DIRECT ALGEBRAIC METHOD FOR SOLVING FRACTIONAL FOKAS EQUATION

by

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Fractional Fokas equation is studied, its exact solution is obtained by the direct algebraic method. The solution process is elucidated step by step, and the fractional complex transform and the characteristic set algorithm are emphasized.

Key words: *direct algebraic method, characteristic set algorithm, space-time fractional Fokas equation*

Introduction

In recent decades, the non-linear fractional PDE have attracted much attention due to their wide applications to various complex phenomena arising in elasticity, plasma physics, solid state physics, gas dynamics, material, and others [1-10]. Searching for their exact solutions is an important topic in both mathematics and engineering. A wealth of methods have been developed for this purpose, for examples, the homotopy perturbation method [11-17], variational iteration method [18-20], the exp-function method [21-25], He-Laplace method [26-28], the symmetry reduction method [29-31], the reproducing kernel method [32, 33], and others [34-36].

In this paper, solitary wave solutions of space-time fractional Fokas equation [37] are considered. The direct algebraic method [38, 39] is used to solve the equation, which leads to a large system of algebraic equations, the characteristic set algorithm [40-42] is adopted to solve the algebraic equations.

The direct algebraic method

There are many types fractional derivative in literature, for example the Jumaries's modification of the Riemann-Liouville derivatives of fractional-order α is defined by [1]:

$$\mathrm{D}_t^{\alpha} f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-\xi)^{-\alpha}[f(\xi)-f(0)]\mathrm{d}\xi \qquad 0<\alpha<1$$

where $f(t)$ is a real and continuous function defined on *R*.

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The He's fractional derivative defined [43, 44]:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (s-t)^{n-\alpha-1} [f_0(s) - f(s)] ds \qquad 0 < \alpha \le 1$$

We outline the main steps of the direct algebraic method [38, 39] with modified Riemann-Liouville derivative for finding exact travelling solutions of fractional-order non-linear PDE.

Consider the fractional-order non-linear PDE:

$$Q(u, D_t^\alpha u, D_{x_1}^\alpha u, D_{x_2}^\alpha u, \cdots, D_t^{2\alpha} u, D_{x_1}^{2\alpha} u, D_{x_2}^{2\alpha} u, \cdots) = 0 \tag{1}$$

where Q is a polynomial of u and its fractional derivatives.

Step 1. The fractional complex transform [45, 46] is used:

$$u(t, x_1, x_2, \cdots, x_n) = u(\xi),$$
$$\xi = \frac{ct^\alpha}{\Gamma(1+\alpha)} + \frac{k_1 x_1^\alpha}{\Gamma(1+\alpha)} + \frac{k_2 x_2^\alpha}{\Gamma(1+\alpha)} + \cdots + \frac{k_n x_n^\alpha}{\Gamma(1+\alpha)} \tag{2}$$

where c, k_1, k_2, …, k_n are arbitrary constants, the eq. (2) transform eq. (1) into an ODE:

$$\tilde{Q}(u, cu', k_1 u', k_2 u', \cdots, c^2 u'', k_1^2 u'', k_2^2 u'', \cdots) = 0 \tag{3}$$

Step 2. We look for exact solution of eq. (3) in the form:

$$u(\xi) = \sum_{i=0}^{N} b_i Q(\xi)^i, \qquad b_N \ne 0 \tag{4}$$

where $b_i (0 \le i \le N)$ are constants to be determined, and $Q(\xi)$ satisfies the ODE in the form [39]:

$$Q'(\xi) = \text{Ln}(A)[\alpha + \beta Q(\xi) + \sigma Q(\xi)^2], \qquad A \ne 0,1 \tag{5}$$

Step 3. By balancing the highest order derivative terms with the non-linear terms of the highest order in eq. (3), we can evaluated the value of the positive integer N.

Step 4. By substituting eqs. (4) and (5) into eq. (3) and equating all the coefficients of same power of $Q(\xi)$ to zero, we obtained a system of algebraic equations. The obtaining system can be solved to find the value of c, k_1, k_2, …, $k_n, b_i (0 \le i \le N)$, substituting these terms into eq. (4), the determination of solutions of eq. (1) will be completed.

