THE PIECEWISE REPRODUCING KERNEL METHOD FOR THE TIME VARIABLE FRACTIONAL ORDER ADVECTION-REACTION-DIFFUSION EQUATIONS

by

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This paper structures some new reproductive kernel spaces based on Legendre polynomials to solve time variable order fractional advection-reaction-diffusion equations. Some examples are given to show the effectiveness and reliability of the method.

Key words: *advection-reaction-diffusion equation, variable fractional derivative, piecewise reproducing kernel method, reproducing kernel space*

Introduction

In this paper, we consider the following time-fractional advection-reaction-diffusion equation:

$$\begin{cases} \mathrm{D}_t^{\alpha(x,t)}u(x,t)+\beta_1(x,t)\dfrac{\partial u(x,t)}{\partial x}+\beta_2\dfrac{\partial^2 u(x,t)}{\partial x^2}=f(x,t),(x,t)\in\Omega=[0,1]\times[0,1] \\ u(x,0)=\phi(x),0\le x\le 1 \\ u(0,t)=\varphi_1(t),u(1,t)=\varphi_2(t),0\le t\le 1 \end{cases} \tag{1}$$

where $\alpha(x,t)$, $\beta_1(x,t)$, $\beta_2(x,t)$, $f(x,t)$, are known functions, and $D_t^{\alpha(x,t)}u(x,t)$ is the variable order Caputo derivative defined:

$$D_t^{\alpha(x,t)}u(x,t)=\frac{1}{\Gamma[1-\alpha(x,t)]}\int_0^t(t-\tau)^{-\alpha(x,t)}\frac{\partial u(x,\tau)}{\partial\tau}\mathrm{d}\tau \tag{2}$$

The time-fractional advection-reaction-diffusion equation [1-5] has wide applications in thermal science, chemical engineering, and mechanics. It is almost impossible to obtain an analytic solution of this equation. In recent years several numerical methods have been proposed, such as the variational iteration method and the homotopy perturbation method [6-11], the reproducing kernel method [12-15], *etc.* In previous work, the Taylor's formula or Delta function was used to construct the reproducing kernel space [16-19], which has been proved to be an effective tool to solving various kinds of differential equations [20-25]. In this paper, we structure some new reproductive kernel spaces based on Legendre polynomials for numerical approach to time-fractional advection-reaction-diffusion equations.

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Structing reproductive kernel space based on Legendre polynomials

The well-known Legendre polynomials is defined on the interval [–1, 1] and its recurrence formula:

$$\begin{cases} p_0(t)=1 \\ p_n(t)=\dfrac{1}{2^n n!}\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}(t^2-1)^n,\ n=1,2,... \end{cases}, t\in[-1,1] \tag{3}$$

Let $t = 2x - 1$, we can get the following formulation:

$$\begin{cases} p_0(x)=1 \\ p_n(2x-1)=\dfrac{1}{n!}\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}[(x^2-x)^n],\ n=1,2,... \end{cases}, x\in[0,1] \tag{4}$$

The Legendre polynomials has following properties:

$$\int_0^1 \sqrt{2n+1}L_n(x)\sqrt{2m+1}L_m(x)\mathrm{d}x=\delta_{mn} \tag{5}$$

where $L_n(x) = p_n(2x - 1)$. The first nine terms of the polynomials of $(2n+1)^{1/2}L_n(x)$ are listed in the tab. 1.

Table 1. The first nine polynomials of $(2n + 1)^{1/2}L_n(x)$

n	$(2n + 1)^{1/2}L_n(x)$
0	1
1	$3^{1/2}(-1 + 2x)$
2	$5^{1/2}(1 - 6x + 6x^2)$
3	$7^{1/2}(-1 + 12x - 30x^2 + 20x^3)$
4	$3(1 - 20x + 90x^2 - 140x^3 + 70x^4)$
5	$11^{1/2}(-1 + 30x - 210x^2 + 560x^3 - 630x^4 + 252x^5)$
6	$13^{1/2}(1 - 42x + 420x^2 - 1680x^3 + 3150x^4 - 2772x^5 + 924x^6)$
7	$15^{1/2}(-1 + 56x - 756x^2 + 4200x^3 - 11550x^4 + 16632x^5 - 12012x^6 + 3432x^7)$
8	$17^{1/2}(1 - 72x + 1260x^2 - 9240x^3 + 34650x^4 - 72072x^5 + 84084x^6 - 51480x^7 + 12870x^8)$

Theorem 1. If

$$\overline{H}_n = Span\left\{L_0(x),\sqrt{3}L_1(x),\sqrt{5}L_5(x),...,\sqrt{2n+1}L_n(x)\right\}$$

the inner product in H_n is given:

$$\langle u(x),v(x)\rangle=\int_0^1 u(x)v(x)\mathrm{d}x, u(x),v(x)\in\overline{H} \tag{6}$$

then

$$K(x,y)=K_x(y)=\sum_{i=0}^{n}L_i(x)L_i(y) \tag{7}$$

is reproducing kernel of $\overline{H}_n$.

