VARIATIONAL THEORY FOR A KIND OF NON-LINEAR MODEL FOR WATER WAVES

by

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The Whitham-Broer-Kaup equation exists widely in shallow water waves, but unsmooth boundary seriously affects the properties of solitary waves and has certain deviations in scientific research. The aim of this paper is to introduce its modification with fractal derivatives in a fractal space and to establish a fractal variational formulation by the semi-inverse method. The obtained fractal variational principle shows conservation laws in an energy form in the fractal space and also hints its possible solution structure.

Key words*: Whitham-Broer-Kaup equation, fractal derivatives, two-scale transform, fractal variational formulation*

Introduction

Nowadays, the study of shallow water waves is a very hot topic [1-4]. This paper, we mainly focus on the Whitham-Broer-Kaup (WBK) equation [5-7], reads:

$$\frac{\partial u}{\partial T}+u\frac{\partial u}{\partial X}+\frac{\partial V}{\partial X}+q\frac{\partial^2 u}{\partial X^2}=0 \tag{1}$$

$$\frac{\partial v}{\partial T}+\frac{\partial uv}{\partial X}+p\frac{\partial^2 u}{\partial X^2}-q\frac{\partial^2 v}{\partial X^2}=0 \tag{2}$$

where p and q are real constants which mean different dispersive powers. In eqs. (1) and (2), if $p = 0$ and $q \neq 0$, the system converts the approximate system of dispersive long wave equation, and if $p = 1$ and $q = 0$, we get the variant Boussinesq equation. At present, there are much literature about the analytical solution of this system, for example the Wu and Zhang elimination method [8], the auxiliary equation method [9], the hyperbolic function method [10], the variational iteration method, the homotopy perturbation method [11, 12] and so on.

Equations (1) and (2) are a complete integrable model, describing the dispersive long wave in shallow water. In practical problems, the unsmooth boundary has a great influence on shallow water waves. Fractal, therefore, provides a very effective manner to deal with discontinuous boundaries. Usually, the smooth space (X, T) should be instead of the fractal space (X^β, T^α), with α and β are fractal dimensional in time and space, respectively.

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So, the governing equations can be modified:

$$\frac{\partial u}{\partial T^{\alpha}}+u\frac{\partial u}{\partial X^{\beta}}+\frac{\partial V}{\partial X^{\beta}}+q\frac{\partial^2 u}{\partial X^{2\beta}}=0 \tag{3}$$

$$\frac{\partial v}{\partial T^{\alpha}}+\frac{\partial uv}{\partial X^{\beta}}+p\frac{\partial^2 u}{\partial X^{2\beta}}-q\frac{\partial^2 v}{\partial X^{2\beta}}=0 \tag{4}$$

in which $\partial u/\partial T^{\alpha}$ and $\partial u/\partial X^{\beta}$ are the fractal derivative defined [13-18]:

$$\frac{\partial u}{\partial T^{\alpha}}(T_0,X)=\Gamma(1+\alpha)\lim_{\substack{T-T_0\to\Delta T\\ \Delta T\neq 0}}\frac{u(T,X)-u(T_0,X)}{(T-T_0)^{\alpha}} \tag{5}$$

$$\frac{\partial u}{\partial X^{\beta}}(T,X_0)=\Gamma(1+\alpha)\lim_{\substack{X-X_0\to\Delta X\\ \Delta X\neq 0}}\frac{u(T,X)-u(T,X_0)}{(X-X_0)^{\beta}} \tag{6}$$

where ΔT and ΔX are the smallest time scale for researching the shallow water waves and the smallest spatial scale of the discontinuous boundaries. If the time scale is large than ΔT, a smooth wave property can be discussed, and if the spatial scale is large than ΔX, the problem studied has become a traditional smooth mechanical problem. In the study of fractal problems, the factors of all problems depend on the scale we used and the fractal dimension defined by an unsmooth boundary.

Variational principle

Variational principle [19-28] is a useful tool in mathematics, as an energy method to describe motion [19], it has wide applications in mathematics [20], mechanics [21], physics [22], economics [23], especially for non-linear problems. In view of the two-scale transform [29-32], eqs. (3) and (4) can be written:

$$u_t+uu_x+v_x+qu_{xx}=0 \tag{7}$$

$$v_t+(uv)_x+pu_{xxx}-qv_{xx}=0 \tag{8}$$

In order to apply the semi-inverse method [33, 34] which was proposed by He to construct a series of variational formulations for the aforementioned equation, eqs. (7) and (8) should be present in the conservation form:

$$u_t+\left(\frac{1}{2}u^2+v+qu_x\right)_x=0 \tag{9}$$

$$v_t+(uv+pu_{xx}-qv_x)_x=0 \tag{10}$$

Two special auxiliary functions ψ and ϕ are introduced, and they are, respectively, satisfying:

$$\psi_t=\frac{1}{2}u^2+v+qu_x \tag{11}$$

$$\psi_x=-u \tag{12}$$

$$\phi_t=uv+pu_{xx}-qv_x \tag{13}$$

$$\phi_x=-v \tag{14}$$

It is intuitive to show that the eqs. (7) and (8) is equivalent to the eqs. (7), (13), and (14) or eqs. (8), (11), and (12). The main research purpose of this paper is to find a variational principle whose stationary conditions satisfy the eqs. (7), (13), and (14) or eqs. (8), (11), and (12):

$$J(u,v,\psi)=\iint L(u,u_t,u_x,u_{xx},v,v_t,v_x,v_{xx},\psi,\psi_t,\psi_x,\psi_{xx})\mathrm{d}x\mathrm{d}t \tag{15}$$

in which L is the trial-Lagrange function.

