THE ULT-HSS HYBRID ITERATION METHOD FOR SYMMETRIC SADDLE POINT PROBLEMS

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This paper proposes a hybrid iteration method for solving symmetric saddle point problem arising in CFD. It is an implicit alternative direction iteration method and named as the ULT-HSS (upper and lower triangular, Hermitian and skew-Hermitian splitting) method. The convergence analysis is provided, and the necessary and sufficient conditions are given for the convergence of the method. Some practical approaches are formulated for setting the optimal parameter of the method. Numerical experiments are given to show its efficiency.

Key words: *Saddle point problem, ULT, HSS, convergence*

Introduction

Consider the following symmetric saddle point problem:

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \tag{1}$$

where $A \in R^{n\times n}$ is a symmetric positive definite, $B \in R^{m\times n}(m \le n)$ is of full row rank, and $f \in R^n$ and $g \in R^m$ are two column vectors. Equation (1) arises widely in computational science and engineering areas, including CFD [1, 2], thermal science [3], parameter identification [4], mixed finite element approximation of elliptic PDE [5, 6], and others. It is well known that Navier-Stokes problem or Oseen problem can be used to model the groundwater contamination transport in porous media [1, 2, 6], and the heat conduction of gas in thermal science [3]. Based on Navier-Stokes or Oseen equations with mixed finite element approximation, it follows the symmetric saddle point problem (1).

In the past decades, a number of iteration methods and their numerical properties have been discussed to solve the saddle point problem (1) in the literature, such as SOR-like methods [7-9], Uzawa-type methods [5, 6, 10-12], Hermitian and skew-Hermitian splitting (HSS) methods [13, 14], and Krylov subspace iteration methods with various preconditioners [15-19]. See the detailed survey by Benzi *et al.* [5]. Bramble *et al.* [6] proposed a non-linear inexact Uzawa algorithm for generalized saddle point problems. Various extensions and improvements of Uzawa-type methods have been widely discussed [10-12]. Based on Hermitian and skew-Hermitian splitting of the coefficient matrix of non-Hermitian positive definite linear system $Ax = b$, Bai *et al.* [13] introduced a shift parameter, and proposed an implicit alter-

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native direction iteration method named as HSS. The HSS method converges unconditionally if the shift parameter is positive. In order to improve the convergence of HSS method, Benzi and Golub [20] proposed a preconditioned Krylov subspace method with HSS preconditioner for generalized saddle point problems. There are also some improved versions of HSS method [14, 21]. Similarly, by means of the upper and lower triangular (ULT) splitting of the coefficient matrix of (1), Zheng and Ma [22] proposed a ULT iteration method. The semi-convergence of ULT iterative method for the singular saddle point problems was considered in [23]. A natural idea is that combining these two splittings may be helpful to improve the convergence of ULT or HSS methods. However, how to apply ULT and HSS splittings results in an efficient iteration method, and how to improve the convergence by choosing the relaxed parameters, both require detailed analysis.

In this paper, we will focus on previous two problems. We will apply the ULT and HSS splittings, and propose an alternative direction iteration method. For simplicity, we call this hybrid method as ULT-HSS method. The convergence conditions are given by analyzing its iteration matrix. We also consider the optimality of the relaxed parameters for some special cases. Numerical experiments are provided to illustrate the efficiency of ULT-HSS method.

Notations. In the rest of this paper, $R^{m\times n}$ denotes the space of real $m \times n$ matrix. For any matrix $X \in R^{n\times n}$, X^T and X^{-1} stand for its transpose and inverse, respectively. The norm $\| \|_2$ is 2-norm of a vector or matrix. Besides, diag(a, b) denotes the diagonal matrix with diagonal elements a and b.

The ULT-HSS hybrid iteration method

The linear system (1) can be equivalently transformed:

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix} \tag{2}$$

We consider two kinds of matrix splitting of the coefficient matrix A. The first case is called as upper and lower block triangular splitting [22]:

$$A = L - U \tag{3}$$

where

$$L = \begin{bmatrix} A & 0 \\ -B & Q \end{bmatrix}, \quad \text{and } U = \begin{bmatrix} 0 & -B^T \\ 0 & Q \end{bmatrix}$$

with a symmetric positive definite matrix $Q \in R^{n\times n}$. The second matrix splitting was based on HSS with a shift parameter:

$$A = (\alpha I + H) - (\alpha I - S) \tag{4}$$

where

$$\alpha > 0,\ H = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0 & -B^T \\ B & 0 \end{bmatrix}. \ \text{ Denote } z_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix} \text{ and } b = \begin{bmatrix} f \\ -g \end{bmatrix}$$