Proof. Using [21], we can prove that $\overline{H}_n$ is a reproducing kernel Hilbert space. Next we proof $K_x(y)$ is a reproducing kernel of $\overline{H}_n$ for $\forall u(y) \in \overline{H}_n$.

Let:

$$u(y)=\sum_{i=0}^{n} c_i \sqrt{2i+1} L_i(y)$$

and we have:

$$\langle u(x,t), K_x(y)\rangle = \left\langle \sum_{i=1}^{n} c_i \sqrt{2i+1} L_i(y), \sum_{j=1}^{n}(2j+1)L_j(y)\right\rangle =$$

$$=\sum_{i=1}^{n} c_i \sqrt{2i+1}\left\langle L_i(y), \sum_{j=1}^{n}(2j+1)L_j(x)L_j(y)\right\rangle = \sum_{i=1}^{n} c_i \sqrt{2i+1} L_i(x) = u(x) \tag{8}$$

so $K_x(y)$ is the reproducing kernel of $\bar{H}_n$.

Using [11] and the reproducing kernel of $\bar{H}_n$, we can get the following reproducing kernel spaces:

– Space $H_2 = u(x)|u(x) \in \bar{H}_4$, $u(0) = 0$, H_2 has the same inner product as $\bar{H}_2$, and it is a reproducing kernel space. Its reproducing kernel:

$$K_2(x,y) = 4xy[12-15y+5x(-3+4y)] \tag{9}$$

– Space $H_3 = u(x)|u(x) \in \bar{H}_5$, $u(0) = 0$, $u(1) = 0$, H_3 has the same inner product as $\bar{H}_3$, and it is a reproducing kernel space. Its reproducing kernel:

$$K_3(x,y) = 60(-1+x)x(-1+y)y[4-7y+7x(-1+2y)] \tag{10}$$

– Space $H(\Omega) = H_2 \otimes H_3 = \{u(x, t)|u(x, t) \in H_2 \otimes H_3, u(0, t) = u(1, t) = u(x, 0) = 0\}$ and its reproducing kernel:

$$K(x,t,y,s) = K_2(x,y)\times K_3(t,s), (x,y),(t,s)\in\Omega \tag{11}$$

where $K_2(x, y)$, $K_3(x, y)$ are given in eqs. (9) and (10), respectively.

Pricewise reproducing kernel method

Put:

$$\mathrm{A}_{(x,t)}u(x,t) \overset{\Delta}{=} D_t^{\alpha(x,t)}u(x,t)+\beta_1(x,t)\frac{\partial u(x,t)}{\partial x}+\beta_2(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} \tag{12}$$

$$v(x,t) = u(x,t) - U(x,t) - u_0(x) + U_0(x) \tag{13}$$

where

$$U(x,t) = \varphi_1(t)(1-x)+\varphi_2(t)x,\ U_0(x) = U(x,0),\ u_0(x) = u(x,0)$$

After homogenization, eq. (1) are converted to the following from:

$$\begin{cases} \mathrm{A}_{(x,t)}v(x,t) = g(x,t), & (x,t)\in\Omega=[0,1]\times[0,1] \\ v(x,0)=0, & 0\le x\le 1 \\ v(0,t)=0, \quad v(1,t)=0, & 0\le t\le 1 \end{cases} \tag{14}$$

where

$$g(x,t) = f(x,t) - \mathrm{A}_{(x,t)}[U(x,t)+u_0(x)-U_0(x)]$$

Let $\{x_i, t_i\}_{i=1}^{\infty}$ be nodes in interval:
$[0,1]\times[0,1], \psi_i(x,t) = L_{(y,s)}K_{(y,s)}(x,t)\Big|_{(y,s)=(x_i,t_i)}$, $i=1,2,...,\infty$

$$\overline{\psi_i}(x,t) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x,t), \ (\beta_{ii} > 0, \ i = 1,2,..,\infty) \tag{15}$$

where β_{ik} are the coefficients resulting from Gram-Schmidt orthonormalization.

Theorem 2. If A^{-1} exists and $\{x_i, t_i\}_{i=1}^{\infty}$ is denumerable dense points in Ω:

$$v(x,t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} g(x_k, t_k)\overline{\psi_i}(x,t) \tag{16}$$

is an analytical solution of eq. (14).