Exact solutions of space-time fractional Fokas equation

Consider the following space-time fractional Fokas equation [37] which could be used to describe various physical phenomena such as fluid mechanics, water wave theory, ocean dynamics and many others.

$$4\frac{\partial^{2\alpha} u}{\partial t^\alpha \partial x_1^\alpha} - \frac{\partial^{4\alpha} u}{\partial x_1^{3\alpha} \partial x_2^\alpha} + \frac{\partial^{4\alpha} u}{\partial x_2^{3\alpha} \partial x_1^\alpha} + 12 \frac{\partial^\alpha u}{\partial x_1^\alpha} \frac{\partial^\alpha u}{\partial x_2^\alpha} + 12u \frac{\partial^{2\alpha} u}{\partial x_1^\alpha \partial x_2^\alpha} - 6\frac{\partial^{2\alpha} u}{\partial y_1^\alpha \partial y_2^\alpha} = 0 \tag{6}$$

The fractional complex transform is:

$$u(x_1, x_2, y_1, y_2, t) = u(\xi),$$
$$\xi = \frac{ct^\alpha}{\Gamma(1+\alpha)} + \frac{k_1 x_1^\alpha}{\Gamma(1+\alpha)} + \frac{k_2 x_2^\alpha}{\Gamma(1+\alpha)} + \frac{l_1 y_1^\alpha}{\Gamma(1+\alpha)} + \frac{l_2 y_2^\alpha}{\Gamma(1+\alpha)} \tag{7}$$

where c, k_1, k_2, …, k_n, are arbitrary constants, and $c, k_1, k_2, \cdots, k_n \neq 0$. Using the wave variable (7), eq. (6) becomes:

$$4ck_1 u'' - k_1^3 k_2 u^{(4)} + k_1 k_2^3 u^{(4)} + 12k_1 k_2 (u')^2 + 12k_1 k_2 uu'' - 6l_1 l_2 u'' = 0 \tag{8}$$

Integrating eq. (8) twice with respect to ξ and setting the integration constant as zero, we get:

$$(4ck_1 - 6l_1 l_2)u + 6k_1 k_2 u^2 + k_1 k_2 (-k_1^2 + k_2^2) u'' = 0 \tag{9}$$

Suppose that the solution of eq. (9) can be expressed:

$$u(\xi) = \sum_{i=0}^{N} b_i Q(\xi)^i \tag{10}$$

where $b_i (0 \le i \le N)$ are constants to be determined, such that $b_N \neq 0$.

Consider the homogeneous balance between the highest order derivative $u^{(3)}$ and non-linear term uu' appearing in (9), we have $N = 2,$ we then suppose that eq. (9) has the following solutions:

$$u(\xi) = b_0 + b_1 Q(\xi) + b_2 Q(\xi)^2, \qquad b_2 \neq 0 \tag{11}$$

Substituting eqs. (11) and (5) into eq. (9) and collecting all the terms with the same power of $Q(\xi)$ together, equating each coefficient to zero, yields a set of algebraic equations, which is large and difficult to solve, with the aid of the characteristic set algorithm [40, 41], we can distinguish the different cases namely:

Case 1.

$$b_2 = \frac{[4ck_1 - 6l_1 l_2 + k_1 k_2 (-k_1^2 + k_2^2)\beta^2 \ln(A)^2]^2}{16k_1^2 k_2^2 (k_1^2 - k_2^2)\alpha^2 \ln(A)^2}$$

$$b_1 = \frac{-4ck_1\beta + 6l_1 l_2 \beta + k_1 k_2 (k_1^2 - k_2^2)\beta^3 \ln(A)^2}{4k_1 k_2 \alpha}$$

$$b_0 = \frac{-4ck_1 + 6l_1 l_2 + k_1 k_2 (k_1^2 - k_2^2)\beta^2 \ln(A)^2}{4k_1 k_2}$$

$$c = \frac{6l_1 l_2 + k_1 k_2 (k_1^2 - k_2^2)(\beta^2 - 4\alpha\sigma)\ln(A)^2}{4k_1} \quad \text{b1} \to \frac{\text{-4ck1β+6l1l2β+k1k2(k1²-k2²)β³Log[A]²}}{\text{4k1k2α}}$$

Case 2.

$$b_2 = \frac{[4ck_1 - 6l_1 l_2 + k_1 k_2 (k_1^2 - k_2^2)\beta^2 \ln(A)^2]^2}{16k_1^2 k_2^2 (k_1^2 - k_2^2)\alpha^2 \ln(A)^2}$$

$$b_1 = \frac{4ck_1\beta - 6l_1l_2\beta + k_1k_2(k_1^2 - k_2^2)\beta^3 \ln(A)^2}{4k_1k_2\alpha}$$

$$b_0 = \frac{4ck_1 - 6l_1l_2 + 3k_1k_2(k_1^2 - k_2^2)\beta^2 \ln(A)^2}{12k_1k_2}$$

$$c = \frac{1}{4}\left[\frac{6l_1l_2}{k_1} + k_2(-k_1^2 + k_2^2)(\beta^2 - 4\alpha\sigma)\ln(A)^2\right]$$

