the equation, the effect of fractal orders on the periodic property is also elucidated.

Key words: *fractal calculus, periodic solution, solitary wave, Duffing oscillator, two-scale mathematics*

Introduction

This paper studies the following fractal modification of the Phi-4 equation [1-4]:

$$D_t^{2\alpha}u - D_x^{2\beta}u + m^2u + wu^3 = 0, \quad 0 < \alpha \le 1, \quad 0 < \beta \le 1 \tag{1}$$

where m and w are constants, D_t^α and D_x^β are fractal derivatives with respect to t and x, respectively. Their definitions are given [5, 6]:

$$D_t^\alpha u(t_0, x) = \Gamma(1+\alpha) \lim_{\substack{t-t_0 \to \Delta t \\ \Delta t \ne 0}} \frac{u(t,x) - u(t_0,x)}{(t-t_0)^\alpha} \tag{2}$$

$$D_x^\beta u(t, x_0) = \Gamma(1+\beta) \lim_{\substack{x-x_0 \to \Delta x \\ \Delta x \ne 0}} \frac{u(t,x) - u(t,x_0)}{(x-x_0)^\beta} \tag{3}$$

The chain role works for the fractal derivatives:

$$D_t^{2\alpha} = D_t^\alpha (D_t^\alpha) \tag{4}$$

$$D_x^{2\beta} = D_x^\beta (D_x^\beta) \tag{5}$$

When $\alpha = \beta = 1$, the classic Phi-4 equation is obtained:

$$u_{tt} - u_{xx} + m^2u + wu^3 = 0 \tag{6}$$

The Phi-4 equation can model many non-linear phenomena arising in optics, thermal science, nanofluid, and non-linear vibration [1-4]. Much literature focused on its solitary solutions, while its periodic solution was rarely studied. Furthermore, the traditional Phi-4 equation cannot figure out the effect of porous structure on the solution property, and a fractal modifica-

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tion is much needed. Now the fractal differential models can reveal many interesting properties which the traditional models cannot reveal. Now fractal calculus has witnessed various applications in various fields [7-20].

Two-scale transform

The two-scale transform [21-23] is to convert eq. (1) on a small scale to an differential equation on a large-scale:

$$T = t^{\alpha} \tag{7}$$

$$X = x^{\beta} \tag{8}$$

We can convert eq. (1) into the following:

$$\frac{\partial^2 u}{\partial T^2} - \frac{\partial^2 u}{\partial X^2} + m^2 u + wu^3 = 0 \tag{9}$$

Introducing a complex variable ξ:

$$u(T, X) = U(\xi), \quad \xi = c(X - vT) \tag{10}$$

Equation (9) becomes:

$$c^2 v^2 U'' - c^2 U'' + m^2 U + wU^3 = 0 \tag{11}$$

where c and v are constants.

We assume the initial c onditions:

$$U(0) = A, U'(0) = 0 \tag{12}$$

Periodic solution

This section applies He's frequency formulation [24-26] to figure out the periodic property of the fractal Phi-4 equation. We re-write eq. (11):

$$U'' + F(U) = 0 \tag{13}$$

where the prime implies the derivative with respect to ξ, $F(U)$ is given:

$$F(U) = \frac{m^2 U + wU^3}{c^2 v^2 - c^2} \tag{14}$$

According to He's frequency formulation [19, 20]:

$$\Omega^2 = F'(U)\Big|_{U=\frac{A}{2}} \tag{15}$$

where Ω is the frequency and A is the amplitude. He's frequency formulation has been caught much attention due to its simplest solution process and relatively high accuracy [27-37].

It is easy to calculate the derivative of $F(U)$ with respect to U:

$$F'(U) = \frac{m^2 + 3wU^2}{c^2 v^2 - c^2} \tag{16}$$

Using He's frequency formulation:

$$\Omega = \sqrt{F'(U)\Big|_{-}} = \sqrt{\frac{m^2 \quad — A^2}{c^2 v^2 \quad c^2}} \tag{17}$$

The criterion for the existence of a periodic solution:

$$c^2v^2 - c^2 > 0 \tag{18}$$

or

$$v > 1 \text{ or } v < -1 \tag{19}$$

In case:

$$\frac{m^2}{c^2v^2 - c^2} = 1 \tag{20}$$

and

$$\frac{w}{c^2v^2 - c^2} = \varepsilon \tag{21}$$

Equation (11) becomes the standard Duffing oscillator:

$$U'' + U + \varepsilon U^3 = 0,\ U(0) = A,\ U'(0) = 0 \tag{22}$$

By the homotopy perturbation [38, 39] method or the variational iteration method, its frequency:

$$\Omega = \sqrt{1 + \frac{3}{4}\varepsilon A^2} \tag{23}$$

The approximate periodic solution of eq. (11):

$$U(\xi) = A\cos\left(\Omega\xi + \varphi\right) \tag{24}$$

where φ is a constant.