Through the semi-inverse method, we can re-write the trial-Lagrange function:

$$L=v\psi_t+(uv+pu_{xx}-qv_x)\psi_x+F(u,v) \tag{16}$$

in which $F(u, v)$ is an undetermined function with respect to u, v and/or their derivatives. The advantage of the Lagrange function that we've constructed in this way is that the stationary condition about ψ is eqs. (8) and (12).

The stationary conditions about u, v are given:

$$v\psi_x+p\psi_{xxx}+\frac{\delta F}{\delta u}=0 \tag{17}$$

$$\psi_t+u\psi_x+q\psi_{xx}+\frac{\delta F}{\delta v}=0 \tag{18}$$

in which $\delta F/\delta u$ and $\delta F/\delta v$ are called the variational derivative presented:

$$\frac{\delta F}{\delta u}=\frac{\partial F}{\partial u}-\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial u_t}\right)-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)+\cdots$$

$$\frac{\delta F}{\delta v}=\frac{\partial F}{\partial v}-\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial v_t}\right)-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right)+\cdots$$

In the following process, we will find out a specific $F(u, v)$ which make eqs. (17) and (18) one of field equations of the governing equations, respectively. From eqs. (11) and (12):

$$\frac{\delta F}{\delta u}=-(v\psi_x+p\psi_{xxx})=uv+pu_{xxx} \tag{19}$$

$$\frac{\delta F}{\delta v}=-(\psi_t+u\psi_x+q\psi_{xx})=\frac{1}{2}u^2-v \tag{20}$$

According to the eqs. (19) and (20), the undetermined function $F(u, v)$ can be uniquely identified:

$$F=\frac{1}{2}u^2v-\frac{1}{2}v^2-\frac{1}{2}pu_x^2 \tag{21}$$

Therefore, we successfully construct the needed variational formulation, which shows:

$$J(u,v,\psi)=\iint\left\{v\psi_t+\left(uv+pu_{xx}-qv_x\right)\psi_x+\frac{1}{2}\left(u^2v-v^2-pu_x^2\right)\right\}\mathrm{d}x\mathrm{d}t \tag{22}$$

Proof: Using the previous variational formulation, the Euler–Lagrange equations can be given in the following form:

$$v\psi_x+uv+pu_{xxx}+p\psi_{xxx}=0 \tag{23}$$

$$\psi_t+u\psi_x+\frac{1}{2}u^2-v+q\psi_x=0 \tag{24}$$

$$-v_t-(uv+pu_{xxx}-qv_x)_x=0 \tag{25}$$

It is very clear that eqs. (23) and (25) are equivalent to eqs. (12) and (8), respectively. According to the constraint eqs. (12), and (24) results in eq. (11).

Under the fractal space (X^β, T^α), the variational formulation can be written in the form:

$$J(u,v,\psi)=\iint\left\{v\frac{\partial\psi}{\partial T^\alpha}+\left(uv+p\frac{\partial^2 v}{\partial X^{2\beta}}-q\frac{\partial v}{\partial X^\beta}\right)\frac{\partial\psi}{\partial X^\beta}+\frac{1}{2}\left[u^2v-v^2-p\left(\frac{\partial u}{\partial X^\beta}\right)^2\right]\right\}\mathrm{d}X^\beta\mathrm{d}T^\alpha \tag{26}$$

We can also construct the other trial-Lagrange function in the form:

$$L^*=u\phi_t+\left(\frac{1}{2}u^2+v+qu_x\right)\phi_x+F^*(u,v) \tag{27}$$

in which $F^*(u, v)$ is an undetermined function. It is crystal clear that the stationary condition about ϕ will lead to eq. (7). Now, the stationary conditions about u, v read:

$$\phi_t+u\phi_x-q\phi_{xx}+\frac{\delta F^*}{\delta u}=0 \tag{28}$$

$$\phi_x+\frac{\delta F^*}{\delta v}=0 \tag{29}$$

According to eqs. (28) and (29), we obtain:

$$\frac{\delta F^*}{\delta u}=-(\phi_t+u\phi_x-q\phi_{xx})=-pu_{xx} \tag{30}$$

$$\frac{\delta F^*}{\delta v}=-\phi_x=-v \tag{31}$$

So, the undetermined function $F^*(u, v)$ can be identified:

$$F^*=\frac{1}{2}pu_x^2+\frac{1}{2}v^2 \tag{32}$$

And the other variational formulation reads:

$$J^*(u,v,\phi)=\iint\left\{u\phi_t+\left(\frac{1}{2}u^2+v+qu_x\right)\phi_x+\frac{1}{2}\left(pu_x^2+v^2\right)\right\}\mathrm{d}x\mathrm{d}t \tag{33}$$

