APPROXIMATE ANALYTICAL SOLUTION FOR PHI-FOUR EQUATION WITH HE'S FRACTAL DERIVATIVE

by

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This paper, for the first time ever, proposes a Laplace-like integral transform, which is called as He-Laplace transform, its basic properties are elucidated. The homotopy perturbation method coupled with this new transform becomes much effective in solving fractal differential equations. Phi-four equation with He's derivative is used as an example to reveal the main merits of the present technology.

Key word: *Phi-four equation, He's fractal derivative, He-Laplace transform, homotopy perturbation method*

Introduction

In this paper, we investigate the following Phi-four equation with He's derivative in a fractal medium:

$$\frac{\partial}{\partial t^{\alpha}}\left[\frac{\partial u(x,t)}{\partial t^{\alpha}}\right]-\frac{\partial}{\partial x^{\beta}}\left[\frac{\partial u(x,t)}{\partial x^{\beta}}\right]+m^2u+\lambda u^3=0, \quad 0<\alpha, \quad \beta\le 1 \tag{1}$$

subject to the initial condition:

$$u(x,0)=\varphi(x) \tag{2}$$

$$\frac{\partial}{\partial^{\alpha}x}u(x,0)=\psi(x) \tag{3}$$

where m and λ are constants, $\varphi(x)$ and $\psi(x)$ are given functions.

In eq. (1), the symbols $\partial u/\partial t^{\alpha}$ and $\partial u/\partial x^{\beta}$ denote the He's fractal derivatives respect to time and space, respectively [1]:

$$\frac{\partial u(x,t_1)}{\partial t^{\alpha}}=\Gamma(1+\alpha)\lim_{\substack{t_1-t\to\Delta t\\ \Delta t\ne 0}}\frac{u(x,t_1)-u(x,t)}{(t_1-t)^{\alpha}} \tag{4}$$

$$\frac{\partial u(x_1,t)}{\partial x^{\beta}}=\Gamma(1+\beta)\lim_{\substack{x_1-x\to\Delta x\\ \Delta x\ne 0}}\frac{u(x_1,t)-u(x,t)}{(x_1-x)^{\beta}} \tag{5}$$

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The differential model with He's fractal derivative has recently proved to be a suitable tool to modeling of many extremely complex phenomena. A number of physical problems in fractal media lead to non-linear models involving He's fractal derivatives [2-14]. Equation (1) has several applications in nuclear and thermal physics and many researchers investigated it in the last decade [15-17].

The problem given in eqs. (1)-(3) is often difficult or impossible to be solved analytically. Usually, scientists use the two-scale transform to solve the differential equation with He's fractal derivative [1]. In the present work, to solve the problem (1)-(3), we first introduce the following Laplace-type integral transform:

$$L_\alpha[f(v)] = \int_0^\infty \exp(-sv^\alpha) f(v)\mathrm{d}v^\alpha \tag{6}$$

We call eq. (6) as He-Laplace transform named after Chinese mathematician Ji-Huan He and French mathematician Pierre-Simon Laplace (1749-1827).

Base on the two-scale transform [18-22], we have:

$$L_\alpha\left[\frac{\mathrm{d}f(v)}{\mathrm{d}v^\alpha}\right] = sL_\alpha(f) - f(0) \tag{7}$$

$$L_\alpha\left[\frac{\mathrm{d}f(v)}{\mathrm{d}v^{2\alpha}}\right] = s^2L_\alpha(f) - sf(0) - \frac{\partial}{\partial v^\alpha}f(0) \tag{8}$$

It is found that He-Laplace transform is very much similar to the traditional Laplace transform, the latter is famous for linear equations, while the former is valid for fractal calculus. This paper will show it is extremely effective for fractal calculus when it is coupled with the homotopy perturbation method (HPM).