By combining the ULT and HSS splittings of A, we can obtain the following ULT-HSS hybrid iteration method:

$$
\begin{aligned}
Lz_{i+\frac{1}{2}} &= Uz_i + b \\
(\alpha I + H)z_{i+1} &= (\alpha I - S)z_{i+\frac{1}{2}} + b
\end{aligned} \tag{5}
$$

The ULT-HSS algorithm

The previous implicit alternating direction iterative method can be rewritten:

$$
\begin{aligned}
y_{i+1} &= y_i + \left(\frac{1}{\alpha}I + Q^{-1}\right)[BA^{-1}(f - B^T y_i) - g] \\
x_{i+1} &= x_i + A^{-1}(f - Ax_i - B^T y_i) - \alpha(\alpha I + \mathrm{A})^{-1}B^T(\alpha I + \mathrm{Q})^{-1}(y_{i+1} - y_i)
\end{aligned} \tag{6}
$$

By iterative eq. (6), we can simplify it:

$$Mz_{i+1} = Nz_i + b$$

with

$$M = \begin{bmatrix} A & \alpha A(\alpha I + A)^{-1}B^T(\alpha I + Q)^{-1} \\ -B & \alpha[Q - B(\alpha I + A)^{-1}B^T](\alpha I + Q)^{-1} \end{bmatrix}, N = \begin{bmatrix} 0 & \alpha A(\alpha I + A)^{-1}B^T(\alpha I + Q)^{-1} - B^T \\ 0 & \alpha[Q - B(\alpha I + A)^{-1}B^T](\alpha I + Q)^{-1} \end{bmatrix}$$

Denote the iteration matrix of ULT-HSS method by T, it follows that:

$$T = M^{-1}N(\alpha I + H)^{-1}(\alpha I - S)L^{-1}U = \begin{bmatrix} 0 & -A^{-1}B^T + (\alpha I + A)^{-1}B^T Q^{-1}BA^{-1}B^T \\ 0 & I - \left(\frac{1}{\alpha}I + Q^{-1}\right)BA^{-1}B^T \end{bmatrix} \tag{7}$$

Convergence analysis of ULT-HSS method

In this section, we consider the convergence analysis of the hybrid ULT-HSS method. By *Lemma* 1 in [14], the ULT-HSS iteration converges for arbitrary initial guesses $x_0 \in R^n$ and $y_0 \in R^m$ to the exact solution $A^{-1}b$ if and only if $\rho(T) < 1$, where $\rho(T)$ is the spectral radius of the iteration matrix T of ULT-HSS.

In the following theorems, we will show the spectral radius of matrix T, and provide the necessary and sufficient conditions for convergence of ULT-HSS method.

Theorem 1. Assume that $A \in R^{n\times n}$ and $Q \in R^{m\times m}$ are symmetric positive definite, and $B \in R^{m\times n}$ is full of row rank. Let $T = (\alpha I + H)^{-1}(\alpha I + S)L^{-1}U$ be the iteration matrix of ULT-HSS method, and let $\theta = (u^T BA^{-1}B^T u)/(u^T u)$ and $\hat{\theta} = (u^T Q^{-1}BA^{-1}B^T u)/(u^T u)$ with non-zero vector $u \in R^m$, then the m eigenvalues of T are zero, and the other n eigenvalues satisfy the following equation:

$$\lambda - 1 + \frac{1}{\alpha}\theta + \hat{\theta} = 0 \tag{8}$$

Furthermore, the spectral radius of T is defined by:

$$\rho(T) = \max\left\{\left|1 - \frac{1}{\alpha}\theta_{\max} - \hat{\theta}_{\max}\right|, \quad \left|1 - \frac{1}{\alpha}\theta_{\min} - \hat{\theta}_{\min}\right|\right\}$$

where $\theta_{\max} = \max_{u\neq 0}\theta$, $\theta_{\min} = \min_{u\neq 0}\theta$, $\hat{\theta}_{\max} = \max_{u\neq 0}\hat{\theta}$, and $\hat{\theta}_{\min} = \min_{u\neq 0}\hat{\theta}$.

Proof. Let us consider an eigenvalue λ of the iteration matrix T, then we have:

$$|\lambda I - T| = \left\| \begin{bmatrix} \lambda I_m & A^{-1}B^T - (\alpha I + A)^{-1} B^T Q^{-1} B A^{-1} B^T \\ 0 & (\lambda - 1) I_n + \left(\frac{1}{\alpha} I + Q^{-1} \right) B A^{-1} B^T \end{bmatrix} \right\| =$$

$$= \lambda^m \left| (\lambda - 1) I_n + \left(\frac{1}{\alpha} I + Q^{-1} \right) B A^{-1} B^T \right| = 0 \tag{9}$$

By (9), we have that $\lambda = 0$ is eigenvalue of T with multiply m, and the other n eigenvalues of T satisfy (8). Then we can obtain the spectral radius of the iteration matrix T, which completes the proof.

