THE MODULATION OF RESPONSE CAUSED BY THE FRACTIONAL DERIVATIVE IN THE DUFFING SYSTEM UNDER SUPER-HARMONIC RESONANCE

by

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The dynamic characteristics of the 3:1 super-harmonic resonance response of the Duffing oscillator with the fractional derivative are studied. Firstly, the approximate solution of the amplitude-frequency response of the system is obtained by using the periodic characteristic of the response. Secondly, a set of critical parameters for the qualitative change of amplitude-frequency response of the system is derived according to the singularity theory and the two types of the responses are obtained. Finally, the components of the 1X and 3X frequencies of the system's time history are extracted by the spectrum analysis, and then the correctness of the theoretical analysis is verified by comparing them with the approximate solution. It is found that the amplitude-frequency responses of the system can be changed essentially by changing the order and coefficient of the fractional derivative. The method used in this paper can be used to design a fractional order controller for adjusting the amplitude-frequency response of the fractional dynamical system.

Key words: *fractional derivative, Duffing oscillator, super-harmonic resonance, critical parameters, spectrum analysis*

Introduction

Fractional calculus is a generalization of the integer order calculus, it extends the order of calculus from the traditional integer order to the case of fractional and complex orders, and has a history of more than 300 years. Compared with the traditional integer order calculus, the fractional calculus has more advantages in modelling complex phenomena. Fractional calculus has attracted wide attention of researchers in different fields and it has become the powerful mathematical tool to studying anomalous diffusion, non-Newton fluid mechanics, viscoelastic mechanics, soft matter physics, and so on. Because the fractional derivative can describe various reaction processes more accurately, many problems can be described better by fractional differential equations, so the studies of the typical mechanical characteristics of

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fractional differential equations and the influence of fractional order parameters on the system are very necessary and have important significances.

At present, most of the research on dynamic system is mainly focused on the analysis of vibration characteristics. Goddard *et al.* [1] discussed a quadrature method for generating bifurcation curves of positive solutions to some autonomous boundary value problems with non-linear boundary condition to some autonomous boundary value problems with non-linear boundary conditions, provided an algorithm for the numerical generation of bifurcation curves and showed its application to selected problems. Cao and Yuan [2] considered a class of neutral functional differential equations (NFDE), described the bifurcation behaviour of the parameterized NFDE by employing the method based on centre manifold reduction and normal form theory. Liebscher [3] investigated the breakdown of normal hyperbolicity of a manifold of equilibria of a flow, provided a description of general systems with a manifold of equilibria of codimension one as a first step towards a classification of bifurcations without parameters by relating the problem to singularity theory of maps. Yu *et al.* [4] investigated the fractional Langevin equation driven by multiplicative colored noise and modulated noise in the over-damped case. The numerical results indicated that the output amplitude presents stochastic resonance driven by periodically modulated noise. Sardar *et al.* [5] studied the analytical approximate solution of the fractionally damped van der Pol equation by the homotopy perturbation method and a numerical method. Li *et al.* [6] analysed the simplified Mathieu equation with fractional derivative term, which is derived by a viscoelastic simply supported beam under axial periodic excitation, and studied the effect of the system's parameters on the stability of solution. Chen and Zhu [7] studied the asymptotic stability of a Duffing oscillator with lightly fractional derivative damping under parametric excitations in the case of primary parametric resonance, and the asymptotic Lyapunov stability with probability one of the original system is determined approximately by using the largest Lyapunov exponent. Zhang *et al.* [8] analysed the response of a Duffing-Rayleigh system with fractional derivative under Gaussian white noise excitation by a stochastic averaging procedure which is developed by using the generalized harmonic functions. Yu and Wang [9] studied a linear HP TiO_2 Memristor model under the fractional order derivative and showed that the material characteristic determines the order of the fractional derivative, and the best memory of the Memristor can be achieved by seeking to the material that can be adapted to the frequency of excitation. Wang and Hu [10] studied the linear oscillator of the single DoF with fractional derivative damping, and found that the fractional derivative not only plays the role of damping force but also acts as an elastic force. Shen *et al.* [11-13] studied the resonance behaviour of Duffing oscillator with fractional order derivative and obtained the first-order approximate solution by using the averaging method, and analysed the influences of the orders and coefficients on the equivalent linear damping and stiffness. Niu *et al.* [14] studied the free vibration of Duffing oscillator with time-delayed fractional order PID controller based on displacement feedback and obtained the second-order approximate analytical solution by KBM asymptotic method. Chen *et al.* [15, 16] and Yang *et al.* [17] equated the fractional derivative term as the linear restoring force and damping force based on the harmonic balance method in order to transform the fractional oscillator systems into the integer order systems, and studied the properties of the related dynamic systems under Gaussian white noise excitation. Yang and Zhu [18] used the harmonic balance method to study the responses of a class of linear systems with fractional order derivative damping under different period signal excitations, and extended the range of the order of fractional derivative damping. Guo and Leung [19] proposed an improved harmonic balance method, and obtained the approximately analytical solution of the fractional van der Pol oscillator. Deng *et al.* [20] studied the mechanical properties of a class of thermo elastic fluid materials by using the law of thermodynamics and the law of conservation of energy, and established a corresponding mathematical model.

