SHARMA-TASSO-OLVER EQUATION INVOLVING A NEW TIME FRACTAL DERIVATIVE

by

Jiang-Jun LIU *

Basic Teaching Department, Huanghe Jiaotong University, Wuzhi, China

Original scientific paper
https://doi.org/10.2298/TSCI200322094L

The Sharma-Tasso-Olver equation with a new time fractal derivative is studied. The fractal Laplace transform, Adomian's decomposition method and He's polynomials are used to solve the equation. The results demonstrate efficiency and reliability of the proposed method.

Key words: *Sharma-Tasso-Olver equation, caputo fractal derivative, fractal Laplace transform, Adomian's decomposition method, He's polynomials*

Introduction

In recent years, many researchers have recognized that a classical differential model is not suitable for problems for anomalous thermal diffusion, transport, and fractal time random walks. To deal with the aforementioned problems, fractal calculus is built by several authors by using different methods [1-7]. Much achievement has been obtained, for examples, Schrodinger equation, Fokker-Planck equation, and Laplace equation on fractal set were derived [8-10]. Also, Hamilton mechanics and Lagrange mechanics were built on fractal sets by using F^{α} calculus [11].

The classical Sharma-Tasso-Olver (STO) equation plays a very important role in studying thermal conductivity of magnetic fluids [12, 13]. However, if we study the motions in a fractal medium, the fractal derivative has to be adopted. The present study is motivated by the desire to obtain an approximate analytical solution of the following STO equation involving a time fractal derivative:

$$\mathrm{D}_t^{\beta} u(x,t) + a\frac{\partial}{\partial x}\left[3u(x,t)\frac{\partial u}{\partial x} + u^3\right] + a\frac{\partial^3 u}{\partial x^3} = 0 \tag{1}$$

with the initial condition:

$$u[x, S_F^{\alpha}(0)] = \varphi(x)$$

where $0 < \beta \leq 1$ is a constant, D_t^{β} – the Caputo time fractal derivative [7], and α is the γ-dimension of a fractal set.

The aim of this work can be achieved by using fractal Laplace transform [7], Adomian's decomposition method (ADM) and He's polynomials [14-17].

* Author's e-mail: liushuxue2004@163.com

Fractal derivative

The fractal derivative is an extension of classical derivative for fractal sets. There are many definitions of fractal derivative. In this section, we recall several definitions, which are frequently used.

Definition 1. Chen's definitions is defined as follows [18, 19]:

$$\frac{\mathrm{d}u}{\mathrm{d}x^{\alpha}}=\lim_{s\to x}\frac{u(x)-u(s)}{x^{\alpha}-s^{\alpha}} \tag{2}$$

where α is the order of the fractal derivative.

Definition 2. Consider a fractal medium and assume the smallest measure is L_0 (any discontinuity less than L_0 is ignored). This fractal derivative has the form [4]:

$$\frac{\partial u}{\partial x^{\alpha}}=\Gamma(1+\alpha)\lim_{x_A-x_B\to L_0}\frac{u(x_A)-u(x_B)}{(x_A-x_B)^{\alpha}} \tag{3}$$

When $\alpha\to 1$ and $L_0\to 0$, eq. (3) turns out to be the ordinary differentiation.

We next recall the Caputo fractal derivative which will be used in this paper. Let a_0 be an arbitrary but fixed real number. The integral staircase function $S_F^{\alpha}(x)$ of order α for a fractal set F is given by [5]:

$$S_F^{\alpha}(x)=\begin{cases}\gamma^{\alpha}(F,a_0,x), & \text{if } x\ge a_0,\\ -\gamma^{\alpha}(F,x,a_0), & \text{otherwise}\end{cases} \tag{4}$$

Definition 3. The F^{α} derivative of $f(x)$ at x is defined as [5]:

$$\mathrm{D}_F^{\alpha}[f(x)]=\begin{cases}F-\lim\limits_{y\to x}\dfrac{f(y)-f(x)}{S_F^{\alpha}(y)-S_F^{\alpha}(x)}, & \text{if } x\in F,\\ 0, & \text{otherwise}\end{cases} \tag{5}$$

