MULTI-COMPLEXITON SOLUTIONS OF THE (2+1)-DIMENSIONAL ASYMMETRICAL NIZHNIK-NOVIKOV-VESELOV EQUATION

by

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In this paper, the (2+1)-dimensional asymmetrical Nizhnik-Novikov-Veselov equation is investigated to acquire the complexiton solutions by the Hirota direct method. It is essential to transform the equation into Hirota bi-linear form and to build N-compilexiton solutions by pairs of conjugate wave variables.

Key words: *(2+1)-dimensional asymmetrical Nizhnik-Novikov-Veselov equation, complexiton solution, pairs of conjugate wave variables, Hirota bi-linear form*

Introduction

Non-linear differential equations (NLDE) represent numerous phenomena in many fields of mathematics and physics [1-6]. Therefore, it has always attracted attentions of mathematicians and physicists to find exact solutions. The Hirota direct method is considered an effective method to find multiple soliton solutions [7-11].

Complexiton solution was first introduced in [12] which means a combination of exponential waves and trigonometric waves and corresponds to complex eigenvalues of associated characteristic problems. Several methods for complexitons have been developed [13-16], among which the Hirota direct method was proved a promising one [17-21].

In this paper, we investigate the (2+1)-D asymmetrical Nizhnik-Novikov-Veselov (ANNV) equation:

$$u_t + u_{xxx} + 3\left(u\int u_x \mathrm{d}y\right)_x = 0 \tag{1}$$

The ANNV equation describes an incompressible fluid, which was first proposed by Boiti *et al.* [22]. Hu [23] obtained the variable separation solutions by Darboux transformations of the eq. (1). This equation has been extensively studied, for examples, Yong and Qi [24] obtained a series of double periodic solutions through the rational elliptic function expansion method, Dai and Zhou [25] constructed separation solutions by the extended tanh-function method, Fan [26] derived the quasi-periodic wave solutions and established the relations between the quasi-periodic wave solutions and soliton solutions, and Zhao *et al.* [27] analyzed the lump soliton, mixed lump stripe and periodic lump solutions.

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With the dependent variable transformation:

$$u = u_0 + 2\ln(f)_{xy} \tag{2}$$

in which u_0 is a constant solution of eq. (1). The Hirota bi-linear form of ANNV is given:

$$(D_x^3 D_y + D_y D_t + 3u_0 D_x^2) ff = 0 \tag{3}$$

where *f* is a real function of *x*, *y* and *t*, D_x^3, D_x^2, D_y and D_t are Hirota bi-linear operators.

Fundamental methods

We consider the bi-linear equation:

$$H(D_{x_1}, D_{x_2}, ..., D_{x_M}) ff = 0 \tag{4}$$

where *H* is a polynomials with *M* variables, and it satisfies $H(0) = 0$ and $H(-x) = H(x)$.

We introduce a complex wave variable defined:

$$\xi_i = \zeta_0 + \sum_{k=1}^{M} \zeta_k x_k \tag{5}$$

where $\zeta_k \in \mathbb{C}$, $k = 0, 1, \dots, M$.

Supposing the expansion $f = 1 + \varepsilon f_{f_1} + \varepsilon^2 f_2 + ...$, letting $f_1 = e^{\xi_1}$, substituting the expansion and f_1 into eq. (4) and collecting all the coefficients about ε, taking coefficient of ε, we have:

$$H(\zeta_1, \zeta_2, ..., \zeta_M) f_1 = 0 \tag{6}$$

Consequently, if a complex function $f = 1 + e^{\xi_1}$ is a solution of eq. (4), they are corresponding to the same dispersion relation:

$$H(\zeta_1, \zeta_2, ..., \zeta_M) = 0, \quad H(\bar{\zeta}_1, \bar{\zeta}_2, ..., \bar{\zeta}_M) = 0 \tag{7}$$

Through the Hirota method, we study $N = 2, \xi_j = \zeta_{j0} + \Sigma_{k=1}^{M} \zeta_{jk} x_k$, $j = 1,2$, which corresponds to the dispersion relation and the complex function:

$$f = 1 + e^{\xi_1} + e^{\xi_2} + \theta_{12} e^{\xi_1 + \xi_2}, \quad \theta_{12} = -\frac{H(\zeta_{21} - \zeta_{11}, ..., \zeta_{2M} - \zeta_{1M})}{H(\zeta_{21} + \zeta_{11}, ..., \zeta_{2M} + \zeta_{1M})} \tag{8}$$

