APPROXIMATE ANALYTIC SOLUTION OF THE FRACTAL KLEIN-GORDON EQUATION

by

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Original scientific paper
https://doi.org/10.2298/TSCI200301051S

The linear and non-linear Klein-Gordon equations are considered. The fractional complex transform is used to convert the equations on a continuous space/time to fractals ones on Cantor sets, the resultant equations are solved by local fractional reduced differential transform method. Three examples are given to show the effectiveness of the technology.

Key words: *fractal Klein-Gordon equation, approximate analytical solutions, fractional complex transform method, local fractional derivative, local fractional reduced differential transform method*

Introduction

The Klein-Gordon equation (KGE) [1] has been applied to solid-state physics, non-linear optics, quantum field theory and thermodynamics. Many methods are available to solve this kind of equations [2, 3]. Recently, a fractional modification with the Caputo fractional derivative was extensively studied [4-7], however, when the studied domain cannot be described by smooth functions, both the classical approach and the fractional approach based on Riemann-Liouville (or Caputo) derivatives are unacceptable. In such case, the local fractional calculus is an efficient technique for modelling these physical problems. For example, the fractal non-linear Burgers' equation was investigated [8], the fractal Harry Dym equations was proposed in [9], a fractal derivative model for a porous structure was discussed [10], and a physical insight into a local fractional KdV-Burgers-Kuramoto (KBK) equation in a fractal space was given in [11].

Using the fractional complex transform [12-14] and local fractional derivative [15, 16], we can transform the classical KGE into its local fractional partner on Cantor sets.

Zhou [17] introduced the differential transform method, which was further improved by Keskin and Oturanc [18], and the method was used to solve local fractional differential equations.

Yang, *et al.* [19] obtained a non-differentiable solution for the local fractional linear KGE on Cantor sets by using the local fractional series expansion method within the local fractional differential operator. Kumar, *et al*. [20] investigated the linear KGE on Cantor sets by employing a mixture of classical homotopy perturbation technique and local fractional Sumudu

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transform technique. But the non-linear case of the local fractional KGE on Cantor sets was not discussed.

In this work, the linear and non-linear KGE with local fractional derivatives on Cantor sets are studied analytically by combining the fractional complex transform and local fractional reduced differential method (LFRDTM).

Mathematics tools

In this section, we recall and review briefly basic definitions of local fractional derivatives and 2-D differential transform on fractal space [16].

Definition 1. The local fractional partial derivative operator of $\psi(x, t)$ of order $\alpha(\alpha \in (0, 1])$ with respect to t at the point (x, t_0) is defined [16]:

$$D^{(\alpha)}\psi(x,t_0)=\frac{\partial^{(\alpha)}\psi(x,t_0)}{\partial t^{\alpha}}=\lim_{t\to t_0}\frac{\Delta^{\alpha}\left[\psi(x,t)-\psi(x,t_0)\right]}{(t-t_0)^{\alpha}} \tag{1}$$

where

$$\Delta^{\alpha}\left[\psi(x,t)-\psi(x,t_0)\right]\cong\Gamma(1+\alpha)\left[\psi(x,t)-\psi(x,t_0)\right] \tag{2}$$

In view of (1), the local fractional partial derivative operator of $\psi(x, t)$ of order is $k\alpha(\alpha \in (0, 1])$ is given [16]:

$$D_t^{(k\alpha)}\psi(x,t)=\frac{\partial^{k\alpha}\psi(x,t)}{\partial t^{k\alpha}}=\overbrace{\frac{\partial^{\alpha}}{\partial t^{\alpha}}\cdots\frac{\partial^{\alpha}}{\partial t^{\alpha}}}^{k\ \ times}\psi(x,t) \tag{3}$$

Definition 2. The 2-D local fractional reduced differential transform $\Psi_k(x)$ of the function $\psi(x, t)$ is defined:

$$\Psi_k(x)=\frac{1}{\Gamma(1+k\alpha)}\left[\frac{\partial^{k\alpha}\psi(x,t)}{\partial t^{k\alpha}}\right]_{t=0} \tag{4}$$

where $k = 0, 1, 2,..., n$ and $\alpha \in (0, 1]$.

Definition 3. The 2-D local fractional reduced differential inverse transform of $\Psi_k(x)$ is defined [16]:

$$\psi(x,t)=\sum_{k=0}^{\infty}\Psi_k t^{k\alpha} \tag{5}$$

where $\alpha \in (0, 1]$.

Local fractional KGE on Cantor sets

In this section, fractal model of the local fractional KGE on Cantor sets is derived with the fractional complex transform via local fractional derivative [15].

The classical KGE:

$$\frac{\partial\psi(X,T)}{\partial T}=\frac{\partial^2\psi(X,T)}{\partial X^2}+a\psi(X,T)+b\psi^2(X,T)+c\psi^3(X,T),\ T>0 \tag{6}$$

subject to the initial condition:

$$\psi(X,0)=\psi_0, X\in\mathbb{R} \tag{7}$$

where a, b, and c are real constants.

