# VARIATIONAL PRINCIPLE FOR NON-LINEAR FRACTIONAL WAVE EQUATION IN A FRACTAL SPACE

by

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*The fractal derivative is adopted to describe the non-linear fractional wave equation in a fractal space. A variational principle is successfully established by the semi-inverse method. The two-scale method and He's exp-function are used to solve the equation, and a good result is obtained.*

Key words: *fractal space, He's fractal derivative, two-scale transform method, exp-function method, variational principle*

## Introduction

The variational principle and conservation laws are extremely important mathematical tools to deep insight of solution structure of a non-linear problem, and have been widely adopted to establish a variety of mathematical models in practical applications, such as nanoscience, thermal science, electrochemistry, physics, optics and mechanics [1-5]. However, little attention was put on the variational principle for a non-smooth problems [6-10], and the condition is now completely changed due to the fast development of fractal calculus [11-19].

When a wave is travelling in a porous space or a space with an unsmooth boundary, the classic model cannot reveal the effect of unsmooth geometry on the wave properties, and a fractal model has to be adopted. In this paper, we mainly consider the time-space fractional non-linear wave equation:

$$\frac{\partial}{\partial t^{\alpha}}\left[f(\Theta)\frac{\partial\Theta}{\partial t^{\alpha}}\right]+a\frac{\partial}{\partial x^{\beta}}\left[g(\Theta)\frac{\partial\Theta}{\partial x^{\beta}}\right]+b\Theta+c\Theta^{3}=0 \tag{1}$$

where $0<\alpha,\ \beta\le 1$, $a$, $b$, and $c$ are constants, $f$, $g$ – are functions of $\Theta$, and $\partial\Theta/\partial t^{\alpha}$ is the He's fractal derivative [11]:

$$\frac{\partial\Theta}{\partial t^{\alpha}}(t_0,x)=\Gamma(1+\alpha)\lim_{t-t_0\to\Delta t}\frac{\Theta(t,x)-\Theta(t_0,x)}{(t-t_0)^{\alpha}} \tag{2}$$

When $\alpha=\beta=1, f(\Theta)=g(\Theta)=1$, eq. (1) can been written:

$$\Theta_{tt}+a\Theta_{xx}+b\Theta+c\Theta^{3}=0 \tag{3}$$

The eq. (3) was researched in [20].

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## The variational principle of non-linear wave equation

In order to establish the variational principle for eq. (1), we introduce two functions $F$ and $G$:

$$\frac{\mathrm{d}F}{\mathrm{d}\Theta}=f(\Theta),\ \frac{\mathrm{d}G}{\mathrm{d}\Theta}=g(\Theta) \tag{4}$$

Therefore, eq. (1) can been written:

$$\frac{\partial}{\partial t^{2\alpha}}\left[F(\Theta)\frac{\partial\Theta}{\partial t^{\alpha}}\right]+a\frac{\partial}{\partial x^{\beta}}\left[g(\Theta)\frac{\partial\Theta}{\partial x^{\beta}}\right]+b\Theta+c\Theta^{3}=0 \tag{5}$$

The variational principle of eq. (5) can been established by semi-inverse method [1-10]:

$$J(\Theta)=\int\left\{\int\left\{-\frac{1}{2}\left(\frac{\partial F}{\partial t^{\alpha}}\right)^{2}-\frac{1}{2}a\left(\frac{\partial G}{\partial x^{\beta}}\right)^{2}+\frac{1}{2}b\Theta^{2}+\frac{1}{4}c\Theta^{4}\right\}\mathrm{d}x^{\beta}\right\}\mathrm{d}t^{\alpha} \tag{6}$$

*Proof:* The Euler-Lagrange equation of eq. (6):

$$\frac{\partial}{\partial t^{\alpha}}\left(\frac{\partial F}{\partial t^{\alpha}}\right)+a\frac{\partial}{\partial x^{\beta}}\left(\frac{\partial G}{\partial x^{\beta}}\right)+b\Theta+c\Theta^{3}=0 \tag{7}$$

Using the chain ruler, we can obtain:

$$\frac{\partial}{\partial t^{\alpha}}\left(\frac{\mathrm{d}F}{\mathrm{d}\Theta}\frac{\partial\Theta}{\partial t^{\alpha}}\right)+a\frac{\partial}{\partial x^{\beta}}\left(\frac{\mathrm{d}G}{\mathrm{d}\Theta}\frac{\partial\Theta}{\partial x^{\beta}}\right)+b\Theta+c\Theta^{3}=0 \tag{8}$$

According to the definitions of eq. (4), we can write eq. (8) into the form:

$$\frac{\partial}{\partial t^{2\alpha}}\left[F(\Theta)\frac{\partial\Theta}{\partial t^{\alpha}}\right]+a\frac{\partial}{\partial x^{\beta}}\left[g(\Theta)\frac{\partial\Theta}{\partial x^{\beta}}\right]+b\Theta+c\Theta^{3}=0 \tag{9}$$

The eq. (9) is the same as eq. (1).

