COMPUTER SIMULATION OF PANTOGRAPH DELAY DIFFERENTIAL EQUATIONS

by

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Ritz method is widely used in variational theory to search for an approximate solution. This paper suggests a Ritz-like method for integral equations with an emphasis of pantograph delay equations. The unknown parameters involved in the trial solution can be determined by balancing the fundamental terms.

Key words: *integral equation, series solution, variational principle, Pantograph delay differential equation*

Introduction

The Ritz method or the Rayleigh-Ritz method is a famous analytical technology in the variational theory [1]. The application of the method requires establishment of a variational formulation for the discussed problem, which might be more complex than the solution process for a non-linear problem. The semi-inverse method [2-8] is a widely used to build up a needed variational principle, a trial solution with some unknown parameters is chosen and the unknowns can be identified by the stationary condition of the variational functional. In this paper we show that the idea can be extended to solving integral equations, and the unknowns in the trial solution can be approximately determined:

$$y''(x) = f[y(x), x] + g[y(x-\tau)] + \sum_{i=1}^{n} y(\sigma_i x), \;\; y(0) = \alpha, \;\; y'(0) = \beta \tag{1}$$

where τ, σ_i, α, and β are the constants, f and g – the functions.

Equations (1) can be solved by various methods, *e. g.*, the Taylor series method [9-13], the variational iteration method [14-19] and the homotopy perturbation method [20-27].

Ritz-like method

Similar to the Ritz method, we can assume that the solution has the form:

$$y(x) = \sum_{n=0}^{N} a_n x^n \tag{2}$$

where a_n ($n = 0 \sim N$) are unknown constants.

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Submitting eq. (2), combining the like terms of x^n, and setting the coefficients of x^n ($n = 0 \sim N - 2$) to zero, we obtain algebraic equations for a_n ($n = 0 \sim N$) by taking into account the initial conditions $y(0) = \alpha$, $y'(0) = \beta$.

Alternatively, we can assume the solution has the form:

$$y(x) = \frac{\sum_{n=0}^{N} B_n(x)}{\sum_{m=0}^{M} A_m(x)} \tag{3}$$

where $A_m(x)$ ($m = 0 \sim M$) and $B_n(x)$ ($n = 0 \sim$ N) are basis functions. If we choose $A_m(x) = a_m x^m$ and $B_n(x) = b_n x^n$, the trial solution has the form:

$$y(x) = \frac{\sum_{n=0}^{N} b_n x^n}{\sum_{m=0}^{M} a_m x^m} \tag{4}$$

Example 1

We consider the following multi-pantograph delay equation [28]:

$$y'(x) = -\frac{5}{6} y(x) + 4y\left(\frac{1}{2}x\right) + 9y\left(\frac{1}{3}x\right) + x^2 - 1, \; y(0) = 1, \; 0 < x \le 1 \tag{5}$$

We assume that the solution has the form:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{6}$$

Submitting eq. (6) into eq. (5):

$$a_1 + 2a_2 x + 3a_3 x^2 = -\frac{5}{6}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right) + 4\left(a_0 + \frac{1}{2}a_1 x + \frac{1}{4}a_2 x^2 + \frac{1}{8}a_3 x^3\right) + $$

$$+ 9\left(a_0 + \frac{1}{3}a_1 x + \frac{1}{9}a_2 x^2 + \frac{1}{27}a_3 x^3\right) + x^2 - 1 \tag{7}$$

It is easy to obtain the following algebra equations:

$$a_1 = -\frac{5}{6}a_0 + 4a_0 + 9a_0 - 1 \tag{8}$$

$$2a_2 = -\frac{5}{6}a_1 + 2a_1 + 3a_1 \tag{9}$$

$$3a_3 = -\frac{5}{6}a_2 + a_2 + a_2 + 1 \tag{10}$$

$$3a_3 = -\frac{5}{6}a_3 + \frac{1}{2}a_3 + \frac{1}{3}a_3 \tag{11}$$

Solving the aforementioned algebra system:

$$a_0 = 1 \tag{12}$$

$$a_1 = \frac{67}{6} \tag{13}$$

$$a_2 = \frac{1675}{72} \tag{14}$$

$$a_3 = \frac{12157}{1296} \tag{15}$$

Therefore we obtain the approximate solution:

$$y(x) = 1 + \frac{67}{6}x + \frac{1675}{72}x^2 + \frac{12157}{1296}x^3 \tag{16}$$

which happens to be the exact one.