Definition 4. Let $f(x)\in C^{\alpha}[a,b]$, then the Caputo fractal derivative is defined by [7]:

$$\mathrm{D}_x^{\beta}f(x)=\frac{1}{\Gamma_F^{\alpha}(1-\beta)}\int_{S_F^{\alpha}(0)}^{S_F^{\alpha}(x)}\frac{\mathrm{D}_F^{\alpha}[f(t)]\mathrm{d}_F^{\alpha}t}{[S_F^{\alpha}(x)-S_F^{\alpha}(t)]^{1-\alpha-\beta}} \tag{6}$$

where the Gamma function with the fractal support is understood:

$$\Gamma_F^{\alpha}(x)=\int_{S_F^{\alpha}(0)}^{S_F^{\alpha}(\infty)}\mathrm{e}^{-S_F^{\alpha}(t)}S_F^{\alpha}(t)^{S_F^{\alpha}(x)-1}\mathrm{d}_F^{\alpha}t \tag{7}$$

Definition 5. Fractal Laplace transform of the function $f(x)$ is defined [7]:

$$\mathrm{L}_F^{\alpha}\left[f(x)\right]=\int_{S_F^{\alpha}(0)}^{S_F^{\alpha}(\infty)}f(x)\mathrm{e}^{-S_F^{\alpha}(s)S_F^{\alpha}(x)}\mathrm{d}_F^{\alpha}x \tag{8}$$

The following Laplace transform formulas hold true [7]:

$$\mathrm{L}_F^\alpha[S_F^\alpha(x)^\beta] = \frac{\Gamma_F^\alpha(1+\beta)}{S_F^\alpha(s)^{1+\beta}} \tag{9}$$

$$L_F^\alpha[D_x^\beta f(x)] = [S_F^\alpha(s)]^\beta L_F^\alpha[f(x) - (S_F^\alpha(s)]^{\beta-1} f[S_F^\alpha(0)] \tag{10}$$

Adomian's decomposition method

In this section, we give a brief presentation of the ADM. The method is usually used for solving non-linear operate equation of the form [14, 15]:

$$u = \varphi + N(u) \tag{11}$$

where N:Ω→Ω is a non-linear mapping from Banach space Ω into itself and $\varphi \in \Omega$ is known.

The ADM admits the use of infinite decomposition series:

$$u = \sum_{n=0}^{\infty} u_n \tag{12}$$

with $u_n \in \Omega, \forall n,$ for the solution $u,$ and the infinite series of:

$$N(u) = \sum_{n=0}^{\infty} H_n(u_0, u_1, \cdots, u_n) \tag{13}$$

for non-linear terms $N(u)$, where the components u_n of the solution u will be determined recurrently, and H_n are He's polynomials, which can be generated by:

$$H_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{n} \lambda^k u_k\right)\right]_{\lambda=0} \tag{14}$$

Thus:

$$H_0 = N(u_0)$$

$$H_1 = u_1(DN)(u_0)$$

$$H_2 = u_2(DN)(u_0) + \frac{u_1^2}{2}(D^2N)(u_0)$$

and so on, where $D^r N(u_0)$ denotes the r^{th} Frechet derivative of N at $u_0 \in \Omega$.

Substitute eqs. (12) and (13) into eq. (11) gives:

$$\sum_{n=0}^{\infty} u_n = \varphi + \sum_{n=0}^{\infty} H_n \tag{15}$$

which is satisfied formally if we take:

$$u_0 = \varphi \tag{16}$$

$$u_{n+1} = H_n \tag{17}$$

In this way, we finally obtain the solution that can be approximated by the partial sum:

$$S_N = \sum_{k=0}^{N-1} u_k$$

Solutions of eq. (1)

Let

$$N(u) = \frac{\partial}{\partial x}\left[3u(x,t)\frac{\partial u}{\partial x} + u^3(x,t)\right] \tag{18}$$

Then eq. (1) can be rewriten:

$$\mathrm{D}_t^\beta u(x,t) + 3aN(u) + a\frac{\partial^3 u}{\partial x^3} = 0 \tag{19}$$

Applying the fractal Laplace transformation on both side of eq. (19), we get:

$$L_F^\alpha\left[\mathrm{D}_t^\beta u(x,t) + 3aN(u) + a\frac{\partial^3 u}{\partial x^3}\right] = 0 \tag{20}$$