Theorem 2. Let:

$$\theta_{\max} = \max \frac{u^T B A^{-1} B^T u}{u^T u},\ \theta_{\min} = \min \frac{u^T B A^{-1} B^T u}{u^T u},\ \hat{\theta}_{\max} = \max \frac{u^T Q^{-1} B A^{-1} B^T u}{u^T u}$$
$$\text{and } \hat{\theta}_{\min} = \min \frac{u^T Q^{-1} B A^{-1} B^T u}{u^T u}$$

with $u \neq 0$, then ULT-HSS method converges if and only if:

$$\frac{1}{\alpha} \theta_{\max} + \hat{\theta}_{\max} < 2 \tag{10}$$

Furthermore, the optimal parameter α^* for $\rho(T)$ is given by:

$$\alpha^* = \frac{\theta_{\max} + \theta_{\min}}{2 - \hat{\theta}_{\max} - \hat{\theta}_{\min}} \tag{11}$$

and the corresponding spectral radius is:

$$\rho^*(T) = \frac{\theta_{\max} - \theta_{\min}}{\theta_{\max} + \theta_{\min}} \tag{12}$$

Proof. The ULT-HSS method converges if and only if $\rho(T) < 1$, where T is the iteration matrix defined by (7). Obviously, $\rho(T) < 1$ is equivalent to $|\lambda| < 1$, where λ represents the eigenvalue of T. By (8), it follows that $\lambda = 1 - (1/\alpha)\theta - \hat{\theta}$.To guarantee the convergence of ULT-HSS method, it requires that $-1 < 1 - (1/\alpha)\theta - \hat{\theta} < 1$. We need to prove that:

$$0 < \frac{1}{\alpha} \theta + \hat{\theta} < 2$$

Since $BA^{-1}B^T$ and Q are symmetric positive definite matrices:

$$\frac{1}{\alpha} \theta + \hat{\theta} > 0$$

holds. Besides, $(1/\alpha)\theta + \hat{\theta} < 2$ is satisfied if $(1/\alpha)\theta_{\max} + \hat{\theta}_{\max} < 2.$ Thus, ULT-HSS converges if and only if (10) is satisfied.

Theoretically, the spectral radius $\rho(T) < 1$ achieves its minimum when:

$$1-\frac{1}{\alpha}\theta_{\max}-\hat{\theta}_{\max}=\frac{1}{\alpha}\theta_{\min}+\hat{\theta}_{\min}-1$$

It is easy to obtain the optimal parameter (11) and the corresponding spectral radius (12), which completes the proof.

Since the convergence of ULT-HSS depends on the preconditioned matrix Q, it requires careful analysis on choosing Q and α*. We further consider the optimality of α when $Q = \alpha I$. In this special case, the iteration matrix T reads as:

$$T=\begin{bmatrix} 0 & -A^{-1}B^T+\frac{1}{\alpha}(\alpha I+A)^{-1}B^T BA^{-1}B^T \\ 0 & I-\frac{2}{\alpha}BA^{-1}B^T \end{bmatrix}$$

By *Theorems 1* and *2*, we have the convergence results for ULT-HSS method with $Q = \alpha I$.

Corollary 1. Assume that $A \in R^{n\times n}$ is symmetric positive definite, $B \in R^{m\times n}$ is full of row rank, and let $\theta_{\max}$ and $\theta_{\min}$ be the largest and smallest eigenvalues of $BA^{-1}B^T$, respectively. Then the ULT-HSS method with $Q = \alpha I(\alpha > 0)$ converges if and only if:

$$\alpha > \theta_{\max} \tag{13}$$

Besides, the spectral radius is given by:

$$\rho(T)=\max\left\{\left|1-\frac{2}{\alpha}\theta_{\max}\right|,\quad \left|1-\frac{2}{\alpha}\theta_{\min}\right|\right\} \tag{14}$$

The optimal α is defined by:

$$\alpha^*=\theta_{\max}+\theta_{\min}$$

and the corresponding spectral radius is:

$$\rho^*(T)=\frac{\theta_{\max}-\theta_{\min}}{\theta_{\max}+\theta_{\min}}$$

Numerical experiments

We test a simulated saddle point problem to show the efficiency of the ULT-HSS method. For comparison, we also test ULT method [22] and preconditioned GMRES (PGMRES) with HSS preconditioner [5, 20]. All the numerical computations are performed by MATLAB software on PC with an Intel Core 2 Duo CPU, 2.3 GHz, and 8 GB RAM.

