THE EXACT SOLUTION OF THE NON-LINEAR SCHRODINGER EQUATION BY THE EXP-FUNCTION METHOD

by

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This paper elucidates the main advantages of the exp-function method in finding the exact solution of the non-linear Schrodinger equation. The solution process is extremely simple and accessible, and the obtained solution contains some free parameters.

Keywords: *non-linear Schrodinger equation, exact solution, exp-function method*

Introduction

The non-linear Schrodinger equation is widely studied in physics, mathematics and engineering [1-9], which is a non-linear parabolic PDE, and it is difficult to solve it explicitly, though there are many analytical methods available in literature to have an approximate solution, such as the variational iteration method [10, 11], the homotopy perturbation [12, 13], the variational approach [14-20], and the others, a complete review on various analytical methods is available in the review articles [21]. An explicit and exact solution is much needed in practical applications, this paper adopts the exp-function method for this purpose.

Exp-function method

In this paper, we consider the following non-linear Schrodinger equation [22]:

$$iq_t + iaq_{xxx} + bq_{xxxx} + F(|q|^2)q = 0 \tag{1}$$

where a and b are coefficients that are real parameters, and F represents a general form of the intensity dependent refractive index.

We introduce a complex variation defined as:

$$q(x,t) = g(\xi)\mathrm{e}^{i\varphi(x,t)} \tag{2}$$

where

$$\xi = x - vt \tag{3}$$

where v is the velocity of the solution. The phase portion of the pulse is given by the form:

$$\varphi(x,t) = -kx + \omega t \tag{4}$$

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Here, k is the soliton frequency and ω is the soliton wave number. The following equation is obtained:

$$\frac{\partial q}{\partial t} = -vg'\mathrm{e}^{i\varphi(x,t)} + i\omega g\mathrm{e}^{i\varphi(x,t)} \tag{5}$$

$$\frac{\partial q}{\partial x} = g'\mathrm{e}^{i\varphi(x,t)} - ikg\mathrm{e}^{i\varphi(x,t)} \tag{6}$$

$$\frac{\partial^2 q}{\partial x^2} = g''\mathrm{e}^{i\varphi(x,t)} - 2ikg'\mathrm{e}^{i\varphi(x,t)} - k^2 g\mathrm{e}^{i\varphi(x,t)} \tag{7}$$

$$\frac{\partial^3 q}{\partial x^3} = g'''\mathrm{e}^{i\varphi(x,t)} - 3ikg''\mathrm{e}^{i\varphi(x,t)} - 3k^2 g'\mathrm{e}^{i\varphi(x,t)} + ik^3 g\mathrm{e}^{i\varphi(x,t)} \tag{8}$$

Substituting eqs. (5)-(8) into eq. (1):

$$bg^{(4)} + (3ak - 6k^2 b)g'' - (bk^4 - \omega - ak^3)g + Fg^3 +$$

$$+ i[(v + 3ak^2 - 4bk^3)g' + (bk - a)g'''] = 0 \tag{9}$$

where

$$a = bk, \qquad v = 4bk^3 - 3ak^2$$

We get an ODE:

$$bg^{(4)} + H_1 g + H_2 g'' + Fg^3 = 0 \tag{10}$$

where $H_1 = ak^3 + \omega - bk^4$, $\quad H_2 = 3ak - 6k^2 b$

The exp-function method was first proposed by Chinese mathematician, He and Wu [23]. We consider the following general PDE to show the solution process [23-28]:

$$P(u, u_t, u_x, u_y, u_{xx}, u_{tt}, u_{yy}) = 0 \tag{11}$$

By the transformation:

$$\xi = kx + \omega t + ly \tag{12}$$

where k, ω and l are unknown constants. Equation (12) is converted into the following non-linear ordinary differential equation:

$$G(u, u', u'', u''', ...) = 0 \tag{13}$$

According to the exp-function method, we assume the solution can be expressed in the following form:

$$u(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_n \exp(m\xi)} \tag{14}$$

where *c, d, p,* and *q* are positive integers that could be freely chosen. By substituting eq. (14) into eq. (13), collecting terms of the same term of exp($i\xi$), and equating the coefficient of each power of exp($i\xi$) to zero, we can get a set of algebraic equations for determining unknown constants.