If ξ_1, ξ_2 are reciprocal conjugation, eq. (8) becomes:

$$f = 1 + e^{\xi_1} + e^{\xi_2} + \theta_{12} e^{2\xi_1} = 1 + e^{\mathrm{Re}(\xi_1)} \cos[\mathrm{Im}(\xi_1)] + \theta_{12} e^{2\,\mathrm{Re}(\xi_1)} \tag{9}$$

When $N \geq 3$, based on [17] the form of *N*-soliton solutions can be shown:

$$\sum \exp\left(\sum_{j=1}^{N} \mu_j \xi_j + \sum_{j<k} \vartheta_{jk} \mu_j \xi_k \right) \tag{10}$$

where $\mu_j = 0 \text{ or } 1$ to $j = 1, 2, ..., N$, $e^{\vartheta_{jk}} = \theta_{jk}$ which equals to:

$$\theta_{jk} = -\frac{H(\zeta_{k1} - \zeta_{j1}, ..., \zeta_{kM} - \zeta_{jM})}{H(\zeta_{k1} + \zeta_{j1}, ..., \zeta_{kM} + \zeta_{jM})}, \quad 1 \leq j < k \leq N \tag{11}$$

Furthermore, H corresponds to the Hirota condition:

$$\sum H\left(\sum_{j=1}^{N}\sigma_j\zeta_{j1},...,\sum_{j=1}^{N}\sigma_j\zeta_{jM}\right)\prod_{k<j}H(\sigma_k\zeta_{k1}-\sigma_j\zeta_{j1},...,\sigma_k\zeta_{kM}-\sigma_j\zeta_{jM})\sigma_k\sigma_j=0 \quad (12)$$

where $\sigma_j=\pm 1,\ \ j,k=1,2,...,N$. According to the *Theorem*, we can derive multi-complexitons.

Theorem [11]. Let H be a real polynomials satisfying $H(0)=0,\ \ H(-x)=H(x)$ for $x\in R^M$, N be a positive integer. Assume that the complex wave variables $\xi_j=\zeta_{j0}+\Sigma_{k=1}^{M}\zeta_{jk}x_k,\ \ j=1,\underline{3},...,2N-1,$ satisfying the dispersion relation and the Hirota condition, and suppose $\xi_{2i}=\bar{\xi}_{2j-1},\ \ j=1,2,...,N,$ then the function:

$$f=1+\sum_{n=1}^{2N}\sum_{\sum_{j=1}^{2N}\mu_j=n}\exp\left(\sum_{j=1}^{2N}\mu_j\xi_j+\sum_{k<j}\vartheta_{kj}\mu_k\mu_j\right) \quad (13)$$

presents a complexiton solutions to eq. (4), where $\mu_j=0$ or 1, for $j=1,2,...,2N,$ and $e^{\vartheta_{jk}}=\theta_{jk}$, $j,k=1,2,...,2N$, determined by eq. (11).

Multi-complexiton solutions to ANNV

In this part, we utilize 2*N*-soliton solutions to builds *N*-compilexiton solutions by pairs of conjugate wave variables and Hirota direct method. One, two, and *N*-complexitons to eq. (3) are respectively obtained in following procedures.

Consider $2N = 2$, according to eq. (5), we take:

$$\xi'(x,y,t)=k'x+l'y+m't+\xi_{*'} \quad (14)$$

where k', l', m' and $\xi_{*'}$ are parameters. Through pairs of conjugate wave variables, we suppose that $\xi'=\xi_1'=\xi_1+I\xi_2$, $\xi_2'=\xi_1-I\xi_2$ (in which $I=\sqrt{-1}$), that means $\xi_1=\mathrm{Re}(\xi')$, $\xi_2=\mathrm{Im}(\xi')$.

If $f=1+e^{\xi'}$ is a complexiton solution of eq. (3), if and only if the following dispersion relation is satisfied:

$$H(k',l',m')=H(\bar{k}',\bar{l}',\bar{m}')=0 \quad (15)$$

Making $k_1 = \mathrm{Re}(k')$, $k_2 = \mathrm{Im}(k')$, $l_1 = \mathrm{Re}(l')$, $l_2 = \mathrm{Im}(l')$, $m_1 = \mathrm{Re}(m')$, and $m_2 = \mathrm{Im}(m')$, we have $\xi_1=k_1x+l_1t+m_1+\xi_{1*}$, $\xi_2=k_2x+l_2t+m_2+\xi_{2*}$. Then, eq. (15) becomes the following form:

$$\begin{aligned} m_1&=\frac{-k_1^3l_1^2-k_1^3l_2^2+3k_1k_2^2l_1^2+3k_1l_2^2k_2^2-3k_1^2l_1u_0-6k_1l_2k_2u_0+3k_2^2l_1u_0}{l_1^2+l_2^2}\\ m_2&=\frac{k_2^3l_1^2+l_2^2k_2^3-3k_1^2k_2l_1^2-3k_1^2l_2^2k_2+3k_1^2l_2u_0-6k_1k_2l_1u_0-3l_2k_2^2u_0}{l_1^2+l_2^2}\end{aligned} \quad (16)$$

