BAYESIAN INFERENCE FOR SOLVING A CLASS OF HEAT CONDUCTION PROBLEMS

by

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This paper considers a heat conduction problem of a common continuum-type stochastic mathematical model in an engineering field. The approximate solution is calculated with the Markov chain Monte-Carlo algorithm for the heat conduction problem. Three examples are given to illustrate the solution process of the method.

Key words: *Bayesian inference, heat conduction problem, Markov chain Monte-Carlo*

Introduction

Complex thermal systems are usually disturbed by random factors and need to be modeled by random differential equations. Scholars have grown interests in numerical methods for stochastic differential equations. Milstein [1] used Taylor's expansion to get a numerical solution of a stochastic differential equation. Higham [2] studied the mean-square stability of the Euler-Maruyama method. Higham *et al.* [3] proved the convergence of the Euler-Maruyama method under the condition of non-total Lipschitz. Hutzenthaler *et al.* [4] proposed an explicit numerical method for solving stochastic differential equation with non-global Lipschitz continuous coefficients and proved its strong convergence. Wang and Gan [5] suggested an explicit strongly convergent numerical scheme for stochastic differential equations with commutative noise.

Various numerical methods have been developed to estimate the parameters of stochastic differential equations, for example, the Gibbs algorithm proposed by German and German [6], and the resampling algorithm given by Gordon *et al.* [7]. Eraker [8] used the Bayesian method to discuss estimation of the parameter in model with a stochastic volatility component. Golightly and Wilknson [9] discussed the parameter estimation of the nonlinear multivariate diffusion models based on the missing data. Miguez, *et al.* [10] discussed the sequential Monte-Carlo method in general state-space models.

Many studies have been conducted in the field of heat conduction. The interdisciplinary study has also developed rapidly and has penetrated to many disciplines. Various methods for solving the thermal conductivity of materials have been proposed, such as the spirit sensitivity method, the least squares method, the regularization method, and the conjugate

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gradient method. Martin-Fernandez and Lanzarone used the Monte-Carlo method to solve the heat conduction problem [11]. Tifkitsis and Skordos [12] developed a modified scheme based on the Markov chain Monte-Carlo (MCMC) for the estimation of unknown stochastic input parameters, such as the heat transfer coefficient. Zeng *et al.* [13] extended the approximate Bayesian computation method to the inverse heat conduction problem and developed two heat conduction solvers.

Recently, many researchers performed excellent studies in the field of the MCMC, and some new expended approaches have been widely used in various engineering fields. Rich information on different aspects of biological mechanisms is encoded by genes, and Ko *et al.* [14] proposed an MCMC method to extract new biological information from the data. Hemantha [15] introduced an MCMC simulation method to engineering economics research. Yousaf *et al.* [16] considered transmuted distribution and compared priors under the squared error loss function. Begona, *et al.* [17] developed a new technique to estimate the SIR model parameters using the MCMC method.

Motivated by the previous ideas, our scheme utilizes MCMC to solve a class of heat conduction problems.

Differential equation of heat conduction is listed:

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}\right)+q_v=\rho c_p\frac{\partial T}{\partial t} \tag{1}$$

where T is the transient temperature, t – the time of the process, k_x, k_y, k_z are thermal coefficients, ρ – the density of materials, c_p – the specific heat capacity at constant pressure, and q_v – the internal heat source strength.

We define the difference operators as:

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \tag{2}$$

Assumption 1: When the material of an object is isotropic, it implies that:

$$k_x=k_y=k_z=k \tag{3}$$

Assumption 2: As there is no heat source in the object, the implication is that:

$$\frac{q_v}{k}=0 \tag{4}$$

Assumption 3: Let $\alpha_T=k(\rho c_p)^{-1}$, the object is in the steady-state temperature field and it implies that:

$$\frac{1}{\alpha_T}\frac{\partial T}{\partial t}=0 \tag{5}$$

Based on eqs. (2) and (3), eq. (1) can be transformed:

$$\nabla^2 T+\frac{q_v}{k}=\frac{1}{\alpha_T}\frac{\partial T}{\partial t} \tag{6}$$

Based on eq. (4), eq. (6) can be transformed:

$$\nabla^2 T=\frac{1}{\alpha_T}\frac{\partial T}{\partial t} \tag{7}$$

Based on eq. (5), eq. (6) can be transformed:

$$\nabla^2 T + \frac{q_v}{k} = 0 \tag{8}$$

with an initial condition $T(x,0) = \rho(x)$, and some boundary conditions $T(0,t) = \varphi(t)$ and $T(x_0,t) = \psi(t)$, where $\varphi(t), \psi(t), x_0$ are known.

