EXACT SOLUTIONS OF SPACE-TIME FRACTIONAL (2+1)-DIMENSIONAL BREAKING SOLITON EQUATION

by

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This paper suggests a direct algebraic method for finding exact solutions of the space-time fractional (2+1)-dimensional breaking soliton equation. The solution procedure is reduced to solve a large system of algebraic equations, which is then solved by Wu's method.

Key words: *fractional (2+1)-dimensional breaking soliton equation, Wu's method, exp-function method, direct algebraic method*

Introduction

The fractional-order non-linear partial differential equations (NPDE) arise in many fields like the elasticity, solid state physics, gas dynamics, material and others, the investigation of the exact solutions is one of the central themes in mathematics and physics. In the past decades, many methods have been developed to obtain exact solutions of fractional-order NPDE. Some of the most important methods are the homotopy perturbation method [1-3], the variational iteration method [4-9], and the exp-function method [10-13].

In this paper, exact solutions of space-time fractional (2+1)-dimensional breaking soliton equation is considered. The solution procedure of the direct algebraic method can be reduced to solve a large system of algebraic equations.

The direct algebraic method with modified Riemann-Liouville derivative

In this section, we outline the main steps of the direct algebraic method with modified Riemann-Liouville derivative for finding exact solutions of fractional-order NPDE. The Jumaries' modified Riemann-Liouville derivatives of fractional-order α is defined by the following expression [14]:

$$\mathrm{D}_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_0^t (t-\xi)^{-\alpha} \left[f(\xi) - f(0)\right] \mathrm{d}\xi & 0 < \alpha < 1 \\ \left[f^{(n)}(t)\right]^{(\alpha - n)} & n \le \alpha < n+1,\ n \ge 1 \end{cases}$$

and three important properties for the modified Riemann-Liouville derivative:

$$\mathrm{D}_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha},\ \mathrm{D}_t^\alpha \left[f(t)g(t)\right] = g(t)\mathrm{D}_t^\alpha f(t) + f(t)\mathrm{D}_t^\alpha g(t)$$

$$\mathrm{D}_t^\alpha f[g(t)] = f_g'[g(t)]\mathrm{D}_t^\alpha g(t) = \mathrm{D}_t^\alpha f[g(t)]\left[g'(t)\right]^\alpha$$

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Consider the fractional-order NPDE in the following form:

$$Q(u,D_t^\alpha u,D_{x_1}^\alpha u,D_{x_2}^\alpha u,\cdots,D_t^{2\alpha}u,D_{x_1}^{2\alpha}u,D_{x_2}^{2\alpha}u,\cdots)=0 \tag{1}$$

Step 1. The fractional complex transform is introduced:

$$\begin{cases} u(t,x_1,x_2,\cdots,x_n)=u(\xi) \\ \xi=\dfrac{ct^\alpha}{\Gamma(1+\alpha)}+\dfrac{k_1x_1^\alpha}{\Gamma(1+\alpha)}+\dfrac{k_2x_2^\alpha}{\Gamma(1+\alpha)}+\cdots+\dfrac{k_nx_n^\alpha}{\Gamma(1+\alpha)} \end{cases} \tag{2}$$

where c, k_1, k_2,..., k_n are arbitrary constants, the eq. (2) transform eq. (1) into an ODE:

$$\bar{Q}(u,cu',k_1u',k_2u',\cdots,c^2u'',k_1^2u'',k_2^2u'',\cdots)=0 \tag{3}$$

Equation (2) is called as the fractional complex transform [14-16], and it can be explained by the two-scale fractal [17-19].

Step 2. Suppose that the solution of eq. (3) can be expressed:

$$u(\xi)=\sum_{i=0}^{N}b_iQ(\xi)^i, \qquad b_N\neq 0 \tag{4}$$

where $Q(\xi)$ satisfies:

$$Q'(\xi)=\ln(\mathrm{A})\left[\alpha+\beta Q(\xi)+\sigma Q(\xi)^2\right], \qquad A\neq 0,1 \tag{5}$$

and $b_i(0\leq i\leq N)$ to be determined later. The N can be determined by balancing the highest order derivative terms with the non-linear terms of the highest order in eq. (3).

Step 3. Substituting eq. (4) along with eq. (5) into eq. (3) and equating all the coefficients of same power of $Q(\xi)$ to zero, we obtained a system of algebraic equations, the obtaining system can be solved to find the value of c, k_1, k_2,..., k_n, $b_i(0\leq i\leq N)$.