Through 2-soliton formulation, we can derive the 1-complexiton solution of eq. (3)

$$f=1+e^{\xi_1'}+e^{\xi_2'}+\theta_{12}e^{\xi_1'+\xi_2'}=1+2e^{\xi_1}\cos(\xi_2)+\theta_{12}e^{2\xi_1} \quad (17)$$

in which:

$$\theta_{12}=-\frac{H(k'-\bar{k}',l'-\bar{l}',m'-\bar{m}')}{H(k'+\bar{k}',l'+\bar{l}',m'+\bar{m}')}=-\frac{H(2ik_2,2il_2,2im_2)}{H(2k_1,2l_2,2m_2)}=$$

$$=-\frac{\left[(l_1^2l_2+l_2^3)k_2-u_0l_2^2\right]k_1^2+2u_0k_1k_2l_1l_2+(l_1^2l_2+l_2^3)k_2^3-u_0k_2^2l_1^2}{(l_1^3+l_1l_2^2)k_1^3+u_0k_1^2l_2^2+\left[(l_1^3+l_1l_2^2)k_2^2-2u_0k_2l_1l_2\right]k_1+u_0k_2^2l_1^2} \tag{18}$$

The 1-complexiton solution of eq. (1) is expressed:

$$u=u_0+\frac{2\left\{4\theta_{12}k_1l_1\mathrm{e}^{2\xi_1}-2[(-k_1l_1+k_2l_2)\cos(\xi_2)+\sin(\xi_2)(k_1l_2+k_2l_1)]\mathrm{e}^{\xi_1}\right\}}{1+2\mathrm{e}^{\xi_1}\cos(\xi_2)+\theta_{12}\mathrm{e}^{2\xi_1}}\cdot$$

$$\cdot\frac{8\left[k_1\mathrm{e}^{\xi_1}\cos(\xi_2)-\mathrm{e}^{\xi_1}k_2\sin(\xi_2)+\theta_{12}k_1\mathrm{e}^{2\xi_1}\right]\left[l_1\mathrm{e}^{\xi_1}\cos(\xi_2)-\mathrm{e}^{\xi_1}l_2\sin(\xi_2)+\theta_{12}l_1\mathrm{e}^{2\xi_1}\right]}{\left[1+2\mathrm{e}^{\xi_1}\cos(\xi_2)+\theta_{12}\mathrm{e}^{2\xi_1}\right]^2} \tag{19}$$

where l_1, l_2, k_1, k_2, ξ_{1*}, ξ_{2*}, t and u_0 are arbitrary constants.

Next, in order to obtain 2-complexiton solution, we consider $2N = 4$, assume the following function:

$$\xi'(x,y,t)=k'x+l'y+m't+\xi_{*'},\quad \xi''(x,y,t)=k''x+l''y+m''t+\xi_{*''} \tag{20}$$

where k', k'', l', l'', m', m'', $\xi_{*'}$, and $\xi_{*''}$ are constants.

Through pairs of conjugate wave variables, we take $\xi'=\xi_1'=\xi_1+I\xi_2$, $\xi_2'=\xi_1-I\xi_2$, $\xi''=\xi_3'=\xi_3+I\xi_4$, $\xi_4'=\xi_3-I\xi_4$, that is to say $\xi_1=\mathrm{Re}(\xi')$, $\xi_2=\mathrm{Im}(\xi')$, $\xi_3=\mathrm{Re}(\xi'')$, $\xi_4=\mathrm{Im}(\xi'')$.

$$\begin{gathered}k_1=\mathrm{Re}(k'),\quad k_2=\mathrm{Im}(k'),\quad k_3=\mathrm{Re}(k''),\quad k_4=\mathrm{Im}(k'')\\ l_1=\mathrm{Re}(l'),\quad l_2=\mathrm{Im}(l'),\quad l_3=\mathrm{Re}(l''),\quad l_4=\mathrm{Im}(l'')\\ m_1=\mathrm{Re}(m'),\quad m_2=\mathrm{Im}(m'),\quad m_3=\mathrm{Re}(m''),\quad m_4=\mathrm{Im}(m'')\end{gathered} \tag{21}$$

we have:

