STUDYING ON KUDRYASHOV-SINELSHCHIKOV DYNAMICAL EQUATION ARISING IN MIXTURES OF LIQUID AND GAS BUBBLES

by

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In this paper, some new exact traveling and oscillatory wave solutions to the Kudryashov-Sinelshchikov non-linear PDE are investigated by using Bernoulli sub-equation function method. Profiles of obtained solutions are plotted.

Keywords: *Kudryashov-Sinelshchikov non-linear PDE, Bernoulli sub-equation function method, traveling wave, oscillatory wave*

Introduction

The analysis of propagation of the pressure waves in a mixture liquid with gas bubbles is one of the most important problems in nature, mathematical physics, and engineering. The Kudryashov-Sinelshchikov (KS) modeled a more general non-linear PDE to explain the pressure waves in a mixture liquid and gas bubbles based on the viscosity of liquid and heat transmission [1, 2]. The KS equation is known:

$$u_t + \gamma u u_x + u_{xxx} - \varepsilon(u u_{xx})_x - \alpha u_x u_{xx} - \lambda u_{xx} - \delta(u u_x)_x = 0 \tag{1}$$

where u is density and models heat transfer and viscosity, and γ, ε, α, λ, and δ are real parameters determining pressure waves in the liquid with gas bubbles taking into account the heat transfer and viscosity. In eq. (1), if $\varepsilon = \alpha = \lambda = \delta = 0$, then:

$$u_t + \gamma u u_x + u_{xxx} = 0 \tag{2}$$

is known as the Korteweg-de Vries equation characterizing pressure waves in gas-liquid mixture. Taking $\varepsilon = \alpha = \delta = 0$ in eq. (1), one gets Korteweg-de Vries-Burgers equation [3]:

$$u_t + \gamma u u_x + u_{xxx} - \lambda u_{xx} = 0 \tag{3}$$

Several methods are used and applied to obtain traveling and oscillating wave solutions. The G'/G-polynomial expansion method [4], modification of truncated expansion method applied [5], Lie symmetry analysis [6, 7], F-expansion method and its improved method

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[8, 9], Riccati-Bernoulli sub-ODE method [10], the method of simplest equation [11, 12], generalized Kudryashov method [13, 14], the approach of dynamical systems [15], Bernoulli sub-equation function method [16, 17], modified method of simplest equation [18, 19], Expansion function method [20], sine-Gordon expansion method [21], Bifurcation method, improved F-expansion method and modified exp-function method [22-24], solutions obtained from a generalized Korteweg-de Vries equation [25], radial basis function method [26], Long wave limit method [27], and modified mathematical method [28].

Bernoulli sub-equation function method

Bernoulli sub-equation function method (BSEFM) is outlined in the succeeding steps.

Step 1. Consider the following non-linear PDE with $u = u(x, t)$:

$$P(u_x, u_t, u_{xt}, u_{xx}, \cdots) = 0 \tag{4}$$

set

$$u(x,t) = U(\zeta), \quad \zeta = kx - ct \tag{5}$$

where k and c are real non-zero constants. By substituting eq. (5) into eq. (4), then the following ODE is obtained:

$$N(U, U', U'', \cdots) = 0 \tag{6}$$

where $U = U(\zeta),\ U' = \dfrac{\mathrm{d}U}{\mathrm{d}\zeta},\ U'' = \dfrac{\mathrm{d}^2U}{\mathrm{d}\zeta^2}, \cdots$.

Step 2. Solution of eq. (6) is assumed to be:

$$U(\zeta) = \sum_{i=0}^{n} a_i F^i = a_0 + a_1 F + a_2 F^2 + \cdots + a_n F^n \tag{7}$$

where

$$F' = bF + dF^M, \quad b \neq 0, \quad d \neq 0, \quad M \in \mathbb{R} - \{0,1\} \tag{8}$$

where $F(\zeta)$ is well known Bernoulli differential equation. Also, b, d, and a_i with $a_n \neq 0$ should be determined later. Substituting eqs. (8) and (7) into eq. (6), one immediately gets:

$$\Psi[F(\zeta)] = \vartheta_\sigma F^\sigma(\zeta) + \cdots + \vartheta_1 F(\zeta) + \vartheta_0 = 0$$

By balancing principles, one starts getting a formula between n and M by comparing the highest order derivatives with highest power of non-linear terms in eq. (6). Let the coefficients of $\Psi[F(\zeta)]$ be equal zero. Then one gets:

$$\vartheta_i = 0, \quad i = 0, \ldots, \sigma \tag{9}$$

By solving eq. (9) with a computerized program, one gets a_i with $i = 0, 1, \ldots, \sigma$.

Step 3. It is trivial that solution to Bernoulli differential eq. (8) are:

$$F(\zeta) = \left[-\frac{d}{b} + \frac{E}{\mathrm{e}^{b(M-1)\zeta}} \right]^{\frac{1}{1-M}}, \quad b \neq d,$$

$$F(\zeta)=\left\{\frac{E-1+(E+1)\tanh\left[\dfrac{b(1-M)\zeta}{2}\right]}{1-\tanh\left[\dfrac{b(1-M)\zeta}{2}\right]}\right\}^{\frac{1}{1-M}},\quad b=d,\quad E\in\mathbb{R}$$

Kudryashov-Sinelshchikov equation solved by BSEFM

In this section, BSEFM is effectively applied to KS equation to find some new complex and real exponential solutions. Firstly, by applying eq. (5) to eq. (1), one right away gets:

$$-cU'+\gamma kUU'+k^3U'''-k^3\varepsilon(UU'''+U'U'')-k^3\alpha U'U''-k^2\lambda U''-k^2\delta(UU''+(U')^2=0 \tag{10}$$

where U, U', U'', and U''' are assumed to be zero at initial points. After integrating eq. (10), one gets:

$$-cU+\frac{\gamma k}{2}U^2+k^3U''-k^3\varepsilon UU''-\frac{k^3\alpha}{2}U'^2-k^2\lambda U'-k^2\delta UU'=0 \tag{11}$$

By applying balance principle in eq. (11), then one reads relationship between n and M as:

$$2M=n+2 \tag{12}$$

From eq. (12), according to the values of M and n the following cases are analyzed.

Case 1. If $M=2$ and $n=2$ in eq. (12), then eq. (7) becomes:

$$\begin{gathered}U=a_0+a_1F+a_2F^2,\quad U'=a_1bF+a_1dF^2+2a_2bF^2+2a_2dF^3\\ U''=b^2a_1F+3bda_1F^2+2d^2a_1F^3+4b^2a_2F^2+10bda_2F^3+6d^2a_4F^2\end{gathered} \tag{13}$$

where each of b, d, and a_2 are non-zero. Putting eq. (13) into eq. (11), a system of algebraic equation is produced. By solving obtained system by computerized computations, then exact solutions are derived as subcases indicated below.