Case 3.

$$b_2 = (k_1^2 - k_2^2)\sigma^2 \ln(A)^2, \quad b_1 = (k_1^2 - k_2^2)\beta\sigma \ln(A)^2$$

$$b_0 = 0, \quad c = \frac{6l_1l_2 + k_1k_2(k_1^2 - k_2^2)\beta^2 \ln(A)^2}{4k_1}, \quad \alpha = 0$$

Case 4.

$$b_2 = (k_1^2 - k_2^2)\sigma^2 \ln(A)^2, \quad b_1 = (k_1^2 - k_2^2)\beta\sigma \ln(A)^2$$

$$b_0 = \frac{-2ck_1 + 3l_1l_2}{3k_1k_2}, \quad c = \frac{6l_1l_2 + k_1k_2(-k_1^2 + k_2^2)\beta^2 \ln(A)^2}{4k_1}, \quad \alpha = 0$$

For the sake of simplicity, we consider only the solution with respect to Case (1), the other solutions can be obtained in a similar way:

– when $\beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$:

$$u_1 = \frac{1}{4}F\left[\frac{FU^2}{\alpha^2\sigma^2 \ln^2(A)(k_1^2 - k_2^2)} - \frac{2\beta U}{\alpha\sigma} + 4\right], \quad u_2 = \frac{1}{4}F\left[\frac{FV^2}{\alpha^2\sigma^2 \ln^2(A)(k_1^2 - k_2^2)} - \frac{2\beta V}{\alpha\sigma} + 4\right]$$

$$u_{3-1} = \frac{1}{4}F\left[\frac{F(W1)^2}{\alpha^2\sigma^2 \ln^2(A)(k_1^2 - k_2^2)} + \frac{2\beta(W1)}{\alpha\sigma} + 4\right]$$

$$u_{3-2} = \frac{1}{4}F\left[\frac{F(W2)^2}{\alpha^2\sigma^2 \ln^2(A)(k_1^2 - k_2^2)} - \frac{2\beta(W2)}{\alpha\sigma} + 4\right]$$

$$u_{4-1} = \frac{1}{4}F\left[\frac{F(X1)^2}{\alpha^2\sigma^2 \ln^2(A)(k_1^2 - k_2^2)} - \frac{2\beta(X1)}{\alpha\sigma} + 4\right]$$

$$u_{4-2} = \frac{1}{4}F\left[\frac{F(X2)^2}{\alpha^2\sigma^2 \ln^2(A)(k_1^2 - k_2^2)} - \frac{2\beta(X2)}{\alpha\sigma} + 4\right]$$

$$u_5 = F\left[\frac{FY^2}{16\alpha^2\sigma^2 \ln^2(A)(k_1^2 - k_2^2)} - \frac{\beta Y}{4\alpha\sigma} + 1\right]$$

where

$$M=4\alpha\sigma-\beta^2\,,F=\frac{\beta^2k_1k_2\ln^2(A)(k_1^2-k_2^2)-4ck_1+6l_1l_2}{4k_1k_2}\,,\quad U=\beta-\sqrt{M}\operatorname{tana}\left(\frac{\sqrt{M}\xi}{2}\right)$$

$$V=\beta+\sqrt{M}\operatorname{cota}\left(\frac{\sqrt{M}\xi}{2}\right),\quad W1=-\beta+\sqrt{Mpq}\operatorname{seca}\left(\sqrt{M}\xi\right)+\sqrt{M}\operatorname{tana}\left(\frac{\sqrt{M}\xi}{2}\right)$$

$$W2=\beta+\sqrt{Mpq}\operatorname{seca}\left(\sqrt{M}\xi\right)-\sqrt{M}\operatorname{tana}\left(\frac{\sqrt{M}\xi}{2}\right)$$

$$X1=\beta+\sqrt{M}\operatorname{cota}\left(\sqrt{M}\xi\right)-\operatorname{csca}\left(\sqrt{M}\xi\right)\sqrt{Mpq}$$

$$X2=\beta+\sqrt{M}\operatorname{cota}\left(\sqrt{M}\xi\right)+\operatorname{csca}\left(\sqrt{M}\xi\right)\sqrt{Mpq}$$

$$Y=2\beta+\sqrt{M}\operatorname{cota}\left(\frac{\sqrt{M}\xi}{4}\right)-\sqrt{M}\operatorname{tana}\left(\frac{\sqrt{M}\xi}{4}\right)$$