In view of eq. (16), an approximate solution of eq. (14) can be expressed:

$$v_N(x,t) = \sum_{i=1}^{N}\sum_{k=1}^{i} \beta_{ik} g(x_k, t_k)\overline{\psi_i}(x,t) \tag{17}$$

However, the direct application of eq. (14) could not have a good numerical accuracy. In order to solve this problem, we use the piecewise reproducing kernel method [21-23].

Numerical simulation

In this section, some numerical tests are given to demonstrate the accuracy of the present method.

Example 1. We consider the following time fractional advection-reaction-diffusion equation:

$$\begin{cases} D_t^{\alpha(x,t)} u(x,t) = \dfrac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \ \ 0 \le x \le 1, \ \ 0 \le t \le 1 \\ u(x,0) = 6x^2(1+2x), \ \ u(0,t) = 0, \ \ u(1,t) = 18(1+t^2) \end{cases} \tag{18}$$

The exact solution is $u_T(x, t) = 6x^2(1 + 2x)(1 + t)^2$. Numerical results of *Example 1* are shown in tab. 2.

Table 2. Comparison of absolute errors obtained by present method for *Example 1* at t = 0.005

x	Exact solution $\alpha = 0.5$	Traditional RKM $\alpha = 0.5$ $h = 1$	Present method $\alpha = 0.5$ $h = 0.0001$	Present method $\alpha = 0.6$ $h = 0.0000000001$
0.1	0.0727218	$5.197 \cdot 10^{-3}$	$4.523 \cdot 10^{-5}$	$1.753 \cdot 10^{-10}$
0.2	0.3393680	$7.315 \cdot 10^{-3}$	$8.426 \cdot 10^{-5}$	$3.502 \cdot 10^{-10}$
0.3	0.8726620	$7.074 \cdot 10^{-3}$	$1.157 \cdot 10^{-4}$	$5.103 \cdot 10^{-10}$
0.4	1.7453200	$5.198 \cdot 10^{-3}$	$1.380 \cdot 10^{-4}$	$6.411 \cdot 10^{-10}$
0.5	3.0300700	$2.407 \cdot 10^{-3}$	$1.498 \cdot 10^{-4}$	$7.281 \cdot 10^{-10}$
0.6	4.7996400	$5.768 \cdot 10^{-3}$	$1.495 \cdot 10^{-4}$	$7.568 \cdot 10^{-10}$
0.7	7.1267400	$3.031 \cdot 10^{-3}$	$1.359 \cdot 10^{-4}$	$7.129 \cdot 10^{-10}$
0.8	10.087100	$4.234 \cdot 10^{-3}$	$1.074 \cdot 10^{-4}$	$5.817 \cdot 10^{-10}$
0.9	13.744400	$3.464 \cdot 10^{-3}$	$6.259 \cdot 10^{-5}$	$3.489 \cdot 10^{-10}$

Example 2. We consider the following time fractional advection-reaction-diffusion equation [1-4]:

$$\begin{cases} D_t^\alpha u(x,t) + x\dfrac{\partial^2 u(x,t)}{\partial x^2} = 2t^\alpha + 2x^2 + 1, \quad 0 \le x \le 1, \quad 0 \le t \le 1 \\ u(x,0) = x^2, \quad u(0,t) = \dfrac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}, \quad u(1,t) = 1 + \dfrac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha} \end{cases} \tag{19}$$

The exact solution is:

$$u_T(x,t) = x^2 + \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\gamma}$$

Numerical results of *Example 2* are shown in fig. 1 and tabs. 3 and 4.

Example 3. We consider the following time fractional time-space fractional diffusion equation [5]:

$$\begin{cases} \dfrac{\partial^\gamma u(x,t)}{\partial t^\gamma} = -x^{0.8}\dfrac{\partial u(x,t)}{\partial x} + \dfrac{x}{2}\Gamma(2.8)\dfrac{\partial^\eta u(x,t)}{\partial x^\eta} + q(x,t), \quad 0 \le x \le 1, \quad 0 < t \le 1 \\ u(x,0) = x^2(1-x), \quad u(0,t) = u(1,t) = 0 \end{cases} \tag{20}$$

The exact solution is $u_T(x, t) = x^2(1 - x)(1 + t^2)$. Numerical solution of *Example 3* are shown tab. 5. Reproducing kernels are shown in figs. 2 and 3.