For the present problem, we assume that its solution has the form:

$$g(\eta)=\frac{a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)}{\exp(\eta)+b_0+b_{-1}\exp(-\eta)} \tag{15}$$

where a_1, a_0, a_{-1}, b_0, and b_{-1} are constants. Substituting eq. (15) into eq. (10), setting the coefficients of $\exp(i\eta),(i=0,\pm1,\pm2,\pm3)$ to zero, we obtain the following algebraic equations:

$$71a_1b+a_1\omega-a_1(3ak-6k^2b)+Fa_1^{\,3}+a_1b^2=0 \tag{16}$$

$$71ba_{-1}b_{-1}^4+\omega a_{-1}b_{-1}^4-(3ak-6k^2b)a_{-1}b_{-1}^4+Fa_1^3b_{-1}^2+b^2a_{-1}b_{-1}^4+ba_{-1}b_{-1}^4(3ak-6k^2b)=0 \tag{17}$$

$$\begin{gathered}73bb_{-1}^3a_0+\omega b_{-1}^3a_0+(3ak-6k^2b)b_{-1}^3a_0+2Fa_{-1}^3b_0+67ba_{-1}b_0b_{-1}^2+4\omega a_{-1}b_0b_{-1}^2-\\-5(3ak-6k^2b)a_{-1}b_0b_{-1}^2+3Fa_{-1}^2b_{-1}a_0+4b^2a_{-1}b_0b_{-1}^2+4(3ak-6k^2b)ba_{-1}b_0b_{-1}^2=0\end{gathered} \tag{18}$$

$$\begin{gathered}73ba_0+\omega a_0+a_0(3ak-6k^2b)+2Fa_1^{\,3}b_0+3Fa_1^2a_0+4a_1b^2b_0+67a_1bb_0+4a_1b_0\omega-\\-5(3ak-6k^2b)a_1b_0+4(3ak-6k^2b)a_1bb_0=0\end{gathered} \tag{19}$$

$$\begin{gathered}2Fa_{-1}^3b_{-1}+87a_1bb_{-1}^4-20ba_{-1}b_{-1}^3+\omega a_1b_{-1}^4+4\omega a_1b_{-1}^3+3(3ak-6k^2b)a_1b_{-1}^4\\-8(3ak-6k^2b)a_{-1}b_{-1}^3+Fa_{-1}^3b_0^2+a_1b^2b_{-1}^4+4b^2a_{-1}b_{-1}^3-7a_{-1}b_0^2b_{-1}^2(3ak-6k^2b)+\\+b^2a_{-1}b_0^2b_{-1}^2+b_0b_{-1}^3a_0(61b+4\omega+3ak-6k^2b)+a_1bb_{-1}^4(3ak-6k^2b)+\\+4ba_{-1}b_{-1}^3(3ak-6k^2b)+3Fb_{-1}^2(a_1a_{-1}^2+a_{-1}a_0^2)+\\+a_{-1}b_0^2b_{-1}^2(5b+6\omega+18ak-36k^2b)+6Fa_{-1}^2b_0b_{-1}a_0=0\end{gathered} \tag{20}$$

$$\begin{gathered}4\omega a_0b_0+(3ak-6k^2b)(a_0b_0-8a_1b_{-1}+4ba_1b_{-1}+6ba_1b_0^2+3a_{-1}-7a_1b_0^2+ba_{-1})+\\+Fa_1(3a_1a_{-1}+3a_0^2+a_1^2b_0^2+6a_1b_0a_0+2a_1^2b_{-1})+87ba_{-1}+\omega a_{-1}+5a_1bb_0^2+\\+6\omega a_1b_0^2+a_1b^2(4b_{-1}+6b_0)-20a_1bb_{-1}+61bb_0b_{-1}+4\omega a_1b_{-1}=0\end{gathered} \tag{21}$$