– when $\beta^2-4\alpha\sigma>0$ and $\sigma\neq0$:

$$u_6=\frac{1}{4}F\left[\frac{FU^2}{\alpha^2\sigma^2\ln^2(A)(k_1^2-k_2^2)}-\frac{2\beta U}{\alpha\sigma}+4\right],u_7=\frac{1}{4}F\left[\frac{FV^2}{\alpha^2\sigma^2\ln^2(A)(k_1^2-k_2^2)}-\frac{2\beta V}{\alpha\sigma}+4\right]$$

$$u_{8-1}=\frac{1}{4}F\left[\frac{F(\mathrm{W1})^2}{\alpha^2\sigma^2\ln^2(A)(k_1^2-k_2^2)}-\frac{2\beta(\mathrm{W1})}{\alpha\sigma}+4\right]$$

$$u_{8-2}=\frac{1}{4}F\left[\frac{F(\mathrm{W2})^2}{\alpha^2\sigma^2\ln^2(A)(k_1^2-k_2^2)}-\frac{2\beta(\mathrm{W2})}{\alpha\sigma}+4\right]$$

$$u_{9-1}=\frac{1}{4}F\left[\frac{F(\mathrm{X1})^2}{\alpha^2\sigma^2\ln^2(A)(k_1^2-k_2^2)}-\frac{2\beta(\mathrm{X1})}{\alpha\sigma}+4\right]$$

$$u_{9-2}=\frac{1}{4}F\left[\frac{F(\mathrm{X2})^2}{\alpha^2\sigma^2\ln^2(A)(k_1^2-k_2^2)}-\frac{2\beta(\mathrm{X2})}{\alpha\sigma}+4\right]$$

$$u_{10}=F\left[\frac{FY^2}{16\alpha^2\sigma^2\ln^2(A)(k_1^2-k_2^2)}-\frac{\beta Y}{4\alpha\sigma}+1\right]$$

where

$$N=\beta^2-4\alpha\sigma\,,\quad F=\frac{\beta^2k_1k_2\ln^2(A)(k_1^2-k_2^2)-4ck_1+6l_1l_2}{4k_1k_2}\,,\quad U=\beta+\sqrt{N}\operatorname{tanha}\left(\frac{\xi\sqrt{N}}{2}\right)$$

$$V=\beta+\sqrt{N}\operatorname{cotha}\left(\frac{\xi\sqrt{N}}{2}\right),\quad W1=\beta-i\sqrt{N}\sqrt{pq}\operatorname{secha}\left(\xi\sqrt{N}\right)+\sqrt{N}\operatorname{tanha}\left(\xi\sqrt{N}\right)$$

$$W2=\beta+i\sqrt{N}\sqrt{pq}\operatorname{secha}\left(\xi\sqrt{N}\right)+\sqrt{N}\operatorname{tanha}\left(\xi\sqrt{N}\right)$$

$$X1=\beta+\sqrt{N}\operatorname{cotha}\left(\xi\sqrt{N}\right)-\sqrt{N}\sqrt{pq}\operatorname{cscha}\left(\xi\sqrt{N}\right)$$

$$X2=\beta+\sqrt{N}\operatorname{cotha}\left(\xi\sqrt{N}\right)+\sqrt{N}\sqrt{pq}\operatorname{cscha}\left(\xi\sqrt{N}\right)$$

$$Y=2\beta+\sqrt{N}\operatorname{cotha}\left(\frac{\xi\sqrt{N}}{4}\right)+\sqrt{N}\operatorname{tanha}\left(\frac{\xi\sqrt{N}}{4}\right)$$

– when $\alpha\sigma>0$ and $\beta=0$:

$$u_{11}=\frac{F^2\operatorname{tana}\left(\xi\sqrt{\alpha\sigma}\right)^2}{\alpha\sigma\ln^2(A)(\mathrm{k1}^2-\mathrm{k2}^2)}+F,\quad u_{12}=\frac{F^2\operatorname{cota}\left(\xi\sqrt{\alpha\sigma}\right)^2}{\alpha\sigma\ln^2(A)(\mathrm{k1}^2-\mathrm{k2}^2)}+F$$

