NON-DIFFERENTIABLE SOLUTIONS OF A FAMILY OF MODIFIED KORTEWEG-de VRIES EQUATIONS WITHIN LOCAL FRACTIONAL DERIVATIVE

by

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In this paper, a family of modified Korteweg-de Vries equations within local fractional derivative are constructed, and their non-differentiable solutions are discussed by using several methods.

Key words: *modified Korteweg-de Vries equation, traveling wave solution, $G^{(\alpha)}(\xi)/G(\xi)$-expansion method*

Introduction

In recent years, many new fractional derivatives have been developed to described the real world problems. For examples, He's fractional derivative [1-4], Atangana-Baleanu fractional derivative [5, 6] and the local fractional derivative [7].

The modified Korteweg-de-Vries (MKdV) equations play an important role in description of the physics problems such as ion-acoustic waves, freak waves, shallow water waves and energy-preserving [8-11]. As for the MKdV in porous media, the local fractional derivatives can better describe the irregularity of holes and the conduction of physical quantities than the classical derivatives. On this basis, a family of local fractional MKdV is established in this paper. In addition, author stumbled upon a common solution to these MKdV. The non-differentiable traveling wave solutions of these MKdV, such as kink solution and periodic solution, are given by several methods.

A family of MKdV equations and their implicit function solutions

The local fractional differential calculus theory can be seen in [7]. Now, we consider the following MKdV equation within the local fractional derivative, which is given in the following form:

$$\gamma u_t^{(\alpha)} + u^2 u_x^{(\alpha)} + \beta u_x^{(3\alpha)} = 0 \tag{1}$$

where $u = u(x,t)$ is the non-differentiable wave function in a porous medium, x and t are considered as fractal space and fractal time respectively, γ and β are all non-zero constants. Moreover, α is the value of fractal dimensions of the porous medium. For $\alpha = 1$, the proposed medium does not have holes.

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Obviously, a generalized form of eq. (1) can be written:

$$\gamma u_t^{(\alpha)} + Vuu_x^{(\alpha)} = \delta\beta u_x^{(3\alpha)} \tag{2}$$

where δ is an arbitrary parameter and $V = V[u, u_x^{(\alpha)}, u_x^{(2\alpha)}, ..., u_x^{(n\alpha)}]$ is an arbitrary function.

We assume that the following traveling wave solution:

$$u(x,t) = f(x - ct) = f(\xi) \tag{3}$$

which satisfies both eqs. (1) and (2) for the same speed *c*. Substituting eq. (3) into eq. (1) and eq. (2), respectively, we obtain:

$$-c\gamma f^{(\alpha)} + f^2 f^{(\alpha)} + \beta f^{(3\alpha)} = 0 \tag{4}$$

and

$$-c\gamma f^{(\alpha)} + Vff^{(\alpha)} - \delta\beta f^{(3\alpha)} = 0 \tag{5}$$

Removing $f^{(3\alpha)}$ from the eqs. (4) and (5) and by noting that $f^{(\alpha)} \neq 0$, we derive:

$$V = (\delta + 1)\frac{c\alpha}{f} - f\delta \tag{6}$$

According to eq. (4), we get:

$$c = \frac{f^2}{\gamma} + \frac{\beta f^{(3\alpha)}}{\gamma f^{(\alpha)}} \tag{7}$$

Differentiating eq. (7) once, twice, three times and *n* times ($n > 3$). We get, respectively:

$$c = \frac{2ff^{(\alpha)2} + f^2 f^{(2\alpha)} + \beta f^{(4\alpha)}}{\gamma f^{(2\alpha)}} \tag{8}$$

$$c = \frac{2f^{(\alpha)3} + 6ff^{(\alpha)} f^{(2\alpha)} + f^2 f^{(3\alpha)} + \beta f^{(5\alpha)}}{\gamma f^{(3\alpha)}} \tag{9}$$

$$c = \frac{12 f^{(\alpha)2} f^{(2\alpha)} + 6ff^{(2\alpha)2} + 8ff^{(\alpha)} f^{(2\alpha)} + f^2 f^{(4\alpha)} + \beta f^{(6\alpha)}}{\gamma f^{(4\alpha)}};... \tag{10}$$

$$c = \frac{(f^3)^{[(n+3)\alpha]} + 3\beta f^{[(n+3)\alpha]}}{3\gamma f^{[(n+1)\alpha]}} = \frac{1}{3\gamma f^{[(n+1)\alpha]}}\left\{\sum_{\substack{r_1+r_2+r_3=n+1 \\ (0\le r_1, r_2, r_3 \le n+1)}} f^{(r_1\alpha)} f^{(r_2\alpha)} f^{(r_3\alpha)} + 3\beta f^{[(n+3)\alpha]}\right\} \tag{11}$$