Denoting:

$$L_F^\alpha[u(x,t)] = U_F^\alpha(x,s)$$

and using the property of the fractal Laplace transformation, we get:

$$S_F^\alpha(s)^\beta U_F^\alpha(x,s) - S_F^\alpha(s)^{\beta-1}\varphi(x) + L_F^\alpha\left[3aN(u) + a\frac{\partial^3 u}{\partial x^3}\right] = 0 \tag{21}$$

or

$$U_F^\alpha(x,s) = \frac{1}{S_F^\alpha(s)}\varphi(x) - \frac{1}{S_F^\alpha(s)^\beta}L_F^\alpha\left[3aN(u) + a\frac{\partial^3 u}{\partial x^3}\right] \tag{22}$$

By computing the inverse fractal Laplace transform, we conclude:

$$u(x,t) = \varphi(x) - L_F^{-\alpha}\left\{\frac{1}{S_F^\alpha(s)^\beta}L_F^\alpha\left[3aN(u) + a\frac{\partial^3 u}{\partial x^3}\right]\right\} \tag{23}$$

We assume that the solution $u(x,t)$ can be expanded as:

$$u = \sum_{n=0}^{\infty} u_n \tag{24}$$

and non-linear term:

$$N(u) = \sum_{n=0}^{\infty} H_n(u_0, u_1, \cdots, u_n) \tag{25}$$

where

$$H_0 = \left(\frac{\partial u_0}{\partial x}\right)^2 + u_0^2 \frac{\partial u_0}{\partial x} + u_0 \frac{\partial^2 u_0}{\partial x^2}$$

$$H_1 = 2\frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial x} + u_0^2 \frac{\partial u_1}{\partial x} + 2u_0 u_1 \frac{\partial u_0}{\partial x} + u_0 \frac{\partial^2 u_1}{\partial x^2} + u_1 \frac{\partial^2 u_0}{\partial x^2}$$

and so on.

From eqs. (16) and (17), it is easy to determine u_n:

$$u_0 = \varphi(x) \tag{26}$$

$$u_n(x,t) = -L_F^{-\alpha}\left\{\frac{1}{S_F^\alpha(s)^\beta} L_F^\alpha\left[3aH_{n-1} + a\frac{\partial^3 u_{n-1}}{\partial x^3}\right]\right\} \tag{27}$$

where $n = 1, 2, 3, \cdots$.

Finally, the solution $u(x,t)$ is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \tag{28}$$

In order to demonstrate the efficiency of the previous method, we consider the following STO equation involving time fractal derivative:

$$\mathrm{D}_t^\beta u(x,t) + \frac{\partial}{\partial x}\left[3u(x,t)\frac{\partial u}{\partial x} + u^3\right] + \frac{\partial^3 u}{\partial x^3} = 0 \tag{29}$$

with the initial condition:

$$u[x, S_F^\alpha(0)] = \frac{2[2+\tanh(x)]}{1+2\tanh(x)} \tag{30}$$

By the aforementioned algorithm, we first assume that the solution $u(x,t)$ can be expanded as an infinite series:

$$u = \sum_{n=0}^{\infty} u_n$$

Then, according to eqs. (26) and (27), we get:

$$u_0(x,t) = \frac{2[2+\tanh(x)]}{1+2\tanh(x)}$$

$$u_1(x,t) = \frac{96\mathrm{e}^{-2x} S_F^\alpha(t)^\beta}{(3-\mathrm{e}^{-2x})^2 \Gamma_F^\alpha(1+\beta)}$$

$$u_2(x,t) = \frac{768\mathrm{e}^{-2x}(3+\mathrm{e}^{-2x}) S_F^\alpha(t)^{2\beta}}{(3-\mathrm{e}^{-2x})^3 \Gamma_F^\alpha(1+2\beta)}$$

and so on.

Finally, we obtain the solution:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

For example, we can use:

$$u(x,t) = \frac{2(3+\mathrm{e}^{-2x})}{3-\mathrm{e}^{2x}} + \frac{96\mathrm{e}^{-2x} S_F^\alpha(t)^\beta}{(3-\mathrm{e}^{-2x})^2 \Gamma_F^\alpha(1+\beta)} + \frac{768\mathrm{e}^{-2x}(3+\mathrm{e}^{-2x}) S_F^\alpha(t)^{2\beta}}{(3-\mathrm{e}^{-2x})^3 \Gamma_F^\alpha(1+2\beta)} \tag{31}$$

as the approximate solution of eq. (1).

When $\alpha = 1$, $\beta = 1$, by direct substituting eq. (31) into eq. (29), we have found that the approximate solution is very close to the exact solution.