Example 2

Consider the following delay equation [28]:

$$y''(x) = \frac{3}{4}y(x) + y\left(\frac{1}{2}x\right) - \frac{\frac{3}{4}}{1+x+x^2} - \frac{4}{4+2x+x^2} + \frac{2(1+2x)^2}{(1+x+x^2)^3} - \frac{2}{(1+x+x^2)^2} \tag{17}$$

with the following initial conditions:

$$y(0) = 1, \;\; y'(0) = -1 \tag{18}$$

We assume the solution has the form:

$$y(x) = \frac{b_0}{a_0 + a_1 x + a_2 x^2 + a_3 x^3} \tag{19}$$

By a similar solution process as that of *Example 1*, the unknown parameters in eq. (19) can be identified:

$$y(x) = \frac{1}{1 + x + x^2 + x^3} \tag{20}$$

which is the exact solution.

Discussion and conclusion

The present method can be easily extended to the case with fractal derivatives [29, 30]:

$$\frac{d^2 y}{dx^{2\eta}} = f[y(x), x] + g[y(x-\tau)] + \sum_{i=1}^{n} y(\sigma_i x), \;\; y(0) = \alpha, \;\; \frac{dy}{dx^{\eta}}(0) = \beta \tag{21}$$

By the two-scale transform [7]:

$$X = x^n \tag{22}$$

Equation (21) is converted:

$$\frac{d^2 y}{dX^2} = f[y(x), x] + g[y(x-\tau)] + \sum_{i=1}^{n} y(\sigma_i x), \;\; y(0) = \alpha, \;\; \frac{dy}{dX}(0) = \beta \tag{23}$$

The solution process for eq. (23) is previously illustrated, so it is easy to solve fractal differential equations.

The non-linear equations with delay terms are difficult to be solved, this paper proposes a simple but effective method for such problems. The results strongly depend upon the initial solution, a suitable choice of the initial solution will always lead to a good result.