Fractional (2+1)-dimensional breaking soliton equation

In this section, we consider the space-time fractional (2+1)-dimensional breaking soliton equation [20]:

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha}+a\dfrac{\partial^{3\alpha}u}{\partial x^{2\alpha}\partial y^\alpha}+4au\dfrac{\partial^\alpha v}{\partial x^\alpha}++4av\dfrac{\partial^\alpha u}{\partial x^\alpha}=0 \\ \dfrac{\partial^\alpha u}{\partial y^\alpha}=\dfrac{\partial^\alpha v}{\partial x^\alpha} \end{cases} \tag{6}$$

By the fractional complex transform [14-19]:

$$\begin{cases} u(x_1,x_2,y_1,y_2,t)=u(\xi) \\ \xi=\dfrac{ct^\alpha}{\Gamma(1+\alpha)}+\dfrac{kx^\alpha}{\Gamma(1+\alpha)}+\dfrac{ly^\alpha}{\Gamma(1+\alpha)} \end{cases} \tag{7}$$

where c, k, l are arbitrary constants with c, k, $l\neq 0$. Equation (6) can be written:

$$\begin{cases} cu'+ak^2lu'''+4akuv'+4akvu'=0 \\ lu'=kv' \end{cases} \tag{8}$$

from $lu'=kv'$, we can obtain $v=(l/k)u$, and eq. (8) can be converted:

$$cu'+ak^2lu'''+8aluu'=0 \tag{9}$$

integrating eq. (9) once, we get:

$$cu + ak^2 lu'' + 4alu^2 = 0 \tag{10}$$

Suppose that the solution of eq. (10) can be expressed:

$$u(\xi) = \sum_{i=0}^{N} b_i Q(\xi)^i \tag{11}$$

where $b_i (0 \le i \le N)$ are constants to be determined, such that $b_N \neq 0$.

Consider the homogeneous balance between the highest order derivative and non-linear term in eq. (10), we have $N = 2$, then eq. (10) has the following solutions:

$$u(\xi) = b_0 + b_1 Q(\xi) + b_2 Q(\xi)^2, \quad b_2 \neq 0 \tag{12}$$

substituting eq. (12) along with eq. (5) into eq. (10) and collecting all the terms with the same power of $Q(\xi)$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving algebraic equations with the aid of Wu's method [21], we have two sets of solutions:

Case 1

$$b_2 = \frac{3\left[c^2 - 2ack^2 l\beta^2 \ln(A)^2 + a^2 k^4 l^2 \beta^4 \ln(A)^4\right]}{32a^2 k^2 l^2 \alpha^2 \ln(A)^2}, \quad b_1 = -\frac{3\left[c\beta - ak^2 l\beta^3 \ln(A)^2\right]}{8al\alpha}$$

$$b_0 = -\frac{3\left[c - ak^2 l\beta^2 \ln(A)^2\right]}{8al}, \quad c = ak^2 l\beta^2 \ln(A)^2 - 4ak^2 l\alpha\sigma \ln(A)^2$$

Case 2

$$b_2 = \frac{3\left[c^2 + 2ack^2 l\beta^2 \ln(A)^2 + a^2 k^4 l^2 \beta^4 \ln(A)^4\right]}{32a^2 k^2 l^2 \alpha^2 \ln(A)^2}, \quad b_1 = \frac{3\left[c\beta + ak^2 l\beta^3 \ln(A)^2\right]}{8al\alpha}$$

$$b_0 = \frac{c + 3ak^2 l\beta^2 \ln(A)^2}{8al}, \quad c = -ak^2 l\beta^2 \ln(A)^2 + 4ak^2 l\alpha\sigma \ln(A)^2$$

We consider only the solution with respect to *Case 1*, the other solution can be obtained in a similar way:

– when $\beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$

$$u_{1-5} = F + \frac{FWi\beta}{\alpha} + \frac{2F^2 Wi^2}{3k^2\alpha^2 \ln(A)^2}, \quad 1 \le i \le 5$$

where

$$M = 4\alpha\sigma - \beta^2, \quad F = -\frac{3\left[c - ak^2 l\beta^2 \ln(A)^2\right]}{8al}, \quad W_1 = -\frac{\beta}{2\sigma} + \frac{\sqrt{M}\ \tan a\left[\dfrac{\sqrt{M}}{2}\xi\right]}{2\sigma}$$

$$W_2 = -\frac{\beta}{2\sigma} - \frac{\sqrt{M}\ \cot a\left[\dfrac{\sqrt{M}}{2}\xi\right]}{2\sigma}, \quad W_3 = -\frac{\beta}{2\sigma} + \frac{\sqrt{M}\left(\pm\sqrt{pq}\ \sec a\left[\dfrac{\sqrt{M}}{2}\xi\right] + \tan a\left[\dfrac{\sqrt{M}}{2}\xi\right]\right)}{2\sigma}$$

$$W_4 = -\frac{\beta}{2\sigma} + \frac{\sqrt{M}\left(-\cot a\left[\sqrt{M}\xi\right] \pm \sqrt{pq}\ \csc a\left[\sqrt{M}\xi\right]\right)}{2\sigma}, \quad W_5 = -\frac{\beta}{2\sigma} + \frac{\sqrt{M}\left(-\cot a\left[\dfrac{\sqrt{M}\xi}{4}\right] + \tan a\left[\dfrac{\sqrt{M}\xi}{4}\right]\right)}{4\sigma}$$