$$\begin{gathered}a_1b^2(b_0^4+b_{-1}^2)+F(a_1^3b_{-1}^2+3a_1a_{-1}^2+3a_{-1}a_0^2+3a_{-1}a_0^2+2b_0a_0^3+6a_1^2a_{-1}b_{-1}+\\+6a_0b_{-1}a_0^2+3a_1b_0^2a_0^2+3a_1^2a_{-1}b_0^2+6a_1^2b_0b_{-1}a_0)+(3ak-6k^2b)(5a_{-1}b_0^2-10a_1b_{-1}^2-b_0^3a_0+\\+a_1bb_0^4+6a_1bb_{-1}^2+6ba_{-1}b_0^2-10a_1b_0^2b_{-1}+4ba_{-1}b_{-1}-13b_0b_{-1}a_0+12a_1bb_0^2b_{-1})+\\+b^2a_{-1}(6b_0^2+4b_{-1})-180ba_{-1}b_{-1}+4\omega a_{-1}b_{-1}+26ba_1b_{-1}^2+5ba_{-1}b_0^2-bb_0^3a_0+\omega a_1b_0^4+\\+6\omega a_1b_{-1}^2+6\omega a_{-1}b_0^2+4\omega b_0^3a_0-70a_1bb_0^2b_{-1}+12\omega a_1b_0^2b_{-1}+12a_1b^2b_0^2b_{-1}-\\-25bb_0a_0b_{-1}+12\omega b_0a_0b_{-1}=0\end{gathered} \tag{22}$$

$$F(a_{-1}^3 + 6a_1a_{-1}^2b_{-1} + 6a_{-1}^2a_0b_0 + 6cb_{-1}a_0^2 + 2b_0b_{-1}a_0^3 + 3a_1a_{-1}^2b_0^2 + 3a_1^2a_{-1}b_{-1}^2 + 3a_1b_{-1}^2a_0^2 +$$
$$+12a_1a_{-1}a_0b_0b_{-1}) + (3ak - 6k^2b)(5a_1b_0^2b_{-1}^2 - 10a_{-1}b_{-1}^2 + ba_{-1}b_0^4 + 6ba_{-1}b_{-1}^2 - 10a_{-1}b_0^2b_{-1} -$$
$$-13a_0b_0b_{-1}^2 - a_0b_0^3b_{-1} + 12ba_{-1}b_0^2b_{-1} + 6a_1bb_0^2b_{-1}^2) + \omega(4a_1b_{-1}^3 + a_{-1}b_0^4 + 6a_{-1}b_{-1}^2 +$$
$$+12a_{-1}b_0^2b_{-1} + 12a_0b_0b_{-1}^2 + 4a_0b_0^3b_{-1} + 6ab_0^2b_{-1}^2) - 180a_1bb_{-1}^3 + 26ba_{-1}b_{-1}^2 + 4a_1b^2b_{-1}^3 +$$
$$+b^2a_{-1}b_0^4 + 6b^2a_{-1}b_{-1}^2 + 6a_1b^2b_0^2b_{-1}^2 - 70ba_{-1}b_0^2b_{-1} + 25a_0bb_0b_{-1}^2 - bb_0^3b_{-1}a_0 +$$
$$+4a_1bb_{-1}^3 + 5a_1bb_0^2b_{-1}^2 = 0 \tag{23}$$

Solving the previous system, we get:

$$a_1 = \sqrt{\frac{3b^2k^2 - 3bk^2 - b^2 - \omega - 71b}{F}}$$

$$a_0 = \frac{(2a\omega^2 - 2a\omega)b_0}{-9b^2k^2 + 9bk^2 + 3b^2 + 3\omega + 213b - 3F} \tag{24}$$
$$a_{-1} = a_1b_{-1}, \quad b_0 = b_0, \quad b_{-1} = b_{-1}$$

The solution of eq. (10) is:

$$u(x,t) = \frac{\sqrt{\dfrac{3b^2k^2 - 3bk^2 - b^2 - \omega - 71b}{F}}\exp(x + bk^3t) + \dfrac{(2a\omega^2 - 2a\omega)b_0}{-9b^2k^2 + 9bk^2 + 3b^2 + 3\omega + 213b - 3F} + a_1b_{-1}\exp(-x - bk^3t)}{\exp(x + bk^3t) + b_0 + b_{-1}\exp(-x - bk^3t)} \tag{25}$$

where b_0, b_{-1}, and a_1 are free parameters, $a_{-1} = a_1b_{-1}$.

Conclusion

We obtain an exact solution for eq. (1) by the exp-function method, the explicit solution given in eq. (25) gives a clear solution property of the non-linear Schrodinger equation, it includes the various solitary wave solutions, b_0b_{-1} and a_1 are free parameters. When:

$$b_{-1} = \sqrt{\frac{3b^2k^2 - 3bk^2 - b^2 - \omega - 71b}{F}} = 1$$

a standard solitary wave solution is obtained.

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