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References

[1] Shahgholian, D., *et al.,* Buckling Analyses of Functionally Graded Graphene-Reinforced Porous Cylindrical Shell Using the Rayleigh-Ritz Method, *Acta Mechanica*, *231* (2020), Feb., pp. 1887-1902
[2] He, J. H., Generalized Variational Principles for Buckling Analysis of Circular Cylinders, *Acta Mechanica, 231* (2020), Dec., pp. 899-906
[3] He, J. H., A Fractal Variational Theory for 1-D Compressible Flow in a Microgravity Space, *Fractals*, *28* (2020), 02, 2050024
[4] He, J.-H., Variational Principle and Periodic Solution of the Kundu-Mukherjee-Naskar Equation, *Results in Physics*, *17* (2020), June, 103031
[5] He, J. H., Lagrange Crisis and Generalized Variational Principle for 3-D Unsteady Flow, *International Journal of Numerical Methods for Heat and Fluid-Flow, 30* (2019), 3, pp. 1189-1196
[6] He, J. H., Sun, C. A., Variational Principle for a Thin Film Equation, *Journal of Mathematical Chemistry*, *57* (2019), 9, pp. 2075-2081
[7] He, J. H., Ain, Q. T., New Promises and Future Challenges of Fractal Calculus: From Two-Scale Thermodynamics to Fractal Variational Principle, *Thermal Science*, *24* (2020), 2A, pp. 659-681
[8] He, J. H., Variational Principle and Periodic Solution of the Kundu-Mukherjee-Naskar Equation, *Results in Physics*, *17* (2020), 103031
[9] He, C.H., *et al*., Taylor Series Solution for Fractal Bratu-Type Equation Arising in eElectrospinning Process, *Fractals*, *28* (2020), 1, 2050011
[10] He, J. H., A Simple Approach to 1-D Convection-Diffusion Equation and Its Fractional Modification for E Reaction Arising in Rotating Disk Electrodes, *Journal of Electroanalytical Chemistry*, *854* (2019), 113565
[11] He, J. H., Ji, F. Y., Taylor Series Solution for Lane-Emden Equation, *Journal of Mathematical Chemistry, 57* (2019), 8, pp. 1932-1934
[12] He, J. H., The Simplest Approach to Non-Linear Oscillators, *Results in Physics, 15* (2019), 102546
[13] He, J. H., Taylor Series Solution for a Third Order Boundary Value Problem Arising in Architectural Engineering, *Ain Shams Engineering Journal*, On-line first, https://doi.org/10.1016/j.asej.2020.01.016, 2020
[14] He, J. H., Latifizadeh, H., A General Numerical Algorithm for Non-Linear Differential Equations by the Variational Iteration Method, *International Journal of Numerical Methods for Heat and Fluid-Flow*, *30* (2020), 11, pp. 4797-4810
[15] Ji, F. Y., *et al.,* A Fractal Boussinesq Equation for Non-Linear Transverse Vibration of a Nanofiber-Reinforced Concrete Pillar, *Applied Mathematical Modelling, 82* (2020), June, pp. 437-448
[16] He, J. H., A Short Review on Analytical Methods for to a Fully Fourth Order Non-Linear Integral Boundary Value Problem with Fractal Derivatives, *International Journal of Numerical Methods for Heat and Fluid-Flow*, *30* (2020), 11, pp. 4933-4943
[17] He, J. H., Jin, X., A Short Review on Analytical Methods for the Capillary Oscillator in a Nanoscale Deformable Tube, *Mathematical Methods in the Applied Sciences,* On-line first, http://dx.doi.org/10.1002/mma.6321, 2020
[18] He, C. H., *et al.,* Fangzhu, An Ancient Chinese Nanotechnology for Water Collection from Air, History, Mathematical Insight, Promises and Challenges, *Mathematical Methods in the Applied Sciences*, (2020), On-line first, https://doi.org/10.1002/mma.6384, 2020
[19] Ahmad, H., Khan, T. A., Variational Iteration Algorithm-I with an Auxiliary Parameter for Wave-Like Vibration Equations, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1113-1124
[20] Wei, C. F., Application of the Homotopy Perturbation Method for Solving Fractional Lane-Emden Type Equation, *Thermal Science*, *23* (2019), 4, pp. 2237-2244
[21] Yang, Y. J., Wang, S. Q., A Local Fractional Homotopy Perturbation Method for Solving the Local Fractional Korteweg-de Vries Equations with Non-Homogeneous Term, *Thermal Science*, *23* (2019), 3A, pp. 1495-1501
[22] Kuang, W. X., *et al.* Homotopy Perturbation Method with an Auxiliary Term for the Optimal Design of a Tangent Non-Linear Packaging System, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1075-1080

[23] Pasha, S. A., *et al*., The Modified Homotopy Perturbation Method with an Auxiliary Term for the Non-Linear Oscillator with Discontinuity, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1363-1373
[24] Li, X. X., He, C. H., Homotopy Perturbation Method Coupled with the Enhanced Perturbation Method, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1399-1403
[25] Yu, D. N., *et al.*, Homotopy Perturbation Method with an Auxiliary Parameter for Non-Linear Oscillators, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, ppp. 1540-1554
[26] He, J. H., The Simpler, the Better: Analytical Methods for Non-Linear Oscillators and Fractional Oscillators, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1252-1260
[27] Bahgat, M. S. M., Approximate Analytical Solution of the Linear and Non-Linear Multi-Pantograph Delay Differential Equations, *Physica Scripta, 95* (2020), 055219
[28] Sedaghat, S., *et al*., Analysis of Spectral Method for Neutral Multi-Pantograph Equations, *Iranian Journal of Science and Technology Transaction A-Science*, *43* (2019), A5, pp. 2261-2268
[29] He, J.-H., Fractal Calculus and its Geometrical Explanation, *Results in Physics*, *10* (2018), Sept., pp. 272-276
[30] Ain, Q. T., He, J.-H., On Two-Scale Dimension and its Applications, *Thermal Science*, *23* (2019) , 3, pp. 1707-1712

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