In view of eq. (10):

$$U(\xi) = A\cos\left(\Omega\xi + \varphi\right) \tag{25}$$

By eqs. (7) and (8), we finally obtain the periodic solution of eq. (1):

$$U(t,x) = A\cos\left[\Omega c(x^\beta - vt^\alpha) + \varphi\right] \tag{26}$$

or

$$U(t,x) = A\cos\left[\sqrt{\frac{m^2 + \frac{3w}{4}A^2}{v^2 - 1}}(x^\beta - vt^\alpha) + \varphi\right] \tag{27}$$

Solution morphology

From eq. (26), we have:

$$\frac{\partial}{\partial t}U(t,x) = \Omega cvA\alpha t^{\alpha-1}\sin\left[\Omega c(x^\beta - vt^\alpha) + \varphi\right] \tag{28}$$

and

$$\frac{\partial}{\partial x}U(t,x) = -\Omega cA\beta x^{\beta-1}\sin\left[\Omega c(x^\beta - vt^\alpha) + \varphi\right] \tag{29}$$

In case of $\alpha = \beta = 1$, we have:

$$\frac{\partial}{\partial} U(t,x) = \Omega cvA \sin\left[\Omega c(x - vt) + \quad\right] \tag{30}$$

$$\frac{\partial}{\partial x} U(t,x) = -\Omega cA \sin\left[\Omega c(x - vt) + \varphi\right] \tag{31}$$

This case leads to the standard periodic property.

In case of $\alpha < 1$ and $\beta < 1$, we have:

$$\frac{\partial}{\partial t} U(0,x) \to \infty \tag{32}$$

$$\frac{\partial}{\partial x} U(t,0) \to \infty \tag{33}$$

Equation (32) sees an extremely large change of U at the initial time, while eq. (33) predicts an extremely large slope at $x = 0$.

From aforementioned analysis, we can see that the fractal orders will greatly affect the solution morphology, see fig. 1 for the case $A = m = w = 1$, $v = 2$, and $\varphi = 0$.