Let an object of length, d, be initialized at the uniform temperature x_1 [°C]. Suppose at x= 0 is heated to x_2 [°C] and at $x = d$ is heated to x_3 [°C]. This problem can be modeled by eq. (1):

$$\begin{aligned} a(t)\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, \quad 0 < x < d, \quad t > 0 \\ T(x,0) = x_1, \qquad 0 < x < d \\ T(0,t) = x_2 \\ T(d,t) = x_3 \end{aligned} \tag{9}$$

Bayesian inference for solving a classical stochastic diffusion problem

Given a diffusion of physical Langevin equation:

$$\mathrm{d}Y_t = b(Y_t,\theta)\mathrm{d}t + \sigma(Y_t,\theta)\mathrm{d}W_t \tag{10}$$

The Y_t is the solution of eq. (10). The transition density:

$$p_t(y\,|\,x) = \frac{\mathrm{d}}{\mathrm{d}y}P(Y_t \le y\,|\,Y_0 = x)$$

further

$$\frac{\partial}{\partial t}p_t(y\,|\,x) = b(y)\frac{\partial}{\partial x}p_t(y\,|\,x) + \frac{1}{2}\sigma^2(y)\frac{\partial^2}{\partial x^2}p_t(y\,|\,x)$$

can be obtained by Fokker-Planck equation. The time interval [0, T] is subdivided equidistant points:

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T, \quad \Delta t = \frac{T}{N}$$

In practice, it is necessary to give the Euler approximation:

$$Y_{t_{i+1}} - Y_{t_i} = b(Y_{t_i},\theta)\Delta t + \sigma(Y_{t_i},\theta)\Delta W_t \tag{11}$$

where Y_{t_i} is the observed at time t_i and $\Delta W_t = W_{t+1} - W_t \sim N(0,\Delta t)$.

From eq. (11), the transition probabilities of $Y_{t_i} \to Y_{t_{i+1}}$ from time t_i to t_{i+1} are:

$$p(Y_{t_{i+1}}\big|Y_{t_i},\theta) \approx N[Y_{t_i} + b(Y_{t_i},\theta)\Delta t, \sigma(Y_{t_i},\theta)\Delta W_t]$$

the posterior distribution of θ is given by:

$$\pi(\theta\big|Y_{t_1},Y_{t_2},\cdots Y_{t_p}) \propto \pi(\theta)\prod_{i=1}^{p} p(Y_{t_i}\big|Y_{t_{i-1}},\theta)$$

where $\pi(\theta)$ is a prior distribution of θ.

Markov chain Monte-Carlo for solving a classical of heat conduction problem

Many heat conduction problems of complex boundary are unable to solve with analytical methods. Instead, related numerical methods have developed rapidly [18-25]. Let's consider:

$$\begin{gathered}\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0, \quad 0\le x\le A, \quad 0\le y\le B \\ T(0,y)=\mathrm{e}^y-\cos y, \quad T(x,0)=\cos x-\mathrm{e}^x \\ T(A,y)=\mathrm{e}^y\cos A-\mathrm{e}^A\cos y \\ T(x,B)=\mathrm{e}^B\cos x-\mathrm{e}^x\cos B\end{gathered} \tag{12}$$

The domain $[0,A]\times[0,B]$ is divided into an $M\times N$ mesh with the step size $h=AM^{-1}$ in the X-direction and the step size $\tau=BN^{-1}$ in the Y-direction. By Taylor's expansion:

$$T_{i+1,j}=T_{i,j}+\frac{\partial T}{\partial x}\bigg|_{i,j}\Delta x+\frac{1}{2}\frac{\partial^2 T}{\partial x^2}\bigg|_{i,j}(\Delta x)^2+\frac{1}{6}\frac{\partial^3 T}{\partial x^3}\bigg|_{i,j}(\Delta x)^3+\cdots$$

$$T_{i-1,j}=T_{i,j}-\frac{\partial T}{\partial x}\bigg|_{i,j}\Delta x+\frac{1}{2}\frac{\partial^2 T}{\partial x^2}\bigg|_{i,j}(\Delta x)^2-\frac{1}{6}\frac{\partial^3 T}{\partial x^3}\bigg|_{i,j}(\Delta x)^3+\cdots$$