Substituting eqs. (8)-(11) into eq. (6), respectively, we get:

$$V_1 = \delta\left[\frac{2ff^{(\alpha)2} + \beta f^{(4\alpha)}}{ff^{(2\alpha)}}\right] + \left[\frac{2f^{(\alpha)2} + f^2 f^{(2\alpha)} + \beta f^{(4\alpha)}}{ff^{(2\alpha)}}\right] \tag{12}$$

$$V_2 = \delta\left[\frac{2f^{(\alpha)3} + 6ff^{(\alpha)} f^{(2\alpha)} + \beta f^{(5\alpha)}}{ff^{(3\alpha)}}\right] + \left[\frac{2f^{(\alpha)3} + 6ff^{(\alpha)} f^{(2\alpha)} + f^2 f^{(3\alpha)} + \beta f^{(5\alpha)}}{ff^{(3\alpha)}}\right] \tag{13}$$

$$V_3=\delta\left[\frac{12f^{(\alpha)2}f^{(2\alpha)}+6ff^{(2\alpha)2}+8ff^{(\alpha)}f^{(2\alpha)}+\beta f^{(6\alpha)}}{ff^{(4\alpha)}}\right]+$$
$$+\left[\frac{12f^{(\alpha)2}f^{(2\alpha)}+6ff^{(2\alpha)2}+8ff^{(\alpha)}f^{(2\alpha)}+f^2f^{(4\alpha)}+\beta f^{(6\alpha)}}{ff^{(4\alpha)}}\right];\ldots \tag{14}$$

$$V_n=\delta\frac{1}{3ff^{((n+1)\alpha)}}\left\{\sum_{\substack{r_1+r_2+r_3=n+1\\(0\le r_1,r_2,r_3\le n+1)}}f^{(r_1\alpha)}f^{(r_2\alpha)}f^{(r_3\alpha)}+3\beta f^{[(n+3)\alpha]}-3f^2f^{[(n+1)\alpha]}\right\}+$$
$$+\frac{1}{3ff^{[(n+1)\alpha]}}\left\{\sum_{\substack{r_1+r_2+r_3=n+1\\(0\le r_1,r_2,r_3\le n+1)}}f^{(r_1\alpha)}f^{(r_2\alpha)}f^{(r_3\alpha)}+3\beta f^{[(n+3)\alpha]}\right\} \tag{15}$$

Afterwards, substituting eqs. (12)-(15) into eq. (2), respectively, and according to $f^{(n\alpha)}=u_x^{(n\alpha)}(x,t)$ we obtain the following MKdV equations:

$$\gamma u_t^{(\alpha)}+\frac{2uu_x^{(\alpha)2}+u^2u_x^{(2\alpha)}+\beta u_x^{(4\alpha)}}{u_x^{(2\alpha)}}u_x^{(\alpha)}+\delta\left[\frac{2uu_x^{(\alpha)2}+\beta u_x^{(4\alpha)}}{u_x^{(2)}}u_x^{(\alpha)}-\beta u_x^{(3\alpha)}\right]=0 \tag{16}$$

$$\gamma u_t^{(\alpha)}+\frac{2u_x^{(\alpha)3}+6uu_x^{(\alpha)}u_x^{(2\alpha)}+u^2u_x^{(3\alpha)}+\beta u_x^{(5\alpha)}}{u_x^{(3)}}u_x+$$
$$+\delta\left[\frac{2u_x^{(\alpha)3}+6uu_x^{(\alpha)}u_x^{(2\alpha)}+\beta u_x^{(5\alpha)}}{u_x^{(3\alpha)}}u_x-\beta u_x^{(3\alpha)}\right]=0 \tag{17}$$