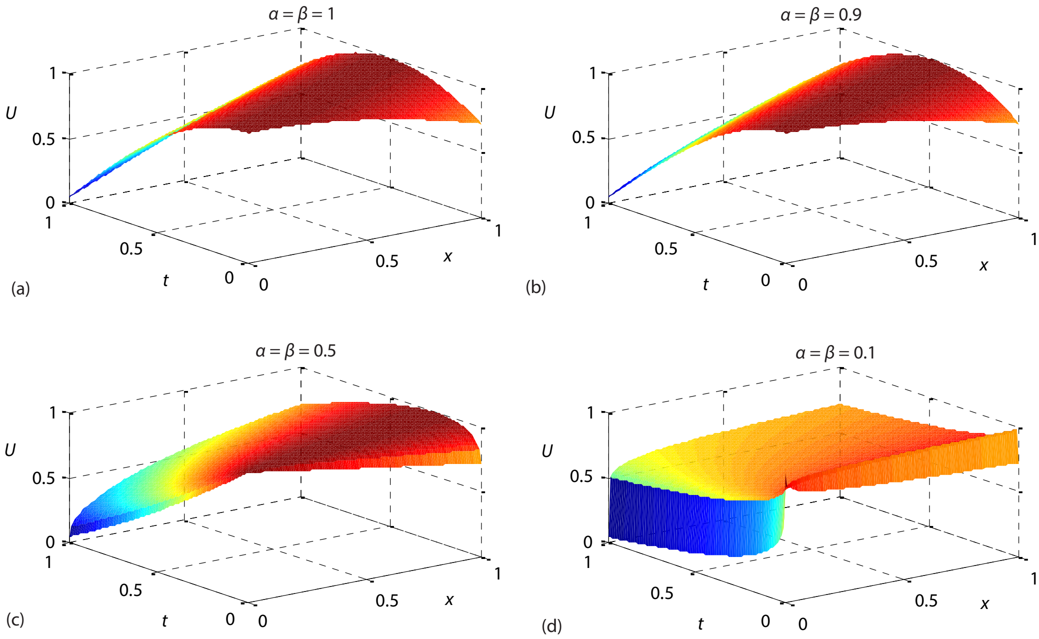

Figure 1. Solution morphology for different values of α and β

Conclusions

In this paper, we use the Hira Tariq method to transform a FPDE to a FODE and it was very effective, however, FODE remains difficult to solve. Fortunately, there is He's frequency effective method to solve it. As Dr. Ji-Huan He has been emphasized, the simpler is the better. In engineering applications, a fast and effective estimation of a nonlinear vibration problem is very needed, and He's frequency formation becomes a universal tool for this purpose.

References

[1] Korpinar, Z., Some Analytical Solutions by Mapping Methods for Non-Linear Conformable Time-Fractional Phi-4 Equation, *Thermal Science*, *23* (2019), S6, pp. S1815-S1822
[2] Tariq, H., Akram, G., New Approach for Exact Solutions of Time Fractional Cahn-Allen Equation and Time Fractional Phi-4 Equation, *Physics A*, *473* (2017), May, pp. 352-362
[3] Alzahrani, A. K., *et al*., The Unsteady Liquid Film Flow of the Carbon Nanotubes Engine Oil Nanofluid over a Non-Linear Radially Extending Surface, *Thermal Scienc*e, *24* (2020), 2A, pp. 951-963
[4] Sirisubtawee, S., *et al*., Exact Traveling Wave Solutions of the Space-Time Fractional Complex Ginzburg-Landau Equation and the Space-Time Fractional Phi-4 Equation Using Reliable Methods, *Advances in Difference Equations*, *2019* (2019), May, 219
[5] He, J. H., Fractal Calculus and Its Geometrical Explanation, *Results in Physics*, *10* (2018), Sept., pp. 272-276
[6] He, J. H., A Short Review on Analytical Methods for to a Fully Fourth Order Non-Linear Integral Boundary Value Problem with Fractal Derivatives, *International Journal of Numerical Methods for Heat and Fluid-Flow*, *30* (2020), 11, pp. 4933-4943
[7] Wang, K. L., *et al*., Physical Insight of Local Fractional Calculus and Its Application Fractional Kdv-Burgers-Kuramoto Equation, *Fractals*, *27* (2019), 7, 1950122
[8] Wang, K. L., He, C. H., A Remark on Wang's Fractal Variational Principle, *Fractal*s, *29* (2019), 8, 1950134
[9] Wang, Y., *et al*., A Fractal Derivative Model for Snow's Thermal Insulation Property, *Thermal Science, 23* (2019), 4, pp. 2351-2354
[10] Liu, H. Y., *et al*., A Fractal Rate Model for Adsorption Kinetics at Solid/Solution Interface, *Thermal Science*, *23* (2019), 4, pp. 2477-2480
[11] Shen, Y., El-Dib, Y. O., A Periodic Solution of the Fractional Sine-Gordon Equation Arising in Architectural Engineering, *Journal of Low Frequency Noise Vibration and Active Control*, On-line first, https://doi.org/10.1177/1461348420917565, 2020
[12] Yang, Z. P., *et al*., A Fractal Model for Pressure Drop through a Cigarette Filter, *Thermal Science*, *24* (2020), 4, pp. 2653-2659
[13] He, J. H., Thermal Science for the Real World: Reality and Challenge, *Thermal Science*, *24* (2020), 4, pp. 2289-2294
[14] He, J. H., A Fractal Variational Theory for 1-D Compressible Flow in a Microgravity Space, *Fractal*s, *28* (2020), 2, 2050024
[15] He, C. H., *et al*., Taylor Series Solution for Fractal Bratu-Type Equation Arising in Electrospinning Process, *Fractals*, *28* (2020), 1, 2050011
[16] He, J. H., A Simple Approach to 1-D Convection-Diffusion Equation and Its Fractional Modification for E Reaction Arising in Rotating Disk Electrodes, *Journal of Electroanalytical Chemistry, 854* (2019), 113565
[17] Zhang, J. J., *et a*l., Some Analytical Methods for Singular Boundary Value Problem in a Fractal Space, A*ppl. Comput. Math*., *18* (2019), 3, pp. 225-235
[18] Ji, F. Y., et al., A Fractal Boussinesq Equation for Non-Linear Transverse Vibration of a Nanofiber-Reinforced Concrete Pillar, *Applied Mathematical Modelling*, *82* (2020), June, pp. 437-448
[19] Shen, Y., and He, J. H., Variational Principle for a Generalized KdV Equation in a Fractal Space, *Fractals*, *28* (2020), 04, 2050069
[20] Li, X. J., *et al*., A Fractal Two-Phase Flow Model for the Fiber Motion in a Polymer Filling Process, *Fractals*, *28* (2020), 05, 2050093
[21] Ain, Q. T., He, J. H., On Two-Scale Dimension and Its Application, *Thermal Science*, *23* (2019), 3B, pp. 1707-1712
[22] He, J. H., Ji, F. Y., Two-Scale Mathematics and Fractional Calculus for Thermodynamics, *Thermal Science*, *23* (2019), 4, pp. 2131-2133
[23] He, J.H., Ain, Q.T. New Promises and Future Challenges of Fractal Calculus: From Two-Scale Thermodynamics to Fractal Variational Principle, *Thermal Science*, *24* (2020), 2A, pp. 659-681
[24] He, J. H., The Simplest Approach to Non-Linear Oscillators, *Results Phys*, *15* (2019), 102546
[25] He, J. H., Some Asymptotic Methods for Strongly Non-Linear Equations, *Int. J. Mod. Phys. B*, *13* (2006), 20, pp. 1141-1199
[26] He, J. H., The Simpler, The Better: Analytical Methods for Non-Linear Oscillators and Fractional Oscillators, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1252-1260
[27] Yao, X., He, J. H., On Fabrication of Nanoscale Non-Smooth Fibers with High Geometric Potential and Nanoparticle's Non-Linear Vibration, *Thermal Science*, *24* (2020), 4, pp. 2491-2497