Hence:
$$T_{i-1,j}+T_{i+1,j}=2T_{i,j}+\frac{\partial^2 T}{\partial x^2}(\Delta x)^2+\cdots$$

we have:

$$\frac{\partial^2 T}{\partial x^2}\bigg|_{i,j}=\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{(\Delta x)^2}+o(\Delta x) \tag{13}$$

Similarly:

$$\frac{\partial^2 T}{\partial y^2}\bigg|_{i,j}=\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{(\Delta y)^2}+o(\Delta y) \tag{14}$$

Let us consider eq. (12) at point (i, j):

$$\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{(\Delta x)^2}+\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{(\Delta y)^2}=0 \tag{15}$$

Equation (15) can be transformed:

$$T_{i,j}=\frac{1}{2+2\alpha}T_{i+1,j}+\frac{1}{2+2\alpha}T_{i-1,j}+\frac{\alpha}{2+2\alpha}T_{i,j+1}+\frac{\alpha}{2+2\alpha}T_{i,j-1}=$$

$$=p(T_{i+1,j}|T_{i,j})T_{i+1,j}+p(T_{i-1,j}|T_{i,j})T_{i-1,j}+p(T_{i,j+1}|T_{i,j})T_{i,j+1}+p(T_{i,j-1}|T_{i,j})T_{i,j-1} \tag{16}$$

where

$$\alpha=(\Delta x)^2(\Delta y)^{-2}, \quad p(T_{i+1,j}|T_{i,j})=(2+2\alpha)^{-1}, \quad p(T_{i+1,j}|T_{i,j})=p(T_{i-1,j}|T_{i,j})=(2+2\alpha)^{-1}$$

$$p(T_{i,j+1}|T_{i,j})=p(T_{i,j-1}|T_{i,j})=\alpha(2+2\alpha)^{-1}$$

Consider a Markov chain:

$$T_{i,j} \to T_{i+1,j} \to T_{i+2,j} \to \cdots;\; T_{i,j} \to T_{i,j+1} \to T_{i,j+2} \to \cdots$$

Let $p(T_{0,j}) = p_j$, and we get and transition matrix $P = (p_{ij})$.

Based on eqs. (13)-(16), a MCMC algorithm is implemented to estimate transition probability.

Numerical results

In this section, numerical tests for the proposed methods are demonstrated.

Example 1. Consider the problem [23]:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad 0 \le x \le A, \quad 0 \le y \le B$$

with boundary conditions $T(0, y) = e^y - \cos y$, $T(A, y) = e^y - \cos A - e^A \cos y$, $T(x, 0) = \cos x - e^x$, and $T(x, B) = e^B \cos x - e^x \cos B$. Results obtained for $T(x, y)$ are presented under $A = 4, B = 5$ in tab. 1.

Table 1. Results with *A* = 4, *B* = 5 for *Example 1*

Point (x, y)	Average number of iterations	Numerical result of $T(x, y)$	Standard deviation	Confidence interval at $\alpha = 0.05$
(0.5, 0.5)	7.476	–0.0678	0.1880	(–0.3012, 0.1656)
(0.5, 1.0)	8.490	1.7058	0.1572	(1.1572, 1.9010)
(1.0, 0.5)	8.618	–1.4581	0.2049	(–1.7125, –1.2037)
(1.0, 1.0)	8.240	0.0129	0.2123	(–0.2506, 0.2765)
(1.5, 1.5)	8.507	–0.0545	0.2176	(–0.3246, 0.2165)

Example 2. Consider the problem [23]:

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2}, \quad x \in [0,1]$$

with an initial condition $T(x, 0) = \sin \pi x$, and two boundary conditions $T(0,t) = T(1,t) = 0$ °C. When a is set to 1, the exact solution of the previous equation is $T(x,t) = \exp(-\pi^2 t) \sin \pi x$. The obtained results are presented in tabs. 2 and 3 and fig. 1.