$$\gamma u_t^{(\alpha)}+\frac{12u_x^{(\alpha)2}u_x^{(2\alpha)}+6uu_x^{(2\alpha)}+8uu_x^{(\alpha)}u_x^{(3\alpha)}+u^2u_x^{(4\alpha)}+\beta u_x^{(6\alpha)}}{u_{x^3}^{(3\alpha)}}u_x^{(\alpha)}+$$
$$+\delta\left[\frac{12u_x^{(\alpha)2}u_x^{(2\alpha)}+6uu_x^{(2\alpha)}+u^2u_x^{(4\alpha)}+\beta u_x^{(6\alpha)}}{u_x^{(3\alpha)}}u_x^{(\alpha)}-\beta u_x^{(3\alpha)}\right]=0,\ldots \tag{18}$$

$$\gamma u_t^{(\alpha)}+\frac{1}{3u_x^{[(n+1)\alpha]}}\left\{\sum_{\substack{r_1+r_2+r_3=n+1\\(0\le r_1,r_2,r_3\le n+1)}}u_x^{(r_1\alpha)}u_x^{(r_2\alpha)}u_x^{(r_3\alpha)}+3\beta u_x^{[(n+3)\alpha]}\right\}u_x^{(\alpha)}+$$
$$+\delta\left(\frac{1}{3u_x^{[(n+1)\alpha]}}\left\{\sum_{\substack{r_1+r_2+r_3=n+1\\(0\le r_1,r_2,r_3\le n+1)}}u_x^{(r_1\alpha)}u_x^{(r_2\alpha)}u_x^{(r_3\alpha)}+3\beta u_x^{[(n+3)\alpha]}-3u^2u_x^{[(n+3)\alpha]}\right\}u_x^{(\alpha)}-\beta u_x^{(3\alpha)}\right)=0 \tag{19}$$

All the eqs. (16)-(19) have the higher order derivatives than the MKdV given in eq. (1).

Supposing that the traveling wave solution has the form:

$$u(x,t)=u(\xi)=u(x-ct) \tag{20}$$

which satisfies eqs. (1), (16)-(19). Analyzing the constitution of eqs. (1), (16)-(19), we can derive that the common solution $u(\xi)$ of MKdV eqs. (1), (16)-(19) satisfies the following equation:

$$-c\gamma u(\xi)+\frac{1}{3}u^3(\xi)+\beta u^{(2\alpha)}(\xi)=0 \tag{21}$$

It is clear that the implicit function solution of eq. (21) is the common solutions of the MKdV eqs. (16)-(19).

In the following, we apply the $G^{(\alpha)}(\xi)/G(\xi)$ expansion method to seek the analytic solutions of eq. (21).

Suppose that the solution $u(\xi)$ of eq. (21) can be expressed by a polynomial in $G^{(\alpha)}(\xi)/G(\xi)$:

$$u(\xi)=\sum_{i=0}^{N}a_i\left[\frac{G^{(\alpha)}(\xi)}{G(\xi)}\right]^i \tag{22}$$

where $G(\xi)$ is the solution of the following equation:

$$G^{(2\alpha)}(\xi)+\lambda G^{(\alpha)}(\xi)+\mu G(\xi)=0 \tag{23}$$

and where $a_i(0\le i\le N),\lambda$ and μ are all constants to be determined.

The positive integral $N=1$ can be determined by balancing the linear term of highest order derivatives with the highest order non-linear term in eq. (21). Thence, we get:

$$u(\xi)=a_0+a_1\frac{G^{(\alpha)}(\xi)}{G(\xi)} \tag{24}$$

Substituting both eqs. (24) and (23) into eq. (21), we yield an algebraic equation involving powers of $G^{(\alpha)}(\xi)/G(\xi)$. Then, equating the coefficients of each power of $G^{(\alpha)}(\xi)/G(\xi)$ to zero, we obtain the following system of algebraic equations for $a_i(0\le i\le 1),\lambda$ and μ:

$$\begin{gathered}a_1^3\beta-2a_1c\gamma=0\\ a_0a_1\beta-c\gamma\lambda=0\\ \frac{a_0}{3}+a_0^3\beta-a_1c\gamma\lambda\mu=0\\ \frac{a_1}{3}+3a_0^2a_1\beta-a_1c\gamma\lambda^2-2a_1c\gamma\mu=0\end{gathered} \tag{25}$$

Then, we obtain the following two solutions of system (25).