$$u_{13-1,2}=\frac{F^2\left[\sqrt{\frac{\alpha pq}{\sigma}}\operatorname{seca}\left(2\xi\sqrt{\alpha\sigma}\right)\pm\sqrt{\frac{\alpha}{\sigma}}\operatorname{tana}\left(2\xi\sqrt{\alpha\sigma}\right)\right]^2}{\alpha^2\ln^2(A)(\mathrm{k1}^2-\mathrm{k2}^2)}+F$$

$$u_{14-1,2}=\frac{F^2\left[\sqrt{\frac{\alpha}{\sigma}}\operatorname{cota}(2\xi\sqrt{\alpha\sigma})\pm\operatorname{csca}(2\xi\sqrt{\alpha\sigma})\sqrt{\frac{\alpha pq}{\sigma}}\right]^2}{\alpha^2\ln^2(A)(\mathrm{k1}^2-\mathrm{k2}^2)}+F$$

$$u_{15}=\frac{F^2\left[\operatorname{cota}\left(\frac{1}{2}\xi\sqrt{\alpha\sigma}\right)-\operatorname{tana}\left(\frac{1}{2}\xi\sqrt{\alpha\sigma}\right)\right]^2}{4\alpha\sigma\ln^2(A)(\mathrm{k1}^2-\mathrm{k2}^2)}+F,\quad\text{and}\quad F=\frac{-4ck_1+6l_1l_2}{4k_1k_2}$$

– when $\alpha\sigma<0$ and $\beta=0$:

$$u_{16}=F-\frac{F^2\operatorname{tanha}(\xi\sqrt{-\alpha\sigma})^2}{\alpha\sigma\ln^2(A)(\mathrm{k1}^2-\mathrm{k2}^2)},\quad u_{17}=F-\frac{F^2\operatorname{cotha}(\xi\sqrt{-\alpha\sigma})^2}{\alpha\sigma\ln^2(A)(\mathrm{k1}^2-\mathrm{k2}^2)}$$

$$u_{18-1,2}=F-\frac{F^2\left[\sqrt{-\frac{\alpha pq}{\sigma}}\operatorname{secha}(2\xi\sqrt{-\alpha\sigma})\pm i\sqrt{-\frac{\alpha}{\sigma}}\operatorname{tanha}(2\xi\sqrt{-\alpha\sigma})\right]^2}{\alpha^2\ln^2(A)(\mathrm{k1}^2-\mathrm{k2}^2)}$$

$$u_{19-1,2}=\frac{F^2\left[\sqrt{-\frac{\alpha}{\sigma}}\operatorname{cotha}(2\xi\sqrt{-\alpha\sigma})\pm\operatorname{cscha}(2\xi\sqrt{-\alpha\sigma})\sqrt{-\frac{\alpha pq}{\sigma}}\right]^2}{\alpha^2\ln^2(A)(\mathrm{k1}^2-\mathrm{k2}^2)}+F$$

$$u_{20}=F-\frac{F^2\left[\text{cotha}\left(\frac{1}{2}\xi\sqrt{-\alpha\sigma}\right)+\text{tanha}\left(\frac{1}{2}\xi\sqrt{-\alpha\sigma}\right)\right]^2}{4\alpha\sigma\ln^2(A)(\text{k}1^2-\text{k}2^2)},\quad \text{and}\quad F=\frac{-4ck_1+6l_1l_2}{4k_1k_2}$$

– when $\beta=0$ and $\sigma=\alpha$:

$$u_{21}=\frac{F^2\text{tana}(\alpha\xi)^2}{\alpha^2\ln^2(A)(k_1^2-k_2^2)}+F,\quad u_{22}=\frac{F^2\text{cota}(\alpha\xi)^2}{\alpha^2\ln^2(A)(k_1^2-k_2^2)}+F$$

$$u_{23-1,2}=\frac{F^2\left[\text{tana}(2\alpha\xi)\pm\sqrt{pq}\text{seca}(2\alpha\xi)\right]^2}{\alpha^2\ln^2(A)(k_1^2-k_2^2)}+F$$

$$u_{24-1,2}=\frac{F^2\left[\text{cota}(2\alpha\xi)\pm\sqrt{pq}\text{csca}(2\alpha\xi)\right]^2}{\alpha^2\ln^2(A)(k_1^2-k_2^2)}+F$$

$$u_{25}=\frac{F^2\left[\text{cota}\left(\frac{\alpha\xi}{2}\right)-\text{tana}\left(\frac{\alpha\xi}{2}\right)\right]^2}{4\alpha^2\ln^2(A)(k_1^2-k_2^2)}+F,\quad \text{and}\quad F=\frac{-4ck_1+6l_1l_2}{4k_1k_2}$$