Homotopy perturbation method

The principals of the HPM are given in [23-27]. For convenience of the audience of *Thermal Science*, in this section, we recall its basic idea. Consider the non-linear differential equation:

$$L(u) + N(u) = f(r), \ \ r \in \Omega \tag{9}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \tag{10}$$

where L is a linear operator and N – a non-linear operator, B – a boundary operator, Γ – the boundary of the domain Ω, and $f(r)$ – a known analytic function.

The homotopy perturbation technique defines the homotopy:

$$v(r, p): \Omega \times [0,1] \to R$$

which satisfies:

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0 \tag{11}$$

or

$$H(v,p)=L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]=0 \tag{12}$$

where $r\in\Omega$ and $p\in[0,1]$ is an impeding parameter, u_0 is an initial approximation which satisfies the boundary conditions. Obviously, from eqs. (11) and (12), we have:

$$H(v,0)=L(v)-L(u_0)=0 \tag{13}$$

$$H(v,1)=L(v)+N(v)-f(r)=0 \tag{14}$$

The changing process of p from zero to unity is just that of $v(r, p)$ from u_0 to $u(r)$. The basic assumption is that the solution of eq. (11) can be expressed as a power series in p:

$$v=v_0+pv_1+p^2v_2+\cdots \tag{15}$$

The approximate solution of eq. (11), therefore, can be obtained:

$$u=\lim_{p\to 1}v=v_0+v_1+v_2+\cdots \tag{16}$$

Solution process with He-Laplace transform

In this section, by using the He-Laplace transform and the HPM, we derive the main algorithms for solving the problem (1)-(3). By the chain rule, we have:

$$\frac{\partial u}{\partial t^{2\alpha}}=\frac{\partial}{\partial t^{\alpha}}\left(\frac{\partial u}{\partial t^{\alpha}}\right),\quad \frac{\partial u}{\partial x^{2\beta}}=\frac{\partial}{\partial x^{\beta}}\left(\frac{\partial u}{\partial x^{\beta}}\right)$$

Consider the following Phi-four equation with He's derivative in a fractal medium:

$$\frac{\partial u(x,t)}{\partial t^{2\alpha}}-\frac{\partial u(x,t)}{\partial x^{2\beta}}+m^2u+\lambda u^3=0,\quad 0<\alpha,\quad \beta\le 1 \tag{17}$$

subject to the initial condition:

$$u(x,0)=\varphi(x) \tag{18}$$

$$\frac{\partial}{\partial^{\alpha}t}u(x,0)=\psi(x) \tag{19}$$

We rewrite the eq. (17) as:

$$\frac{\partial u(x,t)}{\partial t^{2\alpha}}=\frac{\partial u(x,t)}{\partial x^{2\beta}}-m^2u-\lambda u^3 \tag{20}$$

Taking He-Laplace transform on eq. (20), we obtain:

$$L_{\alpha}\left(\frac{\partial u}{\partial t^{2\alpha}}\right)=L_{\alpha}\left(\frac{\partial u}{\partial x^{2\beta}}-m^2u-\lambda u^3\right) \tag{21}$$

By eq. (8), we have:

$$s^2L_{\alpha}[u(x,t)]-s\varphi(x)-\psi(x)=L_{\alpha}\left(\frac{\partial u}{\partial x^{2\beta}}-m^2u-\lambda u^3\right)$$

Namely:

$$L_\alpha[u(x,t)] = \frac{1}{s}\varphi(x) + \frac{1}{s^2}\psi(x) + \frac{1}{s^2}L_\alpha\left(\frac{\partial u}{\partial x^{2\beta}} - m^2 u - \lambda u^3\right) \tag{22}$$

For eq. (22), we can establish the following homotopy:

$$L_\alpha[v(x,t)] = \frac{1}{s}\varphi(x) + \frac{1}{s^2}\psi(x) + \frac{1}{s^2}pL_\alpha\left(\frac{\partial v}{\partial x^{2\beta}} - m^2 v - \lambda v^3\right) \tag{23}$$

where p is the homotopy parameter.