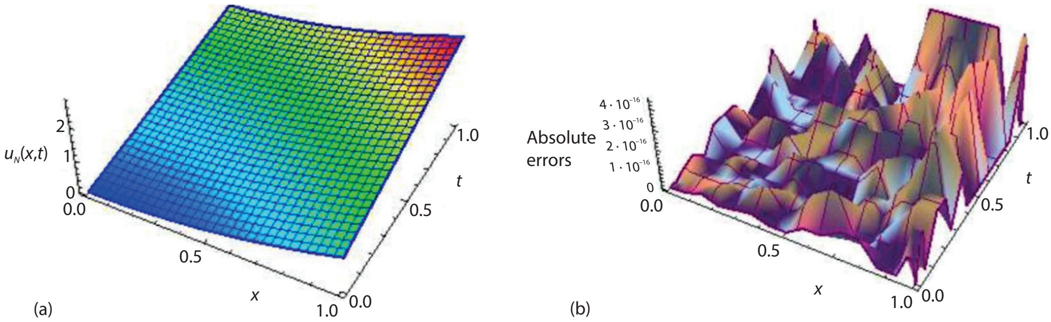

Figure 1. The left shows the approximate solutions of *Example 2* for α = 0.5, *N* = 2, the right shows the absolute errors of *Example 2* for α = 0.5, *N* = 2

Table 3. Comparison of absolute errors of *Example 2* for α = 0.5, *t* = 0.5

x	Present method $N = 2$	[1] $m = 15$	[1] $m = 25$	[2] $m = 32$	[2] $m = 64$	[3] $m = 6$
0.1	$1.110 \cdot 10^{-16}$	$6.994 \cdot 10^{-5}$	$6.462 \cdot 10^{-6}$	$6.093 \cdot 10^{-3}$	$1.210 \cdot 10^{-3}$	$1.110 \cdot 10^{-16}$
0.2	0	$1.721 \cdot 10^{-4}$	$1.578 \cdot 10^{-5}$	$4.843 \cdot 10^{-3}$	$1.259 \cdot 10^{-3}$	$1.110 \cdot 10^{-16}$
0.3	0	$2.472 \cdot 10^{-4}$	$2.272 \cdot 10^{-5}$	$2.750 \cdot 10^{-2}$	$1.865 \cdot 10^{-3}$	$2.220 \cdot 10^{-16}$
0.4	$2.220 \cdot 10^{-16}$	$2.912 \cdot 10^{-4}$	$2.674 \cdot 10^{-5}$	$1.937 \cdot 10^{-2}$	$7.412 \cdot 10^{-3}$	$2.220 \cdot 10^{-16}$
0.5	0	$3.004 \cdot 10^{-4}$	$2.759 \cdot 10^{-5}$	$1.000 \cdot 10^{-6}$	$1.000 \cdot 10^{-6}$	0
0.6	$2.220 \cdot 10^{-16}$	$2.760 \cdot 10^{-4}$	$2.534 \cdot 10^{-5}$	$4.359 \cdot 10^{-2}$	$7.460 \cdot 10^{-3}$	0
0.7	$2.220 \cdot 10^{-16}$	$2.213 \cdot 10^{-4}$	$2.035 \cdot 10^{-5}$	$1.734 \cdot 10^{-2}$	$1.724 \cdot 10^{-3}$	$2.220 \cdot 10^{-16}$
0.8	$2.220 \cdot 10^{-16}$	$1.440 \cdot 10^{-4}$	$1.320 \cdot 10^{-5}$	$7.750 \cdot 10^{-2}$	$4.990 \cdot 10^{-3}$	0
0.9	0	$5.026 \cdot 10^{-5}$	$4.653 \cdot 10^{-6}$	$4.443 \cdot 10^{-2}$	$1.678 \cdot 10^{-2}$	0

Table 4. Comparison of absolute errors obtained by present method for *Example 2*