– when $\beta=0$ and $\sigma=-\alpha$:

$$u_{26}=\frac{F^2\text{tanha}(\alpha\xi)^2}{\alpha^2\ln^2(A)(k_1^2-k_2^2)}+F,\quad u_{27}=\frac{F^2\text{cotha}(\alpha\xi)^2}{\alpha^2\ln^2(A)(k_1^2-k_2^2)}+F$$

$$u_{28-1,2}=F-\frac{F^2\left[\sqrt{pq}\text{secha}(2\alpha\xi)\pm i\text{tanha}(2\alpha\xi)\right]^2}{\alpha^2\ln^2(A)(k_1^2-k_2^2)}$$

$$u_{29-1,2}=\frac{F^2\left[\text{cotha}(2\alpha\xi)\pm\sqrt{pq}\text{cscha}(2\alpha\xi)\right]^2}{\alpha^2\ln^2(A)(k_1^2-k_2^2)}+F$$

$$u_{30}=\frac{F^2\left[\text{cotha}\left(\frac{\alpha\xi}{2}\right)+\text{tanha}\left(\frac{\alpha\xi}{2}\right)\right]^2}{4\alpha^2\ln^2(A)(k_1^2-k_2^2)}+F,\quad \text{and}\quad F=\frac{-4ck_1+6l_1l_2}{4k_1k_2}$$

– when $\beta^2=4\alpha\sigma$:

$$u_{31}=F\left\{-\frac{4}{\beta\xi\ln(A)}+\frac{4F[\beta\xi\ln(A)+2]^2}{\beta^4\xi^2\ln^4(A)(k_1^2-k_2^2)}-1\right\}$$

where

$$F=\frac{\beta^2k_1k_2\ln^2(A)(k_1^2-k_2^2)-4ck_1+6l_1l_2}{4k_1k_2}$$

– when $\beta = k, \alpha = mk(m \neq 0)$ and $\sigma = 0$:

$$u_{32} = \frac{F^2 (A^{k\xi} - m)^2}{k^2 m^2 \ln^2(A)(k_1^2 - k_2^2)} + \frac{F(A^{k\xi} - m)}{m} + F$$

where

$$F = \frac{k^2 k_1 k_2 \ln^2(A)(k_1^2 - k_2^2) - 4ck_1 + 6l_1 l_2}{4k_1 k_2}$$

– when $\beta = \sigma = 0$:

$$u_{33} = \frac{F^2 \xi^2}{k_1^2 - k_2^2} + F$$

where

$$F = \frac{-4ck_1 + 6l_1 l_2}{4k_1 k_2}$$

Remark 1. The generalized hyperbolic and triangular functions are defined [38, 39]:

$$\sinh a(\xi) = \frac{pA^{\xi} - qA^{-\xi}}{2}, \quad \cosh a(\xi) = \frac{pA^{\xi} + qA^{-\xi}}{2}, \quad \tanh a(\xi) = \frac{pA^{\xi} - qA^{-\xi}}{pA^{\xi} + qA^{-\xi}}$$

$$\coth a(\xi) = \frac{pA^{\xi} + qA^{-\xi}}{pA^{\xi} - qA^{-\xi}}, \quad \operatorname{sech} a(\xi) = \frac{2}{pA^{\xi} + qA^{-\xi}}, \quad \operatorname{csch} a(\xi) = \frac{2}{pA^{\xi} - qA^{-\xi}}$$

$$\sin a(\xi) = \frac{pA^{i\xi} - qA^{-i\xi}}{2i}, \quad \cos a(\xi) = \frac{pA^{i\xi} + qA^{-i\xi}}{2i}, \quad \tan a(\xi) = -i\frac{pA^{i\xi} - qA^{-i\xi}}{pA^{i\xi} + qA^{-i\xi}},$$

$$\cot a(\xi) = i\frac{pA^{i\xi} + qA^{-i\xi}}{pA^{i\xi} - qA^{-i\xi}}, \quad \sec a(\xi) = \frac{2}{pA^{i\xi} + qA^{-i\xi}}, \quad \csc a(\xi) = \frac{2i}{pA^{i\xi} - qA^{-i\xi}}$$

where ξ is an independent variable and $p, q > 0$.

Conclusion

In this paper, we use the direct algebraic method combined with characteristic set algorithm to solve the space-time fractional Fokas equation, a abundant of exact solutions are obtained, to the best of our knowledge, the solutions we obtained have not been reported in literature.

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