Table 2. Numerical results and exact solution with *t* = 0.0005, *N* = 50 for *Example 2*

Point (x)	Exact solution	Numerical solution	Numerical result at [20]
0.1	0.3105	0.3085	0.2930
0.2	0.5907	0.5868	0.5621
0.3	0.8130	0.8079	0.7799
0.4	0.9558	0.9595	0.9280
0.5	1.0049	0.9986	0.9931

Table 3. Numerical results and exact solution with t = 0.0005, N = 100 for *Example 2*

Point (x)	Value of accurate	Numerical result	Numerical result [20]
0.1	0.3105	0.3086	0.2941
0.2	0.5907	0.5870	0.5627
0.3	0.8130	0.8078	0.7798
0.4	0.9558	0.9496	0.9278
0.5	1.0049	0.9985	0.9930

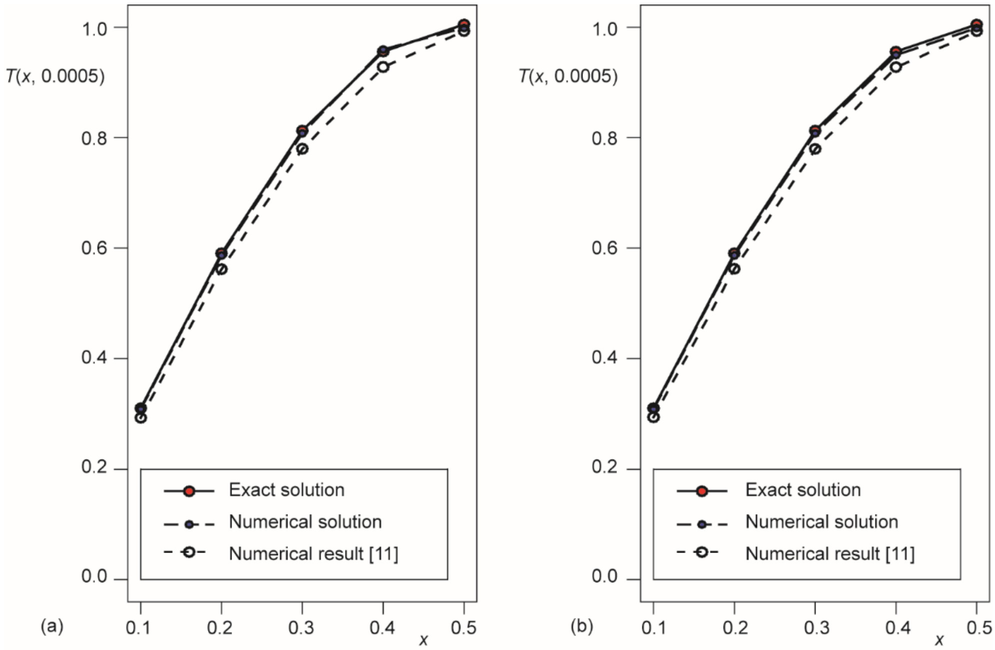

Figure 1. Numerical results and exact solution $T(x, y)$ with t = 0.0005 for *Example 2*; (a) N = 50 and (b) N = 100

Example 3. Consider the problem [26]:

$$\mathrm{d}Y_t = (\theta_1 + \theta_2 Y_t)\mathrm{d}t + \theta_3 \sqrt{Y_t}\mathrm{d}W_t, \quad Y_0 = 10$$

here $\theta_3 = 1$ is known. The explicit estimators for θ_1, θ_2 are presented in [23]:

$$\widehat{\theta}_1 = \frac{\left(\sum_{i=1}^{n} Y_{i-1}\right)^2}{2\left[n\sum_{i=1}^{n} Y_{i-1}^{\ 2} - \left(\sum_{i=1}^{n} Y_{i-1}\right)^2\right]}, \quad \text{and} \quad \widehat{\theta}_2 = \frac{-n\sum_{i=1}^{n} Y_{i-1}}{2\left[n\sum_{i=1}^{n} Y_{i-1}^{\ 2} - \left(\sum_{i=1}^{n} Y_{i-1}\right)^2\right]}$$

Some results for comparison are shown in tab. 4.

Table 4. Results with (θ_1, θ_2) obtained for estimators

N = 500		N = 1000	
Value of accurate	numerical result	Value of accurate	Numerical result
(7, –2)	(7.1534, –1.9652)	(7, –2)	(6.9512, –1.9458)
(8, –2)	(8.7679, –2.1174)	(8, –2)	(8.2297, –2.0267)
(7, –1.9)	(7.4036, –1.9326)	(7, –1.9)	(7.0544, –1.8760)

Conclusion

This paper proposed different methods based on the Bayesian approach to solve a heat conduction problem. The differential equation of the heat conduction is discretized by Taylor's expansion method [27, 28], and MCMC method is applied to estimate the transition probability. Three listed numerical examples validated the suitability and efficiency of the proposed numerical method in solving the heat conduction problem.

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