$$\begin{gathered}\xi_1=k_1x+l_1t+m_1+\xi_{1*},\quad \xi_2=k_2x+l_2t+m_2+\xi_{2*},\\ \xi_3=k_3x+l_3t+m_3+\xi_{3*},\quad \xi_4=k_4x+l_4t+m_4+\xi_{4*}\end{gathered} \tag{22}$$

then the dispersion relation can be converted into:

$$\begin{aligned}m_1&=\frac{-k_1^3l_1^2-k_1^3l_2^2+3k_1k_2^2l_1^2+3k_1l_2^2k_2^2-3k_1^2l_1u_0-6k_1l_2k_2u_0+3k_2^2l_1u_0}{l_1^2+l_2^2}\\ m_2&=\frac{k_2^3l_1^2+l_2^2k_2^3-3k_1^2k_2l_1^2-3k_1^2l_2^2k_2+3k_1^2l_2u_0-6k_1k_2l_1u_0-3l_2k_2^2u_0}{l_1^2+l_2^2}\\ m_3&=\frac{-k_3^3l_3^2-k_3^3l_4^2+3k_3k_4^2l_3^2+3k_3l_4^2k_4^2-3k_3^2l_3u_0-6k_3l_4k_4u_0+3k_4^2l_3u_0}{l_3^2+l_4^2}\\ m_4&=\frac{k_4^3l_3^2+l_4^2k_4^3-3k_3^2k_4l_3^2-3k_3^2l_4^2k_4+3k_3^2l_4u_0-6k_3k_4l_3u_0-3l_4k_4^2u_0}{l_3^2+l_4^2}\end{aligned} \tag{23}$$

Through 4-soliton formulation, we can derive the 2-complexiton solution of eq. (3)

$$f = 1 + e^{\xi_1'} + e^{\xi_2'} + e^{\xi_3'} + e^{\xi_4'} + \theta_{12}e^{\xi_1'+\xi_2'} + \theta_{13}e^{\xi_1'+\xi_3'} + \theta_{14}e^{\xi_1'+\xi_4'} + \theta_{23}e^{\xi_2'+\xi_4'} + \theta_{24}e^{\xi_2'+\xi_4'} +$$
$$+\theta_{34}e^{\xi_3'+\xi_4'} + \theta_{123}e^{\xi_1'+\xi_2'+\xi_3'} + \theta_{124}e^{\xi_1'+\xi_2'+\xi_4'} + \theta_{134}e^{\xi_1'+\xi_3'+\xi_4'} + \theta_{234}e^{\xi_2'+\xi_3'+\xi_4'} + \theta_{1234}e^{\xi_1'+\xi_2'+\xi_3'+\xi_4'} =$$
$$= 1 + 2e^{\xi_1}\cos(\xi_2) + 2e^{\xi_3}\cos(\xi_4) + \theta_{12}e^{2\xi_1} + \theta_{34}e^{2\xi_3} +$$
$$+ 2\mathrm{Re}[\theta_{13}e^{\xi_1+\xi_3+i(\xi_2+\xi_4)} + \theta_{14}e^{\xi_1+\xi_3+i(\xi_2-\xi_4)}] +$$
$$+ 2\mathrm{Re}(\theta_{123}e^{2\xi_1+\xi_3+i\xi_4} + \theta_{134}e^{\xi_1+2\xi_3+i\xi_2}) + \theta_{1234}e^{2\xi_1+2\xi_3} \tag{24}$$

where

$$\theta_{12} = -\frac{H(k'-\bar{k}', l'-\bar{l}', m'-\bar{m}')}{H(k'+\bar{k}', l'+\bar{l}', m'+\bar{m}')}, \quad \theta_{13} = -\frac{H(k'-k'', l'-l'', m'-m'')}{H(k'+k'', l'+l'', m'+m'')}$$
$$\theta_{14} = -\frac{H(k'-\bar{k}'', l'-\bar{l}'', m'-\bar{m}'')}{H(k'+\bar{k}'', l'+\bar{l}'', m'+\bar{m}'')}, \quad \theta_{23} = -\frac{H(\bar{k}'-k'', \bar{l}'-l'', \bar{m}'-m'')}{H(\bar{l}'+k'', \bar{l}'+l'', \bar{m}'+m'')} \tag{25}$$
$$\theta_{24} = -\frac{H(\bar{k}'-\bar{k}'', \bar{l}'-\bar{l}'', \bar{m}'-\bar{m}'')}{H(\bar{l}'+\bar{k}'', \bar{l}'+\bar{l}'', \bar{m}'+\bar{m}'')}, \quad \theta_{34} = -\frac{H(k''-\bar{k}'', l''-\bar{l}'', m''-\bar{m}'')}{H(k''+\bar{k}'', l''+\bar{l}'', m''+\bar{m}'')}$$

and $\theta_{123} = \theta_{12}\theta_{13}\theta_{23}$, $\theta_{124} = \theta_{12}\theta_{14}\theta_{24}$, $\theta_{234} = \theta_{23}\theta_{24}\theta_{34}$, $\theta_{123} = \theta_{12}\theta_{13}\theta_{23}$, $\theta_{1234} = \theta_{12}\theta_{13}\theta_{14}\theta_{23}\theta_{24}\theta_{34}$.