## Two-scale transform for fractional non-linear wave equation

We will adopt the two-scale method [21-24] and the exp-function [25-31] to solve the fractional non-linear wave equation. The two-scale transform [21-24] is an excellent tool to dealing with fractional differential equations in fractal space. The exp-function method [25-31] was proposed by He, it is a powerful tool to finding approximate solutions of fractional differential equations.

When $f(\Theta)=g(\Theta)=1$, $a=c=-1$, $b=1$. The eq. (9):

$$\frac{\partial}{\partial t^{\alpha}}\left(\frac{\partial\Theta}{\partial t^{\alpha}}\right)-\frac{\partial}{\partial x^{\beta}}\left(\frac{\partial\Theta}{\partial x^{\beta}}\right)+\Theta-\Theta^{3}=0 \tag{10}$$

The two-scale transform is to convert eq. (10) with fractal derivatives to an approximate differential equation:

$$T=t^{\alpha} \tag{11}$$

$$X=x^{\beta} \tag{12}$$

According to the definition of He's fractal derivative, eq. (10) becomes:

$$\Theta_{TT}-\Theta_{XX}+\Theta-\Theta^{3}=0 \tag{13}$$

We adopt He's exp-function method [25-31] to solve eq. (13).

Assume:

$$\xi = \lambda X + \mu T \tag{14}$$

where $\lambda$ and $\mu$ are real numbers.

Equation (13) can write into the form:

$$(\mu^2 - \lambda^2)\Theta'' + \Theta - \Theta^3 = 0 \tag{15}$$

We use the exp-function method [25-31], and the solution of eq. (15) can be expressed:

$$\Theta(\xi) = \frac{\sigma_1 e^{\xi} + \sigma_0 + \sigma_{-1} e^{-\xi}}{e^{\xi} + \varsigma_0 + \varsigma_{-1} e^{-\xi}} \tag{16}$$

where $\sigma_j$, $\varsigma_j (j = -1, 0, 1)$ are the unknown parameters, see [20].

By using eqs. (15) and (16), we have the solution of eq. (13):

$$\Theta(X,T) = \frac{e^{\left[\frac{\lambda X}{\Gamma(1+\beta)} + \frac{\mu T}{\Gamma(1+\alpha)}\right]} - \frac{1}{4}\varsigma_0^2 e^{\left[-\frac{\lambda X}{\Gamma(1+\beta)} - \frac{\mu T}{\Gamma(1+\alpha)}\right]}}{e^{\left[\frac{\lambda X}{\Gamma(1+\beta)} + \frac{\mu T}{\Gamma(1+\alpha)}\right]} + \frac{1}{4}\varsigma_0^2 e^{\left[-\frac{\lambda X}{\Gamma(1+\beta)} - \frac{\mu T}{\Gamma(1+\alpha)}\right]} + \varsigma_0} \tag{17}$$

So, the approximate analytical solution of eq. (10) is given:

$$\Theta(x,t) = \frac{e^{\left[\frac{\lambda x^\beta}{\Gamma(1+\beta)} + \frac{\mu t^\alpha}{\Gamma(1+\alpha)}\right]} - \frac{1}{4}\varsigma_0^2 e^{\left[-\frac{\lambda x^\beta}{\Gamma(1+\beta)} - \frac{\mu t^\alpha}{\Gamma(1+\alpha)}\right]}}{e^{\left[\frac{\lambda x^\beta}{\Gamma(1+\beta)} + \frac{\mu t^\alpha}{\Gamma(1+\alpha)}\right]} + \frac{1}{4}\varsigma_0^2 e^{\left[-\frac{\lambda x^\beta}{\Gamma(1+\beta)} - \frac{\mu t^\alpha}{\Gamma(1+\alpha)}\right]} + \varsigma_0} \tag{18}$$

where $\varsigma_0$ is constant.

It is obvious that the fractal orders [32-40] will greatly affect the solution structure.

## Conclusion

In this paper, a variational principle is established for a fractional non-linear wave equation based on He's fractal derivative in a fractal space. The two-scale transform and the exp-function method are used to find the approximate analytical solution of fractional non-linear wave equation.

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