Consider a saddle point problem (1) with the following submatrices:

$$A=\begin{bmatrix} 6I_m-T_m & -I_m \\ -I_m & 6I_m-T_m \end{bmatrix},\quad B=\begin{bmatrix} 4I_m-T_m & O \end{bmatrix}$$

where $T_m = tridiag(1,0,1) \in R^{m\times m}$. We choose the vectors f and g such that the exact solution of (1) is $(x^T, y^T)^T = (1,1,\cdots,1)^T \in R^{3m}$. Both the initial vectors x_0 and y_0 are set to be zero in this example. The iteration schemes are terminated if the current iteration satisfies:

$$\frac{\|r_i\|_2}{\|r_0\|_2} < 10^{-14}$$

where

$$r_i = \begin{bmatrix} f - Ax_i - B^T y_i \\ g - Bx_i \end{bmatrix}$$

is the residual vector of system (1) in the i^{th} iteration.

We let $Q = \alpha I$ for ULT-HSS which can be seen as a preconditioner for the Schur complement $S = BA^{-1}B^T$. By *Corollary 1*, the optimal parameters for ULT-HSS iteration are defined by:

$$\alpha^* = \theta_{\max} + \theta_{\min}$$

where $\theta_{\max}$ and $\theta_{\min}$ are the smallest and the largest non-zero eigenvalues of the Schur complement S, respectively. We perform the ULT method with the optimal:

$$\tau^* = \frac{1}{\theta_{\max} + \theta_{\min}}$$

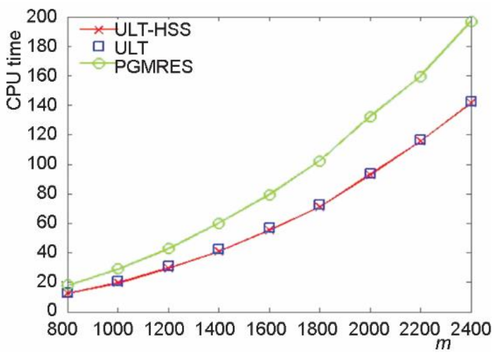

Figure 1. The CPU time of the tested algorithms

As pointed out in [20], the original HSS method converges slowly. Thus, we test preconditioned GMRES with HSS preconditioner $(H + \alpha I)(S + \alpha I)$. In this example, the parameters of ULT-HSS, ULT, and PGMRES for the considered grids are set as 5.6381, 0.1774 and 1.0508, respectively.

Table 1 shows the number of iterations, CPU time, and the relative error for ULT-HSS, ULT and PGMRES. The ULT-HSS method outperforms the other two methods. The ULT-HSS requires the same numbers of iterations as ULT to satisfy the terminated condition. Due to the optimality of α, the total CPU time for ULT-HSS is slightly less than that of ULT. Although the total iteration numbers of PGMRES are less than those of ULT or ULT-HSS, the average computational cost for each iteration is larger than those of the rest two methods. Figure 1 plots the distribution of CPU time for tested algorithms with different grids. We see that the CPU time of these three algorithms increases linearly.

Table 1. Numerical results of tested algorithms

Algorithm	m = 800			m = 1600			m = 2400		
	Outer	CPU	Error	Outer	CPU	Error	Outer	CPU	Error
ULT-HSS	65	12.96	$7.59 \cdot 10^{-15}$	65	54.73	$7.63 \cdot 10^{-15}$	65	141.62	$7.65 \cdot 10^{-15}$
ULT	65	13.52	$6.62 \cdot 10^{-15}$	65	56.52	$6.45 \cdot 10^{-15}$	65	142.33	$6.47 \cdot 10^{-15}$
PGMRES	18	18.69	$6.12 \cdot 10^{-15}$	18	78.15	$6.25 \cdot 10^{-15}$	18	196.74	$6.24 \cdot 10^{-15}$

Conclusions

In this paper, a hybrid ULT-HSS method was proposed to solve the symmetric saddle point problems. Comparing with the convergence or numerical results in the literature, we had the following two improvements:

- Convergence conditions for the ULT-HSS method were given. The optimality of relaxed parameters was also considered for a special case of the ULT-HSS method.
- Numerical results shown the advantages of the ULT-HSS method over the original ULT method.

We can conclude that the ULT-HSS method is an efficient method for solving the symmetric saddle point problems. Discretization of Navier-Stokes or Oseen equations will result in a generalized saddle point problem. We will apply this method for generalized saddle point problems in our future work.

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