Substituting eq. (15) into eq. (2), we can obtain 2-complexiton solution to eq. (1).

Finally, we construct *N*-compilexiton solutions by 2*N*-soliton solutions:

$$\xi'(x,y,t) = k'x + l'y + m't + \xi_{*'}, \quad ..., \quad \xi^N(x,y,t) = k^N x + l^N y + m^N t + \xi_*^N \tag{26}$$

where k', k^N, l', l^N, m', m^N, $\xi_{*'}$, and ξ_*^N are constants. Through the same processing procedure, let:

$$\xi_1 = k_1 x + l_1 t + m_1 + \xi_{1*}, \qquad \xi_2 = k_2 x + l_2 t + m_2 + \xi_{2*}$$
$$\ldots\ldots \qquad \ldots\ldots \tag{27}$$
$$\xi_{2N-1} = k_{2N-1}x + l_{2N-1}t + m_{2N-1} + \xi_{2N-1*}, \quad \xi_{2N} = k_{2N}x + l_{2N}t + m_{2N} + \xi_{2N*}$$

then the dispersion relation can be converted into as following form:

$$m_1 = \frac{-k_1^3 l_1^2 - k_1^3 l_2^2 + 3k_1 k_2^2 l_1^2 + 3k_1 l_2^2 k_2^2 - 3k_1^2 l_1 u_0 - 6k_1 l_2 k_2 u_0 + 3k_2^2 l_1 u_0}{l_1^2 + l_2^2}$$
$$m_2 = \frac{k_2^3 l_1^2 + l_2^2 k_2^3 - 3k_1^2 k_2 l_1^2 - 3k_1^2 l_2^2 k_2 + 3k_1^2 l_2 u_0 - 6k_1 k_2 l_1 u_0 - 3l_2 k_2^2 u_0}{l_1^2 + l_2^2}$$
$$\ldots\ldots \tag{28}$$
$$m_{2l-1} = (-k_{2l-1}^3 l_{2l-1}^2 - k_{2l-1}^3 l_{2l}^2 + 3k_{2l-1}k_{2l}^2 l_{2l-1}^2 + 3k_{2l-1}l_{2l}^2 k_{2l}^2 - 3k_{2l-1}^2 l_{2l-1}u_0$$
$$- 6k_{2l-1}l_2 k_{2l}u_0 + 3k_{2l}^2 l_{2l-1}u_0) / (l_{2l-1}^2 + l_{2l}^2)$$
$$m_{2l} = (k_{2l}^3 l_{2l-1}^2 + l_{2l}^2 k_{2l}^3 - 3k_{2l-1}^2 k_{2l} l_{2l-1}^2 - 3k_{2l-1}^2 l_{2l}^3 k_{2l} + 3k_{2l-1}^2 l_{2l} u_0$$
$$- 6k_{2l-1}k_{2l}l_{2l-1}u_0 - 3l_{2l}k_{2l}^2 u_0) / (l_{2l-1}^2 + l_{2l}^2)$$

with $2 \le l \le N$.

Based on a 2*N*-soliton formulation, we can derive the *N*-complexiton solution of eq. (3):

$$f = 1 + \sum_{i=1}^{2N} e^{\xi_i'} + \sum_{k=2}^{2N} \sum_{1 \le j_1, j_2, \dots, j_k \le 2N}^{2N} \theta_{j_1 j_2 \cdots j_k} e^{\xi_{j_1}' + \xi_{j_2}' + \cdots \xi_{j_k}'} \tag{29}$$

Similarly, substituting eq. (19) into eq. (2), we can obtain *N*-complexiton solution to eq. (1).

Conclusion

In this paper, we construct complexiton solutions of ANNV equation by applying the Hirota direct method and the pairs of conjugate wave variables. The key is to utilize the bi-linear ANNV equation. Using pairs of conjugate wave variables in 2*N*-soliton solutions, we obtain a series of multi-complexiton solutions. It's recommended that this method can be further used to find multi-complexiton solutions of non-linear equations with fractal derivatives [28-35].

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