The first solution is:

$$a_0=0,\quad a_1=\pm\sqrt{\frac{2c\gamma}{\beta}},\quad \lambda=0,\quad \mu=\frac{1}{6c\gamma} \tag{26}$$

The second solution is:

$$a_1 = \pm\sqrt{\frac{2c\gamma}{\beta}}, \quad \lambda = \pm a_0\sqrt{\frac{2\beta}{c\gamma}}, \quad \mu = \frac{1+3a_0^2\beta}{6c\gamma} \tag{27}$$

When $\lambda^2 - 4\mu = 2/(-3c\gamma) > 0$, whether according to eq. (26) or according to eq. (27), we obtain the same solution of eq. (21):

$$u(\xi) = \frac{1}{\sqrt{-3\beta}} \frac{A + B\tanh_\alpha \dfrac{\xi}{\sqrt{-6c\gamma}}}{B + A\tanh_\alpha \dfrac{\xi}{\sqrt{-6c\gamma}}} \tag{28}$$

When $\lambda^2 - 4\mu = 2/(-3c\gamma) > 0$, whether according to eq. (26) or according to eq. (27), we obtain the same solution of eq. (21):

$$u(\xi) = \sqrt{\frac{1}{3\beta}} \frac{B - A\tan_\alpha \sqrt{\dfrac{1}{6c\gamma}}\xi}{A + B\tan_\alpha \sqrt{\dfrac{1}{6c\gamma}}\xi} \tag{29}$$

When the parameters A, B, γ, β, and c are taken as special values, respectively, eqs. (28) and (29) are all special solutions of eq. (21), or the MKdV eqs. (1), (16)-(19). For example:

Case 1: If we choose $A=0, B=1, \gamma=1, \beta=-1/3, c=-1$, then $u(x,t)$ = $\tanh_\alpha x$ + $+t/\sqrt{6}$, which is of the kink type, is a common solution of the KdV-like eqs. (1), (16)-(19).

Case 2: If we choose $A=1, B=0, \gamma=1, \beta=-1/3, c=-1$, then $u(x,t) = \coth_\alpha x + t/\sqrt{6}$, which is singular soliton, is a common solution of the KdV-like eqs. (1), (16)-(19).

Case 3: If we choose $A=1, B=0, \gamma=1, \beta=1/3, c=1$, then $u(x,t) = -\tan_\alpha x - t/\sqrt{6}$ is a common periodic solution of the KdV-like eqs. (1), (16)-(19).

Case 4: If we choose $A=0, B=1, \gamma=1, \beta=1/3, c=1$, then $u(x,t) = \cot_\alpha x - t/\sqrt{6}$ is a common periodic solution of the KdV-like eqs. (1), (16)-(19).

In the following sections, when $\gamma = \beta = c = 1$, we give the same kink solutions and the same periodic solutions of the MKdV eqs. (1), (16)-(19).

The kink solutions

We assume that the solution of eq. (21) has the following form:

$$u(\xi) = A\operatorname{sech}_\alpha(kx - kt) \tag{30}$$

where $\xi = x - t/\Gamma(1+\alpha)$, A and k are all real constants to be determined.

Substituting eq. (30) into eq. (21) and then solving the resulting equation for A and k, we obtain:

$$A = \pm\sqrt{3c\gamma}, \quad k = \pm\sqrt{\frac{-c\gamma}{2\beta}}$$

This in turn gives the kink solution of eq. (21):

$$u(x,t) = \pm\sqrt{3c\alpha}\operatorname{sech}_\alpha\left\lfloor\sqrt{\frac{-c\gamma}{2\beta}}(x-t)\right\rfloor \tag{31}$$

In a like manner, we can also derive other kink solutions of eq. (21):

$$u(x,t)=\pm\sqrt{3c\alpha}c\,\mathrm{sch}_\alpha\left[\sqrt{\frac{-c\gamma}{2\beta}}(x-t)\right] \tag{32}$$

$$u(x,t)=\pm\sqrt{-3c\alpha}th_\alpha\left[\sqrt{\frac{c\gamma}{2\beta}}(x-t)\right] \tag{33}$$

$$u(x,t)=\pm\sqrt{-3c\alpha}cth_\alpha\left[\sqrt{\frac{c\gamma}{2\beta}}(x-t)\right] \tag{34}$$

Obviously, the eqs. (31)-(34) satisfy all the MKdV eqs. (6), (16)-(19) for any δ.