(x,t)	Present method $\alpha = 0.3$	[3] $\alpha = 0.3$	Present method $\alpha = 0.5$	[3] $\alpha = 0.5$	Present method $\alpha = 0.9$	[3] $\alpha = 0.9$
(0.1, 0.1)	0	0	$2.776 \cdot 10^{-17}$	$2.776 \cdot 10^{-17}$	0	$3.122 \cdot 10^{-17}$
(0.2, 0.2)	$1.110 \cdot 10^{-16}$	$1.110 \cdot 10^{-16}$	$5.551 \cdot 10^{-16}$	$1.110 \cdot 10^{-16}$	$2.776 \cdot 10^{-17}$	$2.220 \cdot 10^{-16}$
(0.3, 0.3)	$2.220 \cdot 10^{-16}$	0	$1.110 \cdot 10^{-16}$	$2.220 \cdot 10^{-16}$	$5.551 \cdot 10^{-17}$	$2.220 \cdot 10^{-16}$
(0.4, 0.4)	0	0	0	$1.110 \cdot 10^{-16}$	0	$1.776 \cdot 10^{-15}$
(0.5, 0.5)	0	$2.220 \cdot 10^{-18}$	0	0	$1.110 \cdot 10^{-16}$	$2.564 \cdot 10^{-12}$
(0.6, 0.6)	0	$4.440 \cdot 10^{-16}$	0	0	$1.110 \cdot 10^{-16}$	$2.700 \cdot 10^{-13}$
(0.7, 0.7)	$4.441 \cdot 10^{-16}$	0	0	$2.220 \cdot 10^{-16}$	$2.220 \cdot 10^{-16}$	$6.972 \cdot 10^{-13}$
(0.8, 0.8)	0	0	$4.441 \cdot 10^{-16}$	0	0	$3.648 \cdot 10^{-13}$
(0.9, 0.9)	$4441 \cdot 10^{-16}$	$4.440 \cdot 10^{-16}$	$8.882 \cdot 10^{-16}$	$4.440 \cdot 10^{-16}$	0	$1.209 \cdot 10^{-12}$

Table 5. Comparison of absolute errors obtained by present method for *Example 3* at $\alpha = \beta = 0.5$, $t = 0.2$

x	Exact solution	Present method $N = 2$	Present method ($h = 0.0000000001$) $N = 2$	[5] $N = 6$
0.1	0.0094	$5.983 \cdot 10^{-3}$	$3.600 \cdot 10^{-13}$	$5.373 \cdot 10^{-11}$
0.2	0.0333	$9.996 \cdot 10^{-3}$	$1.280 \cdot 10^{-12}$	$1.021 \cdot 10^{-10}$
0.3	0.0655	$1.228 \cdot 10^{-2}$	$2.520 \cdot 10^{-12}$	$1.621 \cdot 10^{-10}$
0.4	0.0998	$1.307 \cdot 10^{-2}$	$3.840 \cdot 10^{-12}$	$2.335 \cdot 10^{-10}$
0.5	0.1300	$1.262 \cdot 10^{-2}$	$5.000 \cdot 10^{-12}$	$3.174 \cdot 10^{-10}$
0.6	0.1498	$1.115 \cdot 10^{-2}$	$5.760 \cdot 10^{-12}$	$4.142 \cdot 10^{-10}$
0.7	0.1529	$8.919 \cdot 10^{-3}$	$5.880 \cdot 10^{-12}$	$5.242 \cdot 10^{-10}$
0.8	0.1331	$6.156 \cdot 10^{-3}$	$5.120 \cdot 10^{-12}$	$6.464 \cdot 10^{-10}$
0.9	0.0842	$3.103 \cdot 10^{-3}$	$3.240 \cdot 10^{-12}$	$7.787 \cdot 10^{-10}$

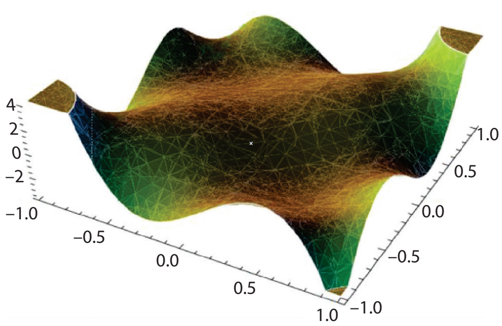

Figure 2. Reproducing kernel of space $\bar{H}_3[0, 1]$

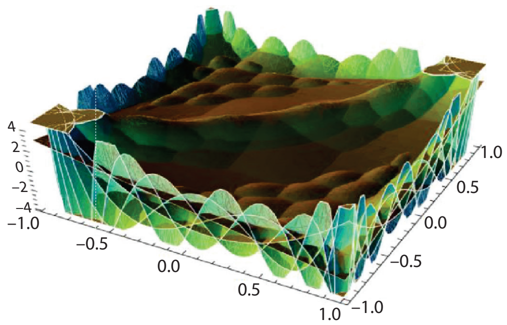

Figure 3. The set of reproducing kernel of space $\bar{H}_n[0, 1]$ with $n = 1, 2, ..., 7$

Conclusion

In this paper, some new reproductive kernels are given. The numerical results show that the present method has high precision compared with traditional reproducing kernel method, and has a better convergence. Besides, the method can also be used to study other fractional advection-dispersion models and fractal advection-dispersion models with fractal derivatives [26-31].

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