We apply the HPM to eq. (23), according to which u can be expanded into power series in p:

$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t) \tag{24}$$

and the non-linear term $N(v) = -m^2 v - \lambda v^3$ can be decomposed:

$$N(v) = \sum_{n=0}^{\infty} p^n H_n(v) \tag{25}$$

where $p \in [0,1]$ is an embedding parameter and $H_n(v)$ – the He's polynomials [28] defined:

$$H_n(v_0, v_1, \cdots, v_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i v_i\right)\right]_{p=0}, \quad n = 0,1,2\cdots \tag{26}$$

Here the first few terms of He's polynomials are given as:

$$H_0 = -m^2 v - \lambda v^3$$

$$H_1 = -m^2 v_1 - 3\lambda v_0^2 v_1$$

$$H_2 = -m^2 v_2 - 3\lambda v_0^2 v_2 - 3\lambda v_0 v_1^2$$

Substituting eqs. (24) and (25) into eq. (23), we have:

$$L_\alpha\left(\sum_{k=0}^{\infty} v_k\right) = \frac{1}{s}\varphi(x) + \frac{1}{s^2}\psi(x) + p\frac{1}{s^2}\left[\frac{\partial}{\partial x^{2\beta}}\left(\sum_{k=0}^{\infty} v_k p^k\right) + \sum_{k=0}^{\infty} H_k\right]$$

Equating the terms with identical powers in p, we get:

$$L_\alpha[v_0(x,t)] = \frac{1}{s}\varphi(x) + \frac{1}{s^2}\psi(x)$$

$$L_\alpha[v_k(x,t)] = \frac{1}{s^2}L_\alpha\left(\frac{\partial v_{k-1}}{\partial x^{2\beta}} + H_{k-1}\right), \quad k = 1,2,3\cdots$$

Operating with the He-Laplace inverse transform, we get:

$$v_0(x,t) = \varphi(x) + \psi(x)\frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$v_k(x,t) = L_\alpha^{-1}\left[\frac{1}{s^2}L_\alpha\left(\frac{\partial v_{k-1}}{\partial x^{2\beta}} + H_{k-1}\right)\right], \quad k = 1,2,3\cdots$$

Finally, the approximate solution can be obtained:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots$$

In order to show the effectiveness of the proposed algorithm, we consider the problem (1)-(3) in the form:

$$\frac{\partial u(x,t)}{\partial t^{2\alpha}} - \frac{\partial u(x,t)}{\partial x^{2\beta}} + u - u^3 = 0 \tag{27}$$

subject to the initial condition:

$$u(x,0) = \sqrt{2} \sec h(2x) \tag{28}$$

$$\frac{\partial}{\partial^\alpha x} u(x,0) = -\sqrt{6}\,\text{sech}(2x) th(2x) \tag{29}$$

Here $m = 1$ and $\lambda = -1$. Let:

$$E = \exp\left[\frac{x^\beta}{\Gamma(1+\beta)}\right], \quad F = E^4 + 1$$

By the previous algorithms, we obtain:

$$v_0(x,t) = \frac{2\sqrt{2}E^2}{F} - \frac{2\sqrt{6}E^2(E^4-1)}{F^2} \frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$v_1(x,t) = \frac{6\sqrt{2}E^2(E^8 - 6E^4 + 1)}{F^3} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$v_2(x,t) = \frac{6\sqrt{6}E^2(E^4-1)(22E^4 - E^8 - 1)}{F^4} \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$$

and so on.

Thus, the $3 - term$ approximate solution of (27) is given by:

$$u(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t)$$

When $\alpha = 1, \beta = 1,$ the reader can easily verify that our solution is an exact solution of the problem (1)-(3).

Discussion and conclusion

This work introduce a Laplace-type integral transform called as the He-Laplace transform. By using the transform and the HPM, we obtain the approximate analytical solution of the Phi-four equation with He's derivative in fractal media. Our example shows that the proposed method is a simple and reliable algorithm.

As the Laplace transform is widely coupled with the HPM or the variational iteration method to solve various non-linear equations including fractional differential equations [29-34], and the He-Laplace transform sheds a bright light on fractal calculus.

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