– when $\beta^2-4\alpha\sigma>0$ and $\sigma\neq 0$

$$u_{6-10}=F+\frac{FWi\beta}{\alpha}+\frac{2F^2Wi^2}{3k^2\alpha^2\ln(A)^2},\quad 6\leq i\leq 10$$

where

$$M=\beta^2-4\alpha\sigma,\ F=-\frac{3\left[c-ak^2l\beta^2\ln(A)^2\right]}{8al},\ W_6=-\frac{\beta}{2\sigma}-\frac{\sqrt{M}\ \tanh a\left[\frac{\sqrt{M}}{2}\xi\right]}{2\sigma}$$

$$W_7=-\frac{\beta}{2\sigma}-\frac{\sqrt{M}\coth a\left[\frac{\sqrt{M}}{2}\xi\right]}{2\sigma},\ W_8=-\frac{\beta}{2\sigma}+\frac{\sqrt{M}\left(\pm i\sqrt{pq}\,\text{sech}\,a\left[\sqrt{M}\xi\right]-\tanh a\left[\sqrt{M}\xi\right]\right)}{2\sigma}$$

$$W_9=-\frac{\beta}{2\sigma}+\frac{\sqrt{M}\left[-\coth a\left[\sqrt{M}\xi\right]\pm\sqrt{pq}\ \text{csch}\,a\left[\sqrt{M}\xi\right]\right]}{2\sigma}$$

$$W_{10}=-\frac{\beta}{2\sigma}-\frac{\sqrt{M}\left(\coth a\left[\frac{\sqrt{M}\xi}{4}\right]+\tanh a\left[\frac{\sqrt{M}\xi}{4}\right]\right)}{4\sigma}$$

– when $\alpha\sigma>0$ and $\beta=0$:

$$u_{11}=F+\frac{2F^2\tan a\left(\xi\sqrt{\alpha\sigma}\right)^2}{3k^2\alpha\sigma\ln(A)^2},\ u_{12}=F+\frac{2F^2\cot a\left(\xi\sqrt{\alpha\sigma}\right)^2}{3k^2\alpha\sigma\ln(A)^2}$$

$$u_{13}=F+\frac{2F^2\left[\pm\sqrt{pq}\ \sec a\left(2\xi\sqrt{\alpha\sigma}\right)+\tan a\left(2\xi\sqrt{\alpha\sigma}\right)\right]^2}{3k^2\alpha\sigma\ln(A)^2}$$

$$u_{14}=F+\frac{2F^2\left[-\cot a\left(2\xi\sqrt{\alpha\sigma}\right)\pm\sqrt{pq}\csc a\left(2\xi\sqrt{\alpha\sigma}\right)\right]^2}{3k^2\alpha\sigma\ln(A)^2}$$

$$u_{15}=F+\frac{F^2\left[-\cot a\left(\frac{1}{2}\xi\sqrt{\alpha\sigma}\right)+\tan a\left(\frac{1}{2}\xi\sqrt{\alpha\sigma}\right)\right]^2}{6k^2\alpha\sigma\ln(A)^2},\ \text{and}\ F=-\frac{3c}{8al}$$

– when $\alpha\sigma<0$ and $\beta=0$

$$u_{16}=F-\frac{2F^2\tanh a\left(\xi\sqrt{-\alpha\sigma}\right)^2}{3k^2\alpha\sigma\ln(A)^2},\quad u_{17}=F-\frac{2F^2\coth a\left(\xi\sqrt{-\alpha\sigma}\right)^2}{3k^2\alpha\sigma\ln(A)^2}$$

$$u_{18}=F-\frac{2F^2\left[\pm i\sqrt{pq}\ \text{sech}\,a\left(2\xi\sqrt{-\alpha\sigma}\right)-\tanh a\left(2\xi\sqrt{-\alpha\sigma}\right)\right]^2}{3k^2\alpha\sigma\ln(A)^2}$$

$$u_{19}=F-\frac{2F^2\left[-\coth a\left(2\xi\sqrt{-\alpha\sigma}\right)\pm\sqrt{pq}\,\text{csch}\,a\left(2\xi\sqrt{-\alpha\sigma}\right)\right]^2}{3k^2\alpha\sigma\ln(A)^2}$$

$$u_{20} = F - \frac{F^2\left[\coth a\left(\frac{1}{2}\xi\sqrt{-\alpha\sigma}\right) + \tanh a\left(\frac{1}{2}\xi\sqrt{-\alpha\sigma}\right)\right]^2}{6k^2\alpha\sigma\ln(A)^2}, \text{ and } F = -\frac{3c}{8al}$$