Conclusion

We extend the classical model of STO to the new model of time fractal STO. By using fractal Laplace transform, Adomian's decomposition method and He's polynomials, we present a general algorithm for the new model. The approximate solution of the new model is obtained, which leads to the classical ones when $\alpha = 1$, $\beta = 1$. The algorithm described in this paper is expected to be further employed to solved other non-linear differential equation involving Caputo fractal derivatives.

References

[1] Barnsley, M. F., Harrington, A. N., The Calculus of Fractal Interpolation Functions, *Journal of Approximation Theory, 57* (1989), 1, pp. 14-34
[2] Zähle, M., Harmonic Calculus on Fractals: A Measure Geometric Approach II., *Potential Analysis, 357* (2005), 9, pp. 3407-3423
[3] Golmankhaneh, A. K., Baleanu, D., Fractal Calculus Involving Gauge Function, *Communications in Non-Linear Science and Numerical Simulation*, *37* (2016), 1, pp. 125-130
[4] He, J. H., Ain, Q. T., New Promises and Future Challenges of Fractal Calculus: from Two-Scale Thermodynamics to Fractal Variational Principle, *Thermal Science*, *24* (2020), 2A, pp. 659-681
[5] Parvate, A., Gangal, A. D., Calculus on Fractal Subsets of Real Line I: Formulation, *Fractals, 17* (2009), 1, pp. 53-81
[6] Parvate, A., Gangal, A. D., Calculus on Fractal Subsets of Real Line II: Conjugacy with Ordinary Calculus, *Fractals, 19* (2011), 3, pp. 271-290
[7] Golmankhaneh, A. K., Baleanu, D., New Derivatives on the Fractal Subset of Real-Line, *Entropy, 18* (2016), 2, Article Number 1
[8] Domany, E., *et al.*, Solutions to the Schrodinger Equation on Some Fractal Lattices, *Physical Review B* , *28* (1983), 6, pp. 1415-1417
[9] Tarasov, V. E., Fractional Fokker-Planck Equation for Fractal Media, *Chaos: An Interdisciplinary Journal of Non-Linear Science*, *2* (2005), 15, pp. 461-478
[10] Maitama, S., Zhao, W., Local Fractional Laplace Homotopy Analysis Method for Solving Non-Differentiable Wave Equations on Cantor Sets, *Computational and Applied Mathematics, 38* (2019), 2, pp. 1-22
[11] Golmankhaneh, A. K., *et al.,* Lagrangian and Hamiltonian Mechanics on Fractals Subset of Real-Line, *International Journal of Theoretical Physics, 52* (2013), 11, pp. 4210-4217
[12] Olver, P. J., Evolution Equations Possessing Infinitely Many Symmetries, *J. Math. Phys., 18* (1977), 6, pp. 1212-1215
[13] Salas, A. H., Exact Solutions to a Generalized Sharma-Tasso-Olver Equation, *Applied mathematics science, 5* (2011), 46, pp. 2289-2295
[14] Abbasbandy, S., Numerical Solutions of the Integral Equations: Homotopy Perturbation Method and Adomian's Decomposition Method, *Applied Mathematics and Computation, 173* (2006), 1, pp. 493-500
[15] Pamuk, S., Solution of the Porous Media Equation by Adomian's Decomposition Method, *Physics Letters A, 344* (2005), 2, pp. 184-188

[16] He, J. H., A Short Review on Analytical Methods for to a Fully Fourth-Order Non-linear Integral Boundary Value Problem with Fractal Derivatives, *Int. J. Numer. Method H., 30* (2020), 11, pp. 4933-4943
[17] He, J. H., Jin, X., A Short Review on Analytical Methods for the Capillary Oscillator in a Nanoscale Deformable Tube, *Math. Method Appl. Sci.,* On-line first, https://doi.org/ 10.1002/mma.6321, 2020
[18] Wen, C., *et al.,* Anomalous Diffusion Modeling by Fractal and Fractional Derivatives, *Computers and Mathematics with Applications, 59* (2010), 5, pp. 1754-1758
[19] Wen, C., *et al.,* Soil Infiltration Rates and Hydrology Model Classifications Based on the Hausdorff Fractal Derivative Richards Equation, *Applied Mathematics and Mechanics, 39* (2018), 1, pp. 77-82

Paper submitted: March 22, 2020
Paper revised: June 26, 2020
Paper accepted: June 26, 2020