[28] He, J. H., Variational Principle and Periodic Solution of the Kund-Mukherjee-Naskar equation, *Results in Physics*, *17* (2020), June, 103031

[29] He, C.H., *et al*., Fangzhu: An Ancient Chinese Nanotechnology for Water Collection from Air: History, Mathematical Insight, Promises and Challenges, *Mathematical Methods in the Applied Sciences*, On-line first, https://doi.org/10.1002/mma.6384, 2020

[30] Liu, C. X., A Short Remark on He's Frequency Formulation, *Journal of Low Frequency Noise*, *Vibration and Active Control*, On-line first, https://doi.org/10.1177/1461348420926331, 2020

[31] Li, X. X., *et al*., Homotopy Perturbation Method Coupled with the Enhanced Perturbation Method, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1399-1403

[32] Ren, Z. F., and Wu, J. B., He's Frequency-Amplitude Formulation for Non-Linear Oscillator with Damping, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1045-1049

[33] Ren, Z. F., and Hu, G. F., He's Frequency-Amplitude Formulation with Average Residuals for Non-Linear Oscillators, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1050-1059

[34] Wang, Q. L., *et al*., A Short Remark on Ren-Hu's Modification of He's Frequency-Amplitude Formulation and the Temperature Oscillation in a Polar Bear Hair, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1374-1377

[35] Ren, Z. F., and Hu, G. F., Discussion on the Accuracies of He's Frequency-Amplitude Formulation and Its Modification with Average Residuals, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1713-1715

[36] He, C. H., et al., A Complement to Period/Frequency Estimation of a Non-Linear Oscillator, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 992-995

[37] Tao, Z. L., *et al*., Approximate Frequency-Amplitude Relationship for a Singular Oscillator, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1036-1040

[38] Yu, D. N., *et al*., Homotopy Perturbation Method with an Auxiliary Parameter for Non-Linear Oscillators, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1540-1554

[39] He, J. H., and Xiao, J., A Short Review on Analytical Methods for the Capillary Oscillator in a Nanoscale Deformable Tube, *Mathematical Methods in the Applied Sciences*, On-line first, https://doi.org/10.1002/mma.6321, 2020

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