Case 1.1. For $b\neq d$, then the following coefficients are obtained.

$$a_1=\frac{-b\varepsilon a_2+\sqrt{\varepsilon}\sqrt{a}_2\omega}{4\varepsilon},\quad k=1,\quad \gamma=\frac{-16a_2-b^2\varepsilon a_2^2+b\sqrt{\varepsilon}a_2^{3/2}\omega}{2a_2^2}$$

$$\alpha=-3\varepsilon,\quad \delta=\frac{ba_2+\sqrt{\varepsilon}\sqrt{a}_2\omega}{2a_2},\quad a_0=0,\quad c=\frac{-24b^2a_2^2\dfrac{40ba_2^{3/2}\omega}{\sqrt{\varepsilon}}+b^3\sqrt{\varepsilon}a_2^{5/2}\omega-\dfrac{ba_2^{3/2}\omega^3}{\sqrt{\varepsilon}}}{16a_2^2} \tag{14}$$

$$\lambda=\frac{40ba_2^2+\dfrac{40a_2^{3/2}\omega}{\sqrt{\varepsilon}}-b^2\sqrt{\varepsilon}a_2^{5/2}\omega+\dfrac{a_2^{3/2}\omega^3}{\sqrt{\varepsilon}}}{16a_2^2}$$

where $\omega=\sqrt{-32+b^2\varepsilon a_2}$. Choosing suitable non-zero values of a_2, b, d, and ε in eq. (14), then following solution is obtained:

$$u_{1.1}(x,t) = -\left\{1+\mathrm{e}^{-2[(6-2i)t+x]}EE\right\}^{-2} + \frac{1+i}{2}\frac{1}{1+\mathrm{e}^{-2[(6-2i)t+x]}EE} \tag{15}$$

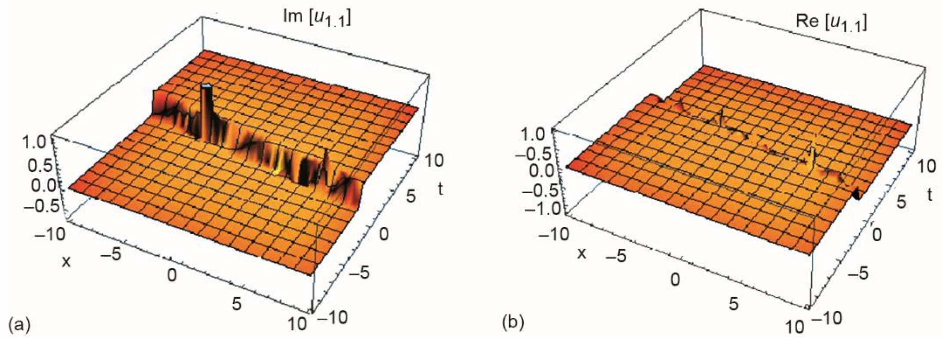

Figure 1. The 3-D graph for eq. (15), with *EE* = 0.6, –10 < *x* < 10, –10 < *t* < 10
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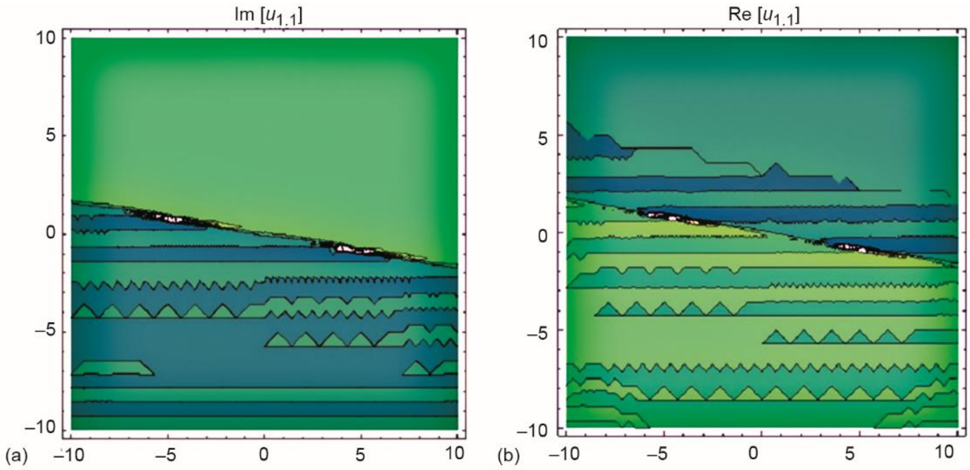

Figure 2. Contour surfaces for eq. (15), with *EE* = 0.6, –10 < *x* < 10, –10 < *t* < 10
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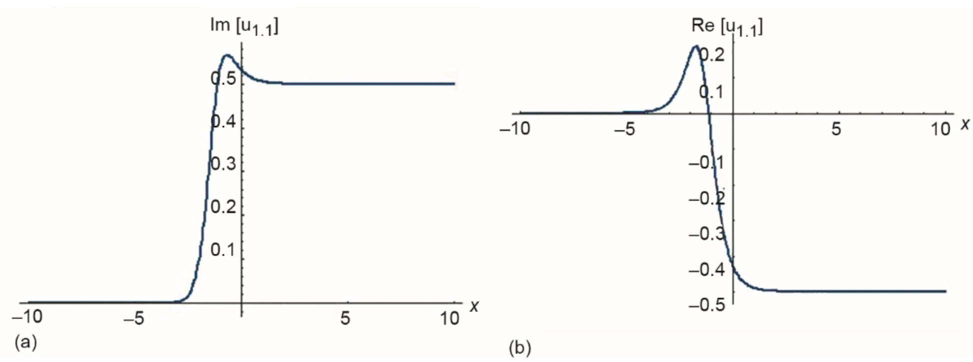

Figure 3. The 2-D graph for eq. (15), with *EE* = 0.6, –10 < x < 10, *t* = 0.2

Case 1.2. For $b \neq d$, then the following coefficients are:

$$a_0 = -\frac{1}{4}b^2 a_2, \quad a_1 = 0, \quad k = 1, \quad c = \frac{3}{2}(4b^2 + b^4 \varepsilon a_2), \quad \delta = b\varepsilon$$
$$\gamma = -\frac{12(4 + b^2 \varepsilon a_2)}{a_2}, \quad \lambda = \frac{1}{2}b(10 + 3b^2 \varepsilon a_2) \tag{16}$$

By determining non-zero values of a_2, b, d, and ε from eq. (16), then the following real exponential solution is:

$$u_{1.2}(x,t) = \frac{1}{4} - \frac{1}{\left(2 + \mathrm{e}^{\frac{9t}{2} - x} EE\right)^2} \tag{17}$$

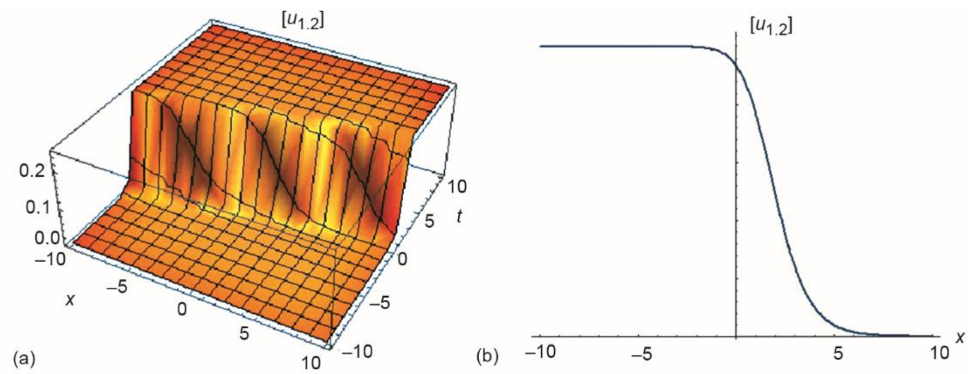

Figure 4. The 3-D and 2-D graphs for (17), with EE = 0.4, –10 < x < 10, –10 < t < 10, t = 0.2
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Case 1.3. Coefficients is given bellow is also generated where $b \neq d$:

$$a_0 = \frac{b^2 \varepsilon a_2^2 + b\sqrt{\varepsilon} a_2^{3/2} \omega}{8\varepsilon a_2}, \quad a_1 = \frac{-3b\varepsilon a_2 - \sqrt{\varepsilon}\sqrt{a_2}\,\omega}{4\varepsilon}, \quad k = 1, \quad \delta = \frac{-b a_2 - \sqrt{\varepsilon}\sqrt{a_2}\,\omega}{2a_2}$$
$$c = \frac{-24b^2 a_2^2 - \dfrac{40 b a_2^{3/2}\omega}{\sqrt{\varepsilon}} + b^3\sqrt{\varepsilon} a_2^{5/2}\omega - \dfrac{b a_2^{3/2}\omega^3}{\sqrt{\varepsilon}}}{16a_2^2}, \quad \gamma = \frac{-16a_2 - b^2 \varepsilon a_2^2 + b\sqrt{\varepsilon} a_2^{3/2}\omega}{2a_2^2} \tag{18}$$
$$\lambda = \frac{-40 b a_2^2 - \dfrac{40 a_2^{3/2}\omega}{\sqrt{\varepsilon}} + b^2\sqrt{\varepsilon} a_2^{5/2}\omega - \dfrac{a_2^{3/2}\omega^3}{\sqrt{\varepsilon}}}{16a_2^2}$$

where $\omega = \sqrt{-32 + b^2 \varepsilon a_2}$. For given suitable values of a_2, b, d, and ε in eq. (18), then one gets solution:

$$u_{1.3}(x,t) = \frac{-9 + 3i\sqrt{87}}{24} - \frac{3}{\left[2 + \mathrm{e}^{\frac{1}{144}\left(-216 + 24i\sqrt{87}\right)t - x} EE\right]^2} + \frac{9 - i\sqrt{87}}{4\left[2 + \mathrm{e}^{\frac{1}{144}\left(-216 + 24i\sqrt{87}\right)t - x} EE\right]} \tag{19}$$

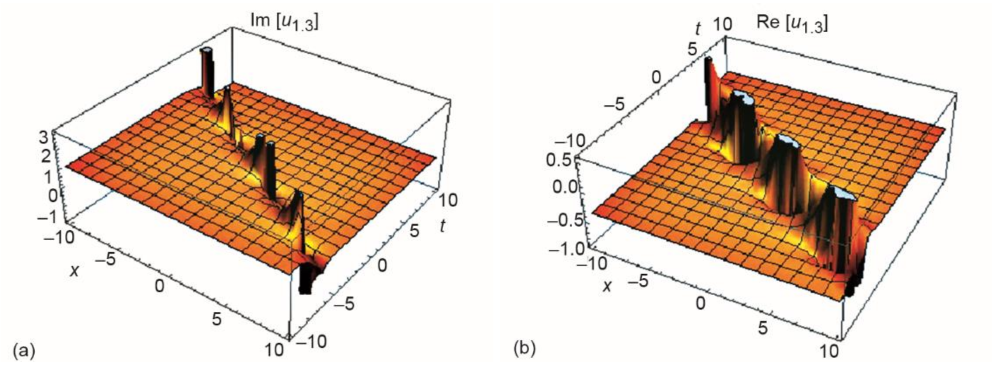

Figure 5. The 3-D graph for eq. (19), with EE = 1.2, –10 < x < 10, –10 < t < 10
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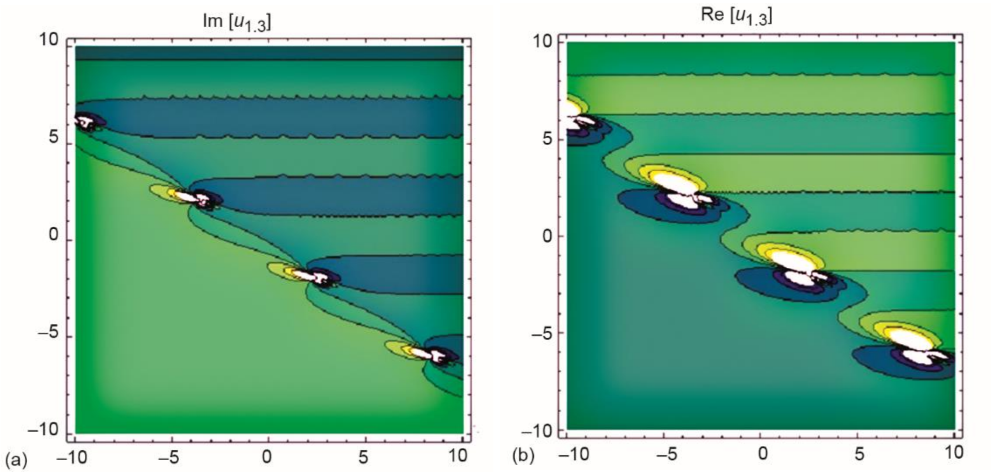

Figure 6. Contour surfaces for eq. (19), with EE = 1.2, –10 < x < 10, –10 < t < 10
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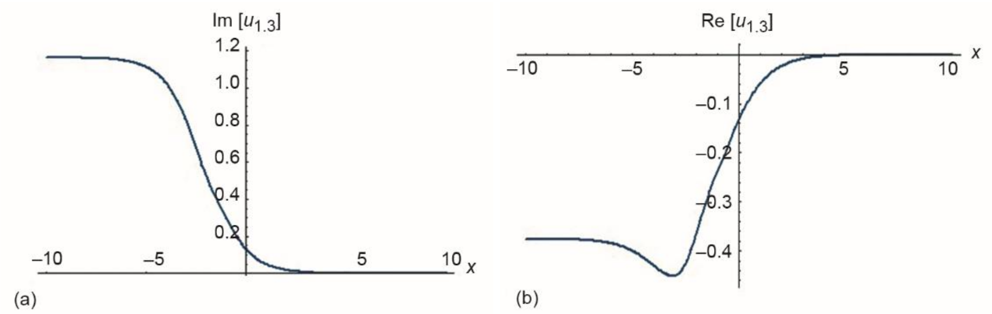

Figure 7. The 2-D graph for eq. (19), with EE = 1.2, –10 < x < 10, t = 0.6

Case 2. If one picks $M = 3$ and $n = 4$ in eq. (12), then eq. (7) reads:

$$
\begin{gathered}
U = a_0 + a_1 F + a_2 F^2 + a_3 F^3 + a_4 F^4 \\
U' = a_1 bF + a_1 dF^3 + 2a_2 bF^2 + 2a_2 dF^4 + 3a_3 bF^3 + 3a_3 dF^5 + 4a_4 bF^4 + 4a_4 dF^6 \\
U'' = b^2 Fa_1 + 4bdF^3 a_1 + 3d^2 F^5 a_1 + 4b^2 F^2 a_2 + 12bdF^4 a_2 + 8d^2 F^6 a_2 + 9b^2 F^3 a_3 + \\
+24bdF^5 a_3 + 15d^2 F^7 a_3 + 16b^2 F^4 a_4 + 40bdF^6 a_4 + 24d^2 F^8 a_4
\end{gathered}
\tag{20}
$$

where each of b, d, and a_4 are non-zero. Substituting eq. (20) into eq. (11) and if obtained system is evaluated by computerized computations, then exact solutions are derived in the following subcases.

Case 2.1. For $b \neq d$, from the previous, the following new coefficients are:

$$
\begin{gathered}
a_0 = 0, \quad a_2 = 0, \quad k = \frac{i\sqrt{\gamma}\sqrt{a_4}}{8\sqrt{3}}, \quad c = -\frac{i\sqrt{3}\sqrt{\gamma}\delta^2\sqrt{a_4}}{4\varepsilon^2} \\
\lambda = \frac{5\delta}{\varepsilon}, \quad b = -\frac{4i\sqrt{3}\delta}{\sqrt{\gamma}\varepsilon\sqrt{a_4}}, \quad \alpha = -3\varepsilon, \quad a_3 = 0, \quad a_1 = 0
\end{gathered}
\tag{21}
$$

by determining non-zero values of γ, δ, ε, d, and a_4, then one finds another complex solution:

$$
u_{2.1}(x,t) = \frac{4}{\left[-i + \mathrm{e}^{4i\left(\frac{3it}{2}+\frac{ix}{4}\right)} EE\right]^2}
\tag{22}
$$

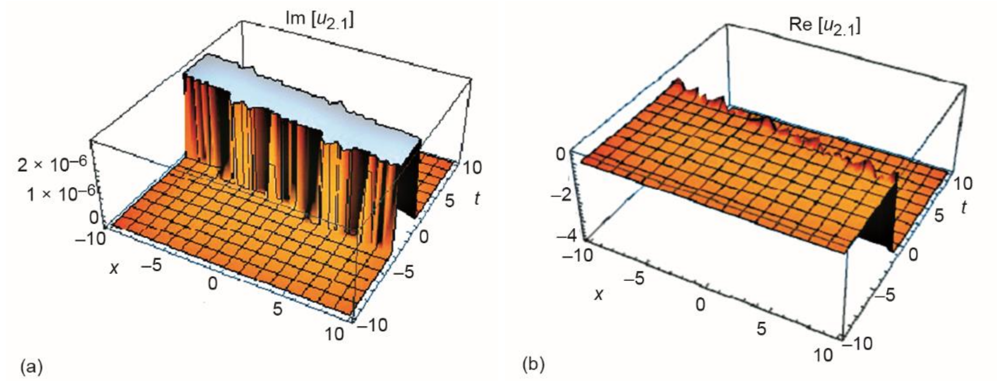

Figure 8. The 3-D graph for eq. (22), with *EE* = 0.6, –10 < *x* < 10, –10 < *t* < 10
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Case 2.2. With $b \neq d$, some new coefficients are:

$$
\begin{gathered}
\alpha = -3\varepsilon, \quad a_3 = 0, \quad a_1 = 0, \quad a_0 = -\frac{12\delta^2}{\gamma\varepsilon^2}, \quad a_2 = \frac{4\omega}{24\gamma^2\delta^2\varepsilon a_4^2 + \gamma^3\varepsilon^2 a_4^2}, \quad k = \frac{i\sqrt{\gamma}\sqrt{a_4}}{8\sqrt{3}} \\
c = -\frac{i\delta\omega}{4(24\gamma\delta^2\varepsilon^2 a_4^2 + \gamma^2\varepsilon^3 a_4^2)}, \quad \lambda = \frac{5\delta}{\varepsilon}, \quad b = \frac{4\omega}{24\gamma^2\delta^2\varepsilon a_4^3 + \gamma^3\varepsilon^2 a_4^3}
\end{gathered}
$$

where $\omega = 24i\sqrt{3}\gamma^{3/2}\delta^3 a_4^{5/2} + i\sqrt{3}\gamma^{5/2}\delta\varepsilon a_4^{5/2}$, by taking suitable positive values of γ, δ, ε, d, and a_4, then one gets following solution:

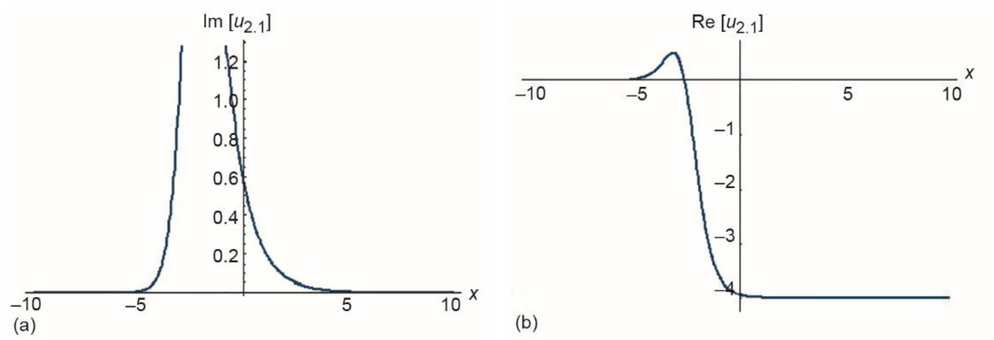

Figure 9. The 2-D graph for eq. (22), with *EE* = 0.6, –10 < *x* < 10, *t* = 0.4

$$u_{2.2}(x,t) = -4 + \frac{4}{\left[-i + \mathrm{e}^{4i\left(\frac{3it}{2}+\frac{ix}{4}\right)}EE\right]^2} - \frac{8i}{\left[-i + \mathrm{e}^{4i\left(\frac{3it}{2}+\frac{ix}{4}\right)}EE\right]} \tag{23}$$

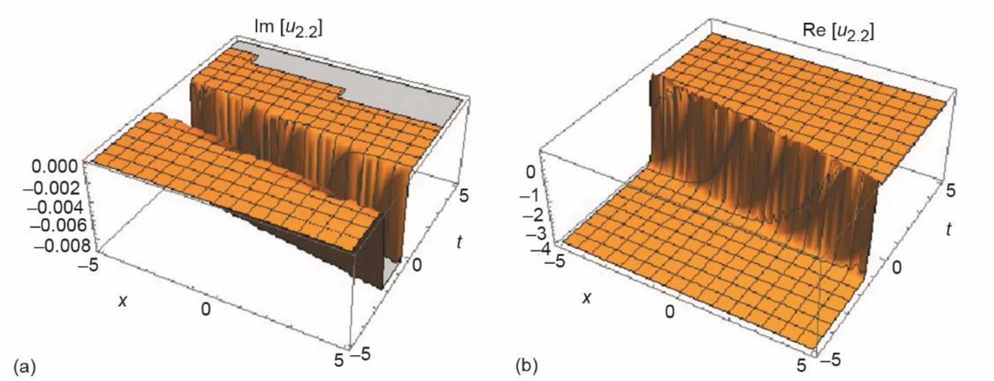

Figure 10. The 3-D graph for eq. (23), with *EE* = 0.5, –5 *x* < 5, –5 < *t* < 5
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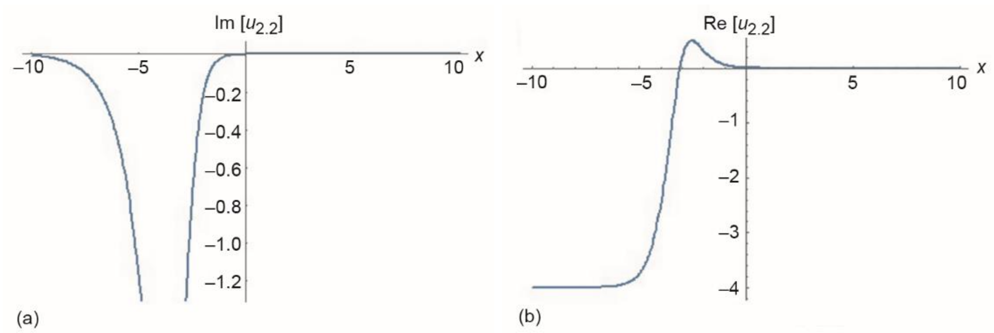

Figure 11. The 2-D graph for eq. (23), with *EE* = 0.5, –5 *x* < 5, *t* = 0.4

Case 2.3. When $b \neq d$, then another coefficients are found:

$$\alpha = -3\varepsilon, \quad a_3 = 0, \quad a_2 = 0, \quad a_1 = 0, \quad a_0 = \frac{12\delta^2}{\varepsilon(12\delta^2 + \gamma\varepsilon)}, \quad k = -\frac{\omega}{8\sqrt{3}\sqrt{\varepsilon}}$$

$$c = \frac{\sqrt{3}\gamma\delta^2\omega}{4\sqrt{\varepsilon}(-12\delta^2\varepsilon - \gamma\varepsilon^2)}, \quad \lambda = \frac{-12\delta^3 + 5\gamma\delta\varepsilon}{\varepsilon(12\delta^2 + \gamma\varepsilon)}, \quad b = \frac{4\sqrt{3}\delta\omega}{\sqrt{\varepsilon}(12\delta^2 a_4 + \gamma\varepsilon a_4)}$$

where $\omega = \sqrt{-(12\delta^2 + \gamma\varepsilon)a_4}$. Picking suitable positive values for *γ*, *δ*, *ε*, *d*, and $a_4 \neq 0$, then the following solution appears:

$$u_{2.3}(x,t) = \frac{12}{23} + \frac{4}{\left[-i\sqrt{\frac{23}{3}} + \mathrm{e}^{4i\sqrt{\frac{3}{23}}\left(-\frac{1}{2}i\sqrt{\frac{3}{23}}t - \frac{1}{4}i\sqrt{\frac{23}{3}}x\right)}EE\right]^2} \tag{24}$$

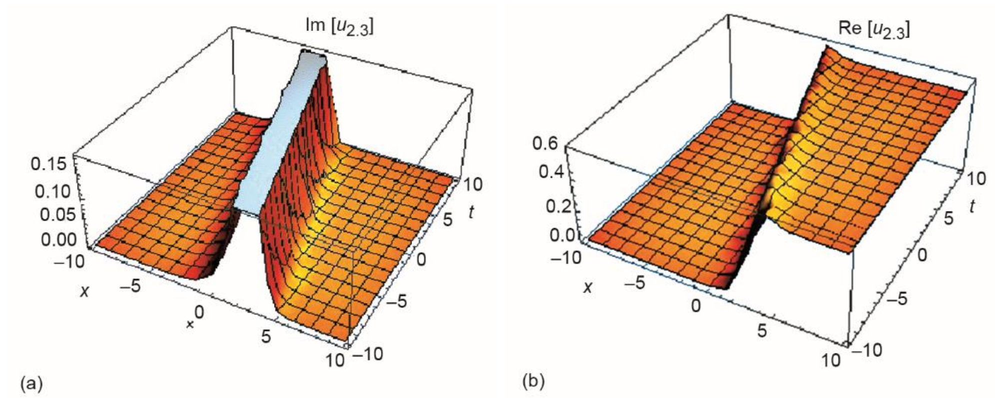

Figure 12. The 3-D graph for eq. (24), with *EE* = 1.2, –10 < *x* < 10, –10 < *t* < 10
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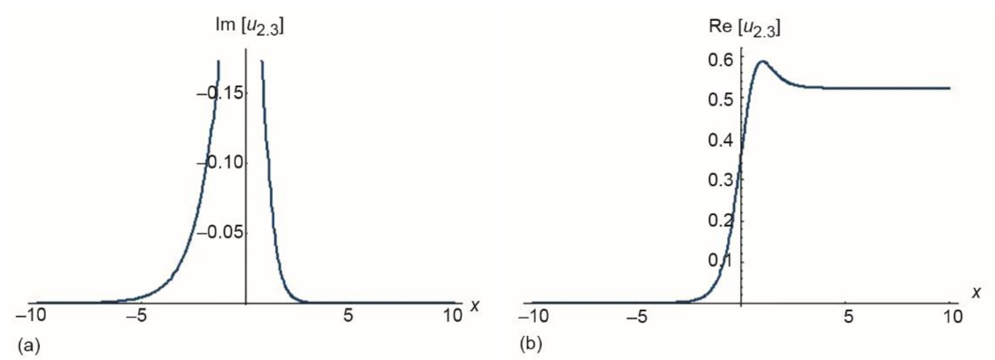

Figure 13. The 2-D graph for eq. (24), with *EE* = 1.2, –10 < *x* < 10, *t* = 1.4

Case 2.4. If $b \neq d$, then coefficients are:

$$\alpha=-3\varepsilon,\quad a_0=\frac{b^2 a_4}{5},\quad a_1=0,\quad a_2=\frac{6ba_4}{5},\quad a_3=0,\quad k=\frac{i\sqrt{\gamma}\sqrt{a_4}}{4\sqrt{3}}$$

$$c=\frac{ib^2\gamma^{3/2}a_4^{3/2}}{40\sqrt{3}},\quad \lambda=-\frac{11ib\sqrt{\gamma}\sqrt{a_4}}{10\sqrt{3}},\quad \delta=-\frac{5i\sqrt{\gamma}}{4\sqrt{3}b\sqrt{a_4}},\quad \varepsilon=\frac{25}{2b^2 a_4}$$

From the previous, by picking positive γ, ε, a_4, b, and d then following solution is occurred:

$$u_{2.4}(x,t)=\frac{4}{5}+\frac{4}{\left[1+\mathrm{e}^{2\left(-\frac{it}{5\sqrt{3}}+\frac{ix}{2\sqrt{3}}\right)}EE\right]^2}-\frac{24}{5\left[1+\mathrm{e}^{2\left(-\frac{it}{5\sqrt{3}}+\frac{ix}{2\sqrt{3}}\right)}EE\right]} \tag{25}$$

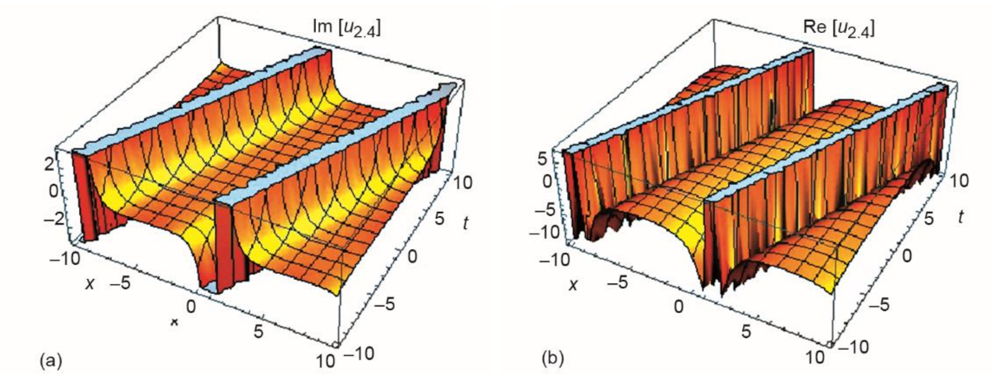

Figure 14. The 3-D graph for eq. (25), with *EE* = 0.6, –10 < *x* < 10, –10 < *t* < 10
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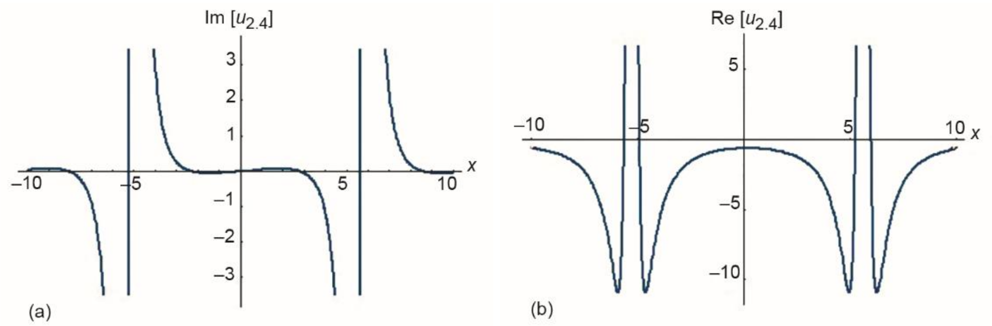

Figure 15. The 2-D graph for eq. (25), with *EE* = 0.6, –10 < *x* < 10, *t* = 0.5

Case 2.5. If $b \neq d$, coefficients of algebraic equations is:

$$a_0=b^2 a_4,\quad a_1=0,\quad a_2=2ba_4,\quad a_3=0,\quad \delta=-\frac{ib\sqrt{\gamma}\varepsilon\sqrt{a_4}}{2\sqrt{3}}$$

$$k=\frac{i\sqrt{\gamma}\sqrt{a_4}}{4\sqrt{3}},\quad c=\frac{ib^2\gamma^{3/2}a_4^{3/2}}{8\sqrt{3}},\quad \lambda=-\frac{5ib\sqrt{\gamma}\sqrt{a_4}}{2\sqrt{3}},\quad \alpha=-3\varepsilon$$

By choosing $\gamma \neq 0$, $\varepsilon \neq 0$, $a_4 \neq 0$, $b \neq 0$, and $d \neq 0$, then following solution is obtained:

$$u_{2.5}(x,t) = 1 + \frac{1}{\left[1 + \mathrm{e}^{2\left(-\frac{it}{\sqrt{3}} + \frac{ix}{2\sqrt{3}}\right)} EE\right]^2} - \frac{2}{\left[1 + \mathrm{e}^{2\left(-\frac{it}{\sqrt{3}} + \frac{ix}{2\sqrt{3}}\right)} EE\right]} \tag{26}$$

Solution $u_{2.5}(x,t)$ is plotted in 3-D, 2-D and its contour surfaces are given.

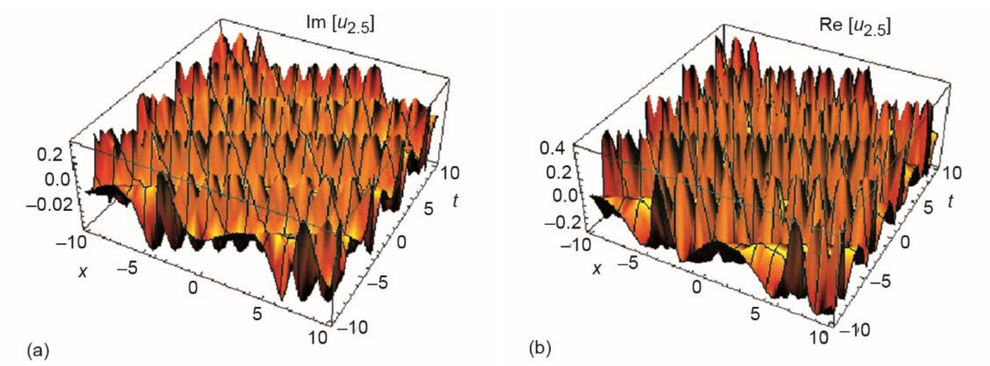

Figure 16. The 3-D graph for $u_{2.5}$ (x, t), with EE = 0.4, –10 < x < 10, –10 < t < 10
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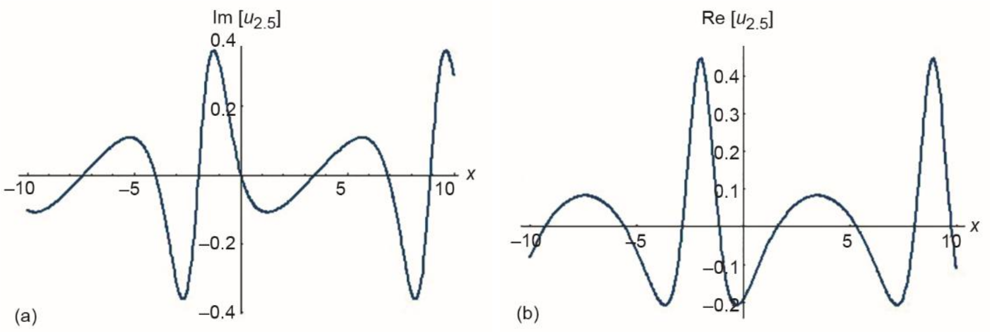

Figure 17. The 2-D graph for $u_{2.5}$ (x, t), with EE = 0.4, –10 < x < 10, t = $\sqrt{3}$

Case 3. If $M = 4$ and $n = 6$ in eq. (12), then eq. (7) becomes:

$$\begin{gathered} U = a_0 + a_1 F + a_2 F^2 + a_3 F^3 + a_4 F^4 + a_5 F^5 + a_6 F^6 \\ U' = a_1 bF + a_1 dF^4 + 2a_2 bF^2 + 2a_2 dF^5 + 3a_3 bF^3 + 3a_3 dF^6 + \\ +4a_4 bF^4 + 4a_4 dF^7 + 5a_5 bF^5 + 5a_5 dF^8 + 6a_6 bF^6 + 6a_6 dF^9 \\ U'' = a_1 b^2 F + 5a_1 bdF^4 + 4a_1 d^2 F^7 + 4a_2 b^2 F^2 + 14a_2 bdF^5 + 10a_2 d^2 F^8 + 9a_3 b^2 F^3 + \\ +27a_3 bdF^6 + 18a_3 d^2 F^9 + 16a_4 b^2 F^4 + 44a_4 bdF^7 + 28a_4 d^2 F^{10} + 25a_5 b^2 F^5 + 65a_5 bdF^8 + \\ +40a_5 d^2 F^{11} + 36a_6 bdF^6 + 54a_6 bdF^9 + 36a_6 d^2 F^9 + 54a_6 d^2 F^{12} \end{gathered} \tag{27}$$

where each of *b*, *d*, and a_6 are not zero. Substituting eq. (27) into eq. (11) and solving system of generated algebraic equation, then the following five cases are occurred to be analyzed.

Case 3.1. For $b \neq d$, then one gets the following coefficients:

$$\alpha=-3\varepsilon,\quad a_5=0,\quad a_4=0,\quad a_2=0,\quad a_1=0,\quad a_3=0,\quad a_0=0,\quad b=-\frac{i\sqrt{c}}{3\sqrt{6}},\quad d=-\frac{\sqrt{a_6}}{3\sqrt{6}}$$

$$\lambda=-\frac{1}{6}i\left(3\sqrt{6}\sqrt{c}-2i\sqrt{6}\sqrt{a_6}\right),\quad \delta=\frac{1}{6}\left(-3i\sqrt{6}\sqrt{c}\varepsilon+2\sqrt{6}\varepsilon\sqrt{a_6}\right)$$

where *k, c, γ, ε,* and a_6 are non-zero parameters. With appropriate values of these parameters the following solution is obtained:

$$u_{3.1}(x,t)=1-\left[-1+\mathrm{e}^{\frac{i(-t+x)}{\sqrt{6}}}EE\right]^{-2}+2\left[-1+\mathrm{e}^{\frac{i(-t+x)}{\sqrt{6}}}EE\right]^{-1} \tag{28}$$

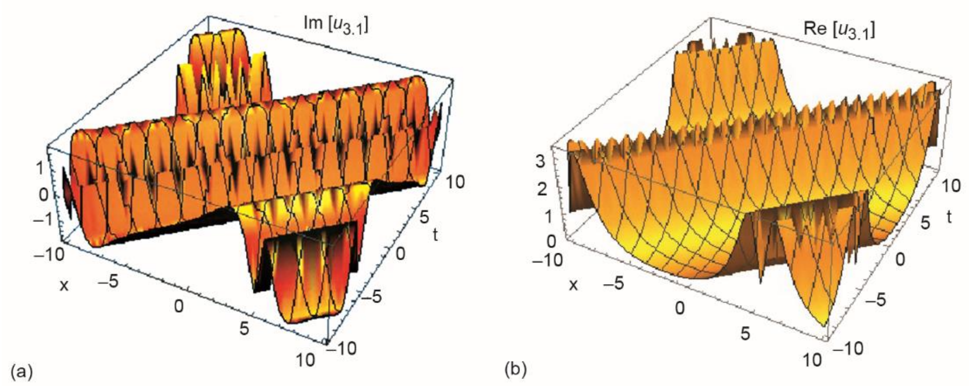

Figure 18. The 3-D graph for $u_{3.1}$ (*x, t*), with *EE* = 1.5, –10 < *x* < 10, –10 < *t* < 10
(*for color image see journal web site*)

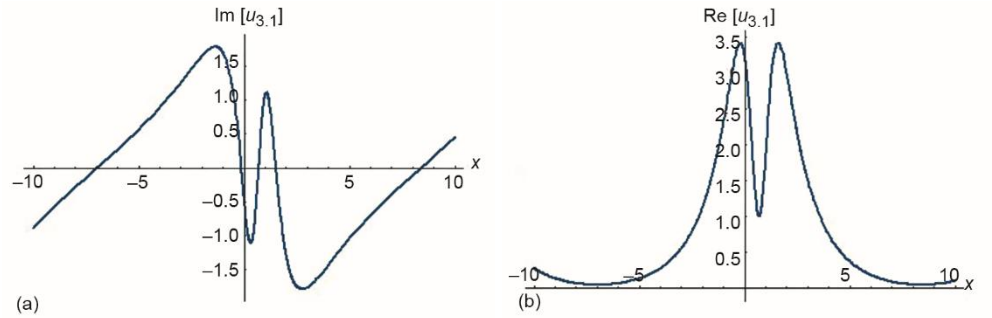

Figure 19. The 2-D graph for $u_{3.1}$ (*x, t*), with *EE* = 1.5, –10 < *x* < 10, *t* = 0.6

Case 3.2. For $b \neq d$, then the following coefficients are obtained:

$$\alpha=-3\varepsilon,\quad a_5=0,\quad a_4=0,\quad a_2=0,\quad a_1=0,\quad a_3=0,\quad b=-id\omega,\quad \varepsilon=\frac{54d^2-a_6}{54d^2a_0},\quad c=-a_0$$

$$\delta=\frac{-108d^2-162id^2\omega+3i\sqrt{a_0}\sqrt{a_6}+2a_6}{18da_0},\quad \lambda=-\frac{-18d^2-27id^2\omega+i\sqrt{a_0}\sqrt{a_6}}{3d}$$

where $\omega=\sqrt{a_0}/\sqrt{a_6}$, *k, γ, d,* a_0, and a_6 are non-zero parameters to be assigned so that solution is:

$$u_{3.2}(x,t) = 2 + \left[-\frac{i}{\sqrt{2}} + \mathrm{e}^{-3i\sqrt{2}(2t+x)} EE \right]^{-2} \tag{29}$$

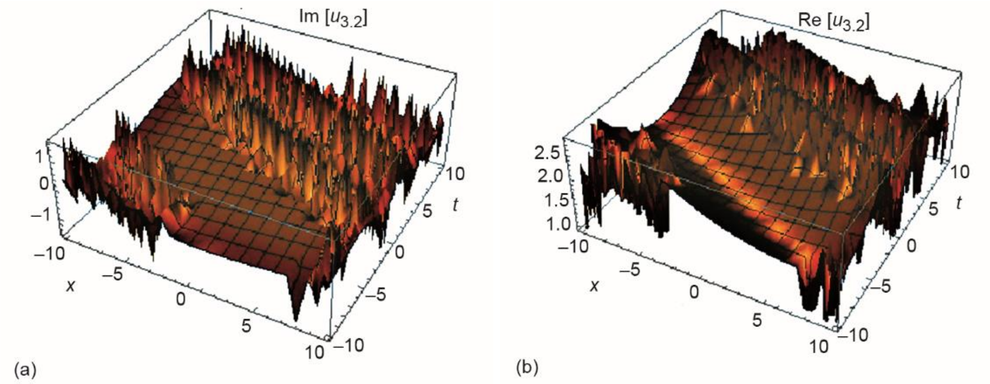

Figure 20. The 3-D graph for $u_{3.2}$ (x, t), with EE = 1.5, –10 < x < 10, –10 < t < 10
(*for color image see journal web site*)

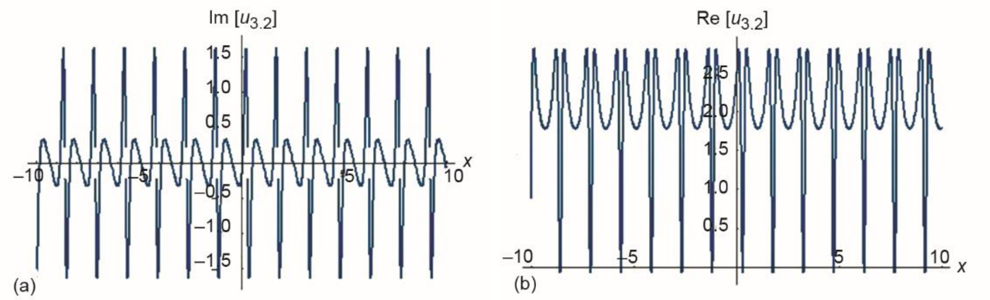

Figure 21. The 2-D graph for $u_{3.2}$ (x, t), with EE = 1.5, –10 < x < 10, t = 0.4

Case 3.3. If $b \neq d$, then one derives following new coefficients:

$$\alpha = -3\varepsilon, \quad a_5 = 0, \quad a_4 = 0, \quad a_2 = 0, \quad a_1 = 0, \quad a_0 = 0, \quad a_6 = 54d^2$$

$$\delta = -i\sqrt{\frac{3}{2}}\sqrt{c}\varepsilon - 6d\varepsilon, \quad \lambda = \frac{1}{2}\left(-i\sqrt{6}\sqrt{c} + 12d\right), \quad b = -\frac{i\sqrt{c}}{3\sqrt{6}}$$

Choosing appropriate non-zero values for *k, c, d, γ, ε,* and a_0, then the following solution is obtained:

$$u_{3.3}(x,t) = \frac{27}{8}\left[-\frac{3}{4}i\sqrt{\frac{3}{2}} + \mathrm{e}^{i\sqrt{\frac{2}{3}}(-4t+x)} EE \right]^{-2} \tag{30}$$

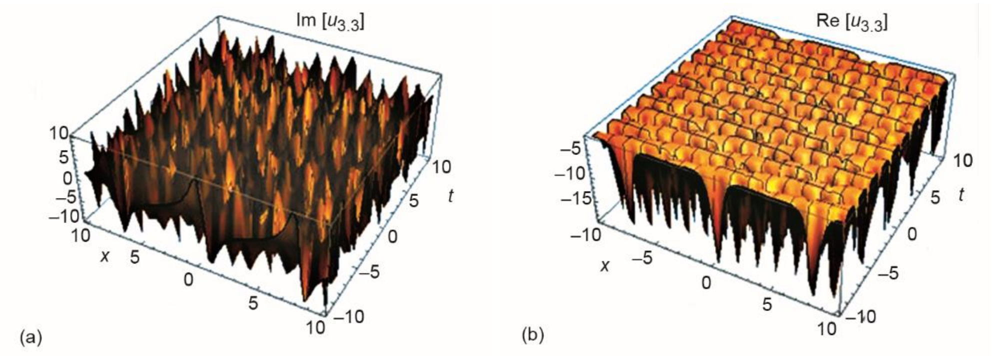

Figure 22. The 3-D graph for $u_{3.3}$ (*x*, *t*), with *EE* = 1.4, –10 < *x* < 10, –10 < *t* < 10
(*for color image see journal web site*)

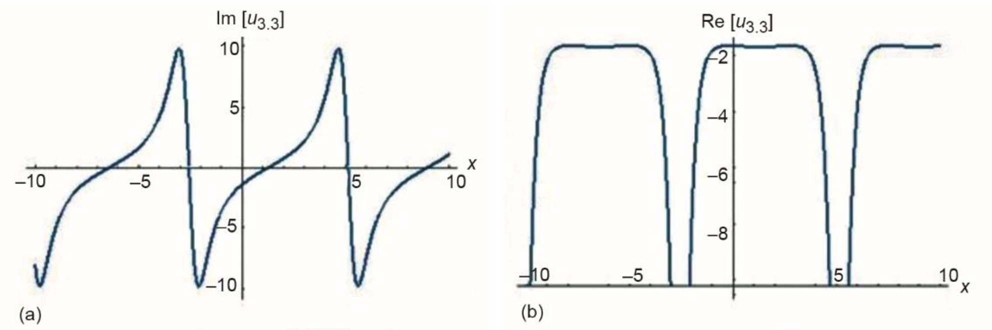

Figure 23. The 2-D graph for $u_{3.3}$ (*x*, *t*), with *EE* = 0.5, –10 < *x* < 10, *t* = 0.6

Case 3.4. In final case, the following new coefficients are appeared when $b \neq d$:

$$b = -\frac{i\sqrt{c}}{3\sqrt{6}}, \quad d = \frac{1}{12}\left(i\sqrt{6}\sqrt{c} + 2\lambda\right), \quad a_6 = -\frac{3}{4}\left(3c - 2i\sqrt{6}\sqrt{c}\lambda - 2\lambda^2\right),$$

$$a_0 = 0, \quad a_2 = 0, \quad a_1 = 0, \quad \delta = -\sqrt{6}\sqrt{c}\varepsilon - \varepsilon\lambda, \quad \alpha = -3\varepsilon, \quad a_5 = 0, \quad a_4 = 0$$

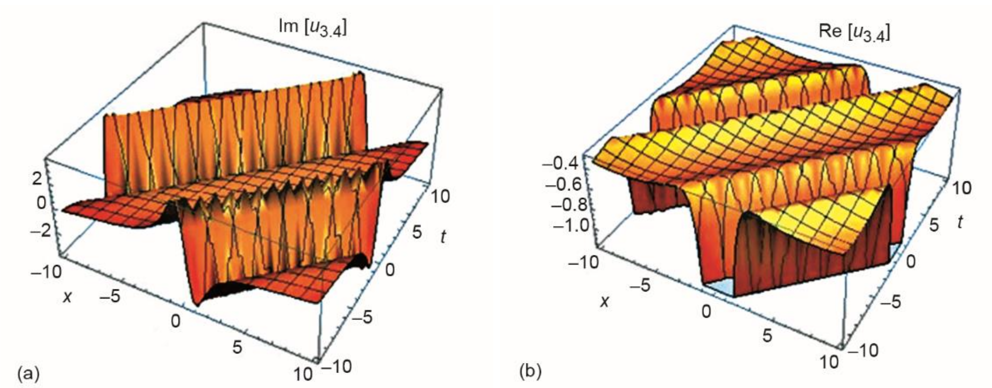

Figure 24. The 3-D graph for $u_{3.4}$ (*x*, *t*), with *EE* = 1.8, –10 < *x* < 10, 10 < *t* < 10
(*for color image see journal web site*)

By assigning suitable values for non-zero constants k, c, λ, γ, ε, and a_3, then the new solution is:

$$u_{3.4}(x,t) = -\frac{3\left(-5+4i\sqrt{6}\right)}{4}\left[-\frac{1}{2}i\sqrt{\frac{3}{2}}\left(-4+i\sqrt{6}\right)+\mathrm{e}^{\frac{i(-t+x)}{\sqrt{6}}}EE\right]^{-2} \tag{31}$$

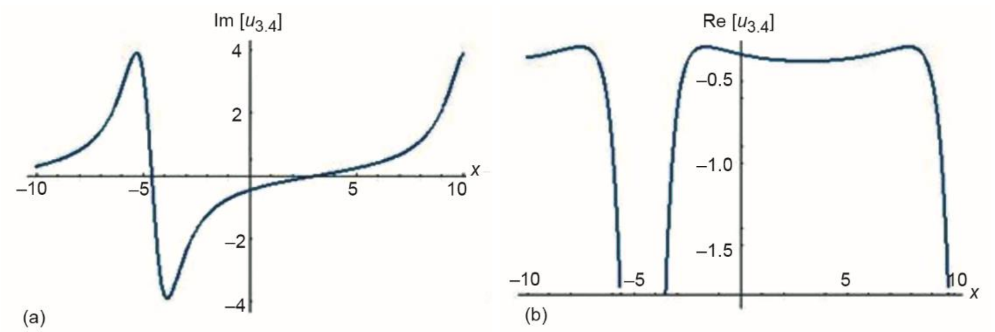

Figure 25. The 2-D graph for $u_{3.4}$ (x, t), with EE = 1.8, –10 < x < 10, t = 0.6

Conclusion and discussion

The BSEFM is first time successfully applied to KS non-linear PDE. Several new exponentially exact solutions, traveling and oscillatory wave solutions are obtained compare to available literature. Taking advantage of the computer software, profiles of obtained new solutions are plotted in 2-D and 3-D. Moreover, contour surfaces are exhibited. New exact solutions signify density, heat transfer and viscosity of liquids while generated coefficients denoted for compression of waves in liquid with gas-bubbles. It is seen that BSEFM is quite effective and reliable to solve non-linear differential equations.

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