The periodic solutions

We assume that the solution of eq. (21) has the form:

$$u(\xi)=A\sec_\alpha\left[k(x-t)\right] \tag{35}$$

where $\xi=x-t/\Gamma(1+\alpha)$, A and k are all real constants to be determined.

Substituting eq. (35) into eq. (21) and then solving the resulting equation for A and k, we find:

$$A=\pm\sqrt{6c\alpha},\quad k=\pm\sqrt{\frac{c\alpha}{\beta}}$$

This in turn gives the solution of eq. (21):

$$u(x,t)=\pm\sqrt{6c\alpha}\sec_\alpha\left[\sqrt{\frac{c\gamma}{\beta}}(x-t)i\right] \tag{36}$$

In a like manner, we can also derive other periodic solutions of eq. (21):

$$u(x,t)=\pm\sqrt{6c\alpha}\csc_\alpha\left[\sqrt{\frac{c\gamma}{\beta}}(x-t)i\right] \tag{37}$$

$$u(x,t)=\pm\sqrt{3c\alpha}\cot_\alpha\left[\sqrt{\frac{-c\gamma}{\beta}}(x-t)\right] \tag{38}$$

$$u(x,t)=\pm\sqrt{3c\alpha}\tan_\alpha\left[\sqrt{\frac{-c\gamma}{\beta}}(x-t)\right] \tag{39}$$

Obviously, the solutios (36)-(39) satisfy all the MKdV eqs. (1), (16)-(19) for any δ.

In order to clarify the constructed MKdV eqs. (16)-(19) have distinct solutions for different δ, in the following section, we give some kink solutions that satisfy only one kind of the eq. (19) for the specific values of δ.

Solutions for MKdV eqs. (16)-(19) only

We assume that the solution of eq. (19) has the form:

$$u(x,t)=RE_\alpha(kx-rt) \tag{40}$$

where *R*, *k*, and *r* are all real constants to be determined. Substituting eq. (40) into eq. (19), we get:

$$-\gamma r + \frac{1}{3}R^2 3^n kE_\alpha(2kx-2rt) + \beta k^3 +$$

$$+\delta\left[\frac{1}{3}R^2 3^n kE_\alpha(2kx-2rt) + \beta k^3 - R^2 kE_\alpha(2kx-2rt) - \beta k^3\right] = 0 \quad (41)$$

Supposing:

$$\begin{aligned} -\gamma r + \beta k^3 + \delta\beta k^3 - \delta\beta k^3 = 0 \\ \frac{1}{3}R^2 3^n k + \frac{1}{3}R^2 3^n k\delta - R^2 k = 0 \end{aligned} \quad (42)$$

and then solving eq. (42),we derive.

$$\delta = 3^{1-n} - 1, \quad \gamma r = \beta k^3 \quad (43)$$

Consequently, eq. (19) has the following solution:

$$u(x,t) = RE_\alpha\left(kx - \frac{\beta}{\gamma}k^3 t\right) \quad (44)$$

Obviously, the solution (44) only satisfies the MKdV eqs. (16)-(19) with the corresponding parameter $\delta = 3^{1-n} - 1$.

Solutions for MKdV eq. (1) only

In the following, we give two non-constant solutions, which satisfy the eq. (1) only and do not satisfy any other kind of the MKdV eqs. (16)-(19):

$$u(x,t) = \pm\sqrt{-6\beta k^2}\,\sec_\alpha\left(kx - \frac{\beta}{\gamma}k^3 t\right) \quad (45)$$

$$u(x,t) = \pm\sqrt{-6\beta k^2}\,\sec h_\alpha\left(kx - \frac{\beta}{\gamma}k^3 t\right) \quad (46)$$

Conclusion

In this paper, a family of MKdV equations are constructed, and their solution properties are discussed. The results show that all these MKdV equations even with the terms of higher derivatives have common solutions. This property is important in non-linear scienence and solitary theory, and the results can be extended to other fractional differential equations of fractal differential equations [12-19].

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