The methods of approximate analysis of fractional system mainly include the averaging method and the harmonic balance method and so on. Although the amplitude-frequency response equation of the system can be obtained by these methods, however, the singularity analysis of the amplitude-frequency response equation is difficult due to its complexity. Additionally, the study on the vibration characteristics of the parameters can only be qualitatively analysed, and the critical conditions of the parameters' influences cannot be found, which affects the analysis and design of such systems. In view of the aforementioned situation, the non-linear vibration of the typical Duffing equation is considered as an example, the transition set of the fractional order system are obtained numerically, thus the critical parameter conditions for the bifurcation of the system are obtained. Meanwhile, the types of amplitude-frequency response curves of the system in each region of the parameter plane divided by the transition set are analysed. By the method of spectrum analysis, the peak values at the corresponding frequencies are extracted from the spectrogram and are compared with the approximately analytical solutions obtained in this paper. Results show that the numerical solutions are in good agreement with the approximate solutions, thus the correctness of the theoretical analysis in this paper is verified.

Amplitude-frequency response equation of the system

There are many definitions of fractional derivatives, and the Riemann-Liouville derivative and Caputo derivative are commonly used. However, the initial conditions corresponding to the Riemann-Liouville derivative have no physical meanings, the initial conditions of the systems described by the Caputo derivative have clear physical meanings and their forms are the same as the initial conditions for the differential equations of integer order. So in this paper, the Caputo-type fractional derivative is adopted:

$$ {}_a^C\mathrm{D}^p[x(t)] = \frac{1}{\Gamma(m-p)} \int_a^t \frac{x^{(m)}(u)}{(t-u)^{1+p-m}} \mathrm{d}u \tag{1} $$

where $m-1 < p \le m,\ m \in N,\ t \in [a,b], \Gamma(m)$ is the Euler Gamma function, and $x^{(m)}(t) -$ the m order derivative of $x(t)$.

For a given physical system, due to the initial moment of the oscillator is $t = 0$, so the following form of the Caputo derivative is often used:

$$ {}_0^C\mathrm{D}^p[x(t)] = \frac{1}{\Gamma(m-p)} \int_0^t \frac{x^{(m)}(u)}{(t-u)^{1+p-m}} \mathrm{d}u \tag{2} $$

where $m-1 < p \le m,\ m \in N$.

In this paper, the Duffing oscillator system with fractional derivative under periodic signal excitation:

$$ m\ddot{x}(t) + kx(t) + c\dot{x} + \alpha_1 x^3(t) + K_1\, {}_0^C\mathrm{D}^p[x(t)] = F_1 \cos(wt) \tag{3} $$

is studied, where m, k, c, and α_1 represent the quality, the linear stiffness coefficient, the linear damping coefficient, and the cubic term's coefficient of the system, respectively. The F_1 and w are the amplitude and frequency of the periodic excitation, respectively. The ${}_0^C\mathrm{D}^p[x(t)]$ is the p-order Caputo derivative of $x(t)$, which is defined by eq. (2). The order of the fractional derivative is taken as $0 \le p \le 2$ in this paper, and K_1 is the coefficient of the fractional derivative term.

Under the definition of Caputo derivative, according to the formulas in [12], the results can be obtained:

$$\begin{aligned} {}_0^C\mathrm{D}^p[\cos(wt+\theta)] &= w^p\cos\left(wt+\theta+\frac{p\pi}{2}\right) \\ {}_0^C\mathrm{D