Using the fractional complex transform method [15] via local fractional derivatives, we can obtain:

$$\frac{\partial^\alpha \psi(x,t)}{\partial t^\alpha} = \frac{\partial^{2\alpha}\psi(x,t)}{\partial x^{2\alpha}} + a\psi(x,t) + b\psi^2(x,t) + c\psi^3(x,t), \quad t>0 \tag{8}$$

Approximate analytical solutions of fractal KGE

In this section, local fractional KGE on Cantor sets are discussed.

Firstly, we research the linear case when $a = 1$, $b = 0$ and $c = 0$:

$$\frac{\partial^\alpha \psi(x,t)}{\partial t^\alpha} = \frac{\partial^{2\alpha}\psi(x,t)}{\partial x^{2\alpha}} + \psi(x,t), \quad t>0,\ 0<\alpha\le 1 \tag{9}$$

with the initial conditions (IC)

$$\psi(x,0) = 1+\sin_\alpha(x^\alpha) \tag{10}$$

By taking LFRDTM of eq. (9), we have:

$$\Psi_{k+1}(x) = \frac{\Gamma(1+k\alpha)}{\Gamma(1+k\alpha+\alpha)}\left(\frac{\partial^{2\alpha}\Psi_k}{\partial x^{2\alpha}} + \Psi_k\right) \tag{11}$$

From the IC eq. (10), it can be obtained:

$$\Psi_0(x) = 1+\sin_\alpha(x^\alpha) \tag{12}$$

By using eq. (12) in eq. (11), we can obtain the following $\Psi_k(x)$ values successively:

$$\Psi_1(x) = \frac{1}{\Gamma(1+\alpha)},\ \Psi_2(x) = \frac{1}{\Gamma(1+2\alpha)},\cdots$$
$$\Psi_k(x) = \frac{1}{\Gamma(1+k\alpha)},\cdots \tag{13}$$

Using the inverse LFRDTM, the approximation solution of eq. (9) can be written:

$$\psi(x,t) = \sum_{k=0}^{\infty}\Psi_k(x)t^{k\alpha} = \Psi_0(x) + \Psi_1(x)t^\alpha + \Psi_2(x)t^{2\alpha} + \Psi_3(x)t^{3\alpha} + \cdots =$$
$$= 1+\sin_\alpha(x^\alpha) + \sum_{k=1}^{\infty}\frac{t^{k\alpha}}{\Gamma(1+k\alpha)} \tag{14}$$

In particular, for $\alpha \to 1$, eq. (14) reduced:

$$\psi(x,t) = 1+\sin(x) + \sum_{k=1}^{\infty}\frac{t^k}{\Gamma(1+k)} \tag{15}$$

which is the exact solution of the classical KGE eq. (9) with $\alpha \to 1$.

It is evident that the aforementioned result is in complete agreement with the results provided by Tamsir and Srivastava [7] using FRDTM.

Secondly, we consider a non-linear KGE case when $a = 0$, $b = -1$, and $c = 0$:

$$\frac{\partial^\alpha \psi(x,t)}{\partial t^\alpha} = \frac{\partial^{2\alpha}\psi(x,t)}{\partial x^{2\alpha}} - \psi^2(x,t), \quad t>0,\quad 0<\alpha\le 1 \tag{16}$$

with the IC:

$$\psi(x,0) = 1+\sin_\alpha(x^\alpha) \tag{17}$$

Applying LFRDT to eq. (16), we obtain:

$$\Psi_{k+1}(x)=\frac{\Gamma(1+k\alpha)}{\Gamma(1+k\alpha+\alpha)}\left(\frac{\partial^{2\alpha}\Psi_k}{\partial x^{2\alpha}}-\sum_{r=0}^{k}\Psi_r\Psi_{k-r}\right) \tag{18}$$

From the IC eq. (17), we can get:

$$\Psi_0(x)=1+\sin_\alpha\left(x^\alpha\right) \tag{19}$$

By using eq. (19) into eq. (18), the following $\Psi_k(x)$ values are obtained successively:

$$\begin{gathered}\Psi_1(x)=-\frac{1}{\Gamma(1+\alpha)}\left[1+3\sin_\alpha(x^\alpha)+\sin_\alpha^2\left(x^\alpha\right)\right]\\ \Psi_2(x)=\frac{1}{\Gamma(1+2\alpha)}\left[11\sin_\alpha(x^\alpha)+12\sin_\alpha^2\left(x^\alpha\right)+2\sin_\alpha^3\left(x^\alpha\right)\right]\\ \Psi_3(x)=\frac{1}{\Gamma(1+3\alpha)}\left[18-57\sin_\alpha(x^\alpha)-160\sin_\alpha^2\left(x^\alpha\right)-\right.\\ \left.-82\sin_\alpha^3\left(x^\alpha\right)-10\sin_\alpha(4x^\alpha)\right]\\ \vdots\end{gathered} \tag{20}$$