In view of the aforementioned functional, the Euler-Lagrange equations:

$$\phi_t+u\phi_x-q\phi_{xx}-pu_{xx}=0 \tag{34}$$

$$\phi_x+v=0 \tag{35}$$

$$-u_t-\left(\frac{1}{2}u^2+v+qu_x\right)_x=0 \tag{36}$$

It is obvious that eqs. (34)-(36) are equivalent to eqs. (13), (14), and (7). Similarly, in the fractal space (X^β, T^α), the variational formulation eq. (33) reads:

$$J^*(u,v,\phi)=\iint\left\{u\frac{\partial\phi}{\partial T^\alpha}+\left(\frac{1}{2}u^2+v+q\frac{\partial u}{\partial X^\beta}\right)\frac{\partial\phi}{\partial X^\beta}+\frac{1}{2}\left[p\left(\frac{\partial u}{\partial X^\beta}\right)^2+v^2\right]\right\}\mathrm{d}X^\beta\mathrm{d}T^\alpha \tag{37}$$

Moreover, on the basis of the generalized variational formulations eqs. (22) and (33), many constrained variational principles can be directly given. Accordingly, the corresponding variational formula in fractal space can be obtained. For instance, we substitute eq. (12) into eq. (22), the constrained functional:

$$J^{**}(u,\psi)=\iint\left\{v\psi_t-(pu_{xx}-qv_x)u-\frac{1}{2}\left(u^2v+v^2+pu_x^2\right)\right\}\mathrm{d}x\mathrm{d}t \tag{38}$$

with the constraint of eq. (12). The corresponding constrained variational formulation in fractal space:

$$J^{**}(u,\psi)=\iint\left\{v\frac{\partial\psi}{\partial T^\alpha}-\left(p\frac{\partial^2u}{\partial X^{2\beta}}-q\frac{\partial v}{\partial X^\beta}\right)u-\frac{1}{2}\left[u^2v+v^2+p\left(\frac{\partial u}{\partial X^\beta}\right)^2\right]\right\}\mathrm{d}X^\beta\mathrm{d}T^\alpha \tag{39}$$

with the constraint of $\partial\psi/\partial X^\beta = -u$. Further constraining eq. (39) by eq. (11), we have:

$$J^{***}(\psi)=\iint\left\{\frac{1}{2}v^2+q\left(uv\right)_x-puu_{xx}-\frac{1}{2}pu_x^2\right\}\mathrm{d}x\mathrm{d}t \tag{40}$$

The constrained variational formulation in fractal space of aforementioned functional can be written:

$$J^{***}(\psi)=\iint\left\{\frac{1}{2}v^2+q\frac{\partial(uv)}{\partial X^\beta}-pu\frac{\partial^2u}{\partial X^{2\beta}}-\frac{1}{2}p\left(\frac{\partial u}{\partial X^\beta}\right)^2\right\}\mathrm{d}X^\beta\mathrm{d}T^\alpha \tag{41}$$

with the constraint of $\partial\psi/\partial T^\alpha = 1/2u^2 + v + qu_x$. Integrating by parts and ignoring its boundary items of eq. (40), we get:

$$J^{***}(\psi)=\frac{1}{2}\iint\left\{v^2-puu_{xx}\right\}\mathrm{d}x\mathrm{d}t=\frac{1}{2}\iint\left\{\left(\psi_t-\frac{1}{2}\psi_x^2+q\psi_{xx}\right)^2-p\psi_x\psi_{xxx}\right\}\mathrm{d}x\mathrm{d}t \tag{42}$$

The final constrained variational formulation in fractal space is given:

$$J^{***}(\psi)=\frac{1}{2}\iint\left\{\left[\frac{\partial\psi}{\partial T^\alpha}-\frac{1}{2}\left(\frac{\partial\psi}{\partial X^\beta}\right)^2+q\frac{\partial^2\psi}{\partial X^{2\beta}}\right]^2-p\frac{\partial\psi}{\partial X^\beta}\frac{\partial^3\psi}{\partial X^{3\beta}}\right\}\mathrm{d}X^\beta\mathrm{d}T^\alpha \tag{43}$$

which is subject to $\partial\psi/\partial X^\beta = -u$ and $\partial\psi/\partial T^\alpha = 1/2u^2 + v + qu_x$.

Conclusion

In this work, we research variational principle of the WBK equation in the fractal space (X^β, T^α) through the semi-inverse method. This method was first proposed by He and is an effective theory to establish the variational formulation from the field equations. The variational principle can be construct conservation laws and solution structures.

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