– when $\beta = 0$ and $\sigma = \alpha$

$$u_{21} = F + \frac{2F^2\tan a(\alpha\xi)^2}{3k^2\alpha^2\ln(A)^2}, \; u_{22} = F + \frac{2F^2\cot a(\alpha\xi)^2}{3k^2\alpha^2\ln(A)^2}$$

$$u_{23} = F + \frac{2F^2\left[\tan a(2\alpha\xi) \pm \sqrt{pq}\sec a(2\alpha\xi)\right]^2}{3k^2\alpha^2\ln(A)^2}, \; u_{24} = F + \frac{F^2\left[-\cot a(2\alpha\xi) \pm \sqrt{pq}\csc a(2\alpha\xi)\right]^2}{3k^2\alpha^2\ln(A)^2}$$

$$u_{25} = F + \frac{F^2\left[-\cot a\left(\frac{\alpha\xi}{2}\right) + \tan a\left(\frac{\alpha\xi}{2}\right)\right]^2}{6k^2\alpha^2\ln(A)^2}, \text{ and } F = \frac{3c}{8al}$$

– when $\beta = 0$ and $\sigma = -\alpha$

$$u_{26} = F + \frac{2F^2\tanh a(\alpha\xi)^2}{3k^2\alpha^2\ln(A)^2}, \; u_{27} = F + \frac{2F^2\coth a(\alpha\xi)^2}{3k^2\alpha^2\ln(A)^2}$$

$$u_{28} = F + \frac{2F^2\left[\pm i\sqrt{pq}\,\mathrm{sech}\, a(2\alpha\xi) - \tanh a(2\alpha\xi)\right]^2}{3k^2\alpha^2\ln(A)^2}$$

$$u_{29} = F + \frac{2F^2\left[\coth a(2\alpha\xi) \pm \sqrt{pq}\,\mathrm{csch}\, a(2\alpha\xi)\right]^2}{3k^2\alpha^2\ln(A)^2}$$

$$u_{30} = F + \frac{F^2\left[\coth a\left(\frac{\alpha\xi}{2}\right) + \tanh a\left(\frac{\alpha\xi}{2}\right)\right]^2}{6k^2\alpha^2\ln(A)^2}, \text{ and } F = -\frac{3c}{8al}$$

Remark 1. The generalized hyperbolic and triangular functions are defined [22, 23]:

$$\sinh a(\xi) = \frac{pA^{\xi} - qA^{-\xi}}{2}, \cosh a(\xi) = \frac{pA^{\xi} + qA^{-\xi}}{2}, \tanh a(\xi) = \frac{pA^{\xi} - qA^{-\xi}}{pA^{\xi} + qA^{-\xi}}, \coth a(\xi) = \frac{pA^{\xi} + qA^{-\xi}}{pA^{\xi} - qA^{-\xi}}$$

$$\mathrm{sech}\, a(\xi) = \frac{2}{pA^{\xi} + qA^{-\xi}}, \mathrm{csch}\, a(\xi) = \frac{2}{pA^{\xi} - qA^{-\xi}}, \sec a(\xi) = \frac{2}{pA^{i\xi} + qA^{-i\xi}}, \csc a(\xi) = \frac{2i}{pA^{i\xi} - qA^{-i\xi}}$$

$$\sin a(\xi) = \frac{pA^{i\xi} - qA^{-i\xi}}{2i}, \; \cos a(\xi) = \frac{pA^{i\xi} + qA^{-i\xi}}{2i}$$

$$\tan a(\xi) = -i\frac{pA^{i\xi} - qA^{-i\xi}}{pA^{i\xi} + qA^{-i\xi}}, \; \cot a(\xi) = i\frac{pA^{i\xi} + qA^{-i\xi}}{pA^{i\xi} - qA^{-i\xi}}$$

where ξ is an independent variable and $p, q > 0$.

Conclusion

In this paper, we use the direct algebraic method combined with Wu's method to solve the space-time fractional (2+1)-dimensional breaking soliton equation, this process can be reduced to solve a large system of algebraic equations, which is hard to solve, then we use Wu's method to solve the algebraic equations. The results show the effectiveness of this method, which can be also extended to other fractional differential equations with different definitions for fractional derivative, especially He's fractional derivative [24-27], and fractal calculus [28-30]. Additionally Lie symmetry and conservation laws for fractional partial differential equations [31-33] and integro-differential equations [34], and quenching phenomenon [35, 36] will be the research frontier in future.

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