Using the inverse LFRDTM, we get the approximation solution of eq. (16):

$$\begin{gathered}\psi(x,t)=\sum_{k=0}^{\infty}\Psi_k(x)t^{k\alpha}=1+\sin_\alpha\left(x^\alpha\right)-\frac{t^\alpha}{\Gamma(1+\alpha)}\cdot\\ \cdot\left[\left(1+3\sin_\alpha\left(x^\alpha\right)+\sin_\alpha^2\left(x^\alpha\right)\right]+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\cdot\\ \cdot\left[11\sin_\alpha\left(x^\alpha\right)+12\sin_\alpha^2\left(x^\alpha\right)+2\sin_\alpha^3\left(x^\alpha\right)\right]+\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\cdot\\ \cdot\left[18-57\sin_\alpha\left(x^\alpha\right)-160\sin_\alpha^2-\left(x^\alpha\right)82\sin_\alpha^3\left(x^\alpha\right)-10\sin_\alpha\left(4x^\alpha\right)\right]+\cdots\end{gathered} \tag{21}$$

Equation (21) is the series solution for local fractional non-linear KGE on Cantor sets, clearly in complete agreement with the results given by Tamsir and Srivastava [7] using FRDTM with $\alpha \to 1$.

Thirdly, when $a = -1$, $b = 0$, and $c = 1$, we have non-linear KGE:

$$\frac{\partial^\alpha\psi(x,t)}{\partial t^\alpha}=\frac{\partial^{2\alpha}\psi(x,t)}{\partial x^{2\alpha}}-\psi(x,t)+\psi^3(x,t),\ t>0,\ 0<\alpha\le 1 \tag{22}$$

with the IC:

$$\psi(x,0)=1+\sin_\alpha\left(x^\alpha\right) \tag{23}$$

Similarly the following recurrence relation is obtained :

$$\Psi_{k+1}(x)=\frac{\Gamma(1+k\alpha)}{\Gamma(1+k\alpha+\alpha)}\left(\frac{\partial^{2\alpha}\Psi_k}{\partial x^{2\alpha}}-\Psi_k-\sum_{r=0}^{k}\sum_{s=0}^{r}\Psi_s\Psi_{r-s}\Psi_{k-r}\right) \tag{24}$$

From the IC eq. (23), we get:

$$\Psi_0(x)=-\operatorname{sech}_\alpha(x^\alpha) \tag{25}$$

By using eq. (25) into eq. (24), the following $\Psi_k(x)$ values are obtained successively:

$$\Psi\ (x)=-\frac{}{\Gamma(\ +\ )}\left[2\operatorname{sech}_\alpha(x^\alpha)-3\operatorname{sech}_\alpha(x^\alpha)\right]$$

$$\Psi\ (x)=-\frac{}{\Gamma(1+2\ )}\left[3\operatorname{sech}_\alpha(x^\alpha)-34\operatorname{sech}_\alpha^3(x^\alpha)-18\operatorname{sech}_\alpha^5(x^\alpha)\right] \tag{26}$$

$$\Psi\ (x)=-\frac{}{\Gamma(1+3\ )}\left[64\operatorname{sech}_\alpha^3(x^\alpha)-288\operatorname{sech}_\alpha^5(x^\alpha)+240\operatorname{sech}_\alpha^7(x^\alpha)\right]$$

Using the inverse LFRDTM, we obtain the approximation solution of eq. (22):

$$\psi(x,t)=\sum_{k=0}^{\infty}\Psi_k(x)t^{k\alpha}=-\operatorname{sech}_\alpha(x^\alpha)-\frac{t^\alpha}{\Gamma(1+\alpha)}\left[2\operatorname{sech}_\alpha(x^\alpha)-3\operatorname{sech}_\alpha^3(x^\alpha)\right]-$$
$$-\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\left[3\operatorname{sech}_\alpha(x^\alpha)-34\operatorname{sech}_\alpha^3(x^\alpha)-18\operatorname{sech}_\alpha^5(x^\alpha)\right]- \tag{27}$$
$$-\frac{1}{\Gamma(1+3\alpha)}\left[64\operatorname{sech}_\alpha^3(x^\alpha)-288\operatorname{sech}_\alpha^5(x^\alpha)+240\operatorname{sech}_\alpha^7(x^\alpha)\right]-\cdots$$

Equation (27) is the series solution for local fractional non-linear KGE on Cantor sets, it is in a complete agreement with the results given by Tamsir and Srivastava [7] using FRDTM with $\alpha \to 1$.

Conclusion

In this paper, the KGE on Cantor sets within the local fractional differential operator had been analyzed by coupling the fractional complex transform and the local fractional differential transform method. The non-differentiable solutions for local fractional KGE are obtained. The present method is a powerful mathematical tool for solving the local fractional linear differential equations, and can be extended easily to other fractional differential equations [21, 22].

Acknowledgment

This work is supported by the high level scientific research project cultivation fund of Jiaozuo Teachers College and the key scientific research projects of Henan higher education institutions (Grant No. 20B110009 and Grant No. 17A880017)

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Paper submitted: March 1, 2020
Paper revised: May 28, 2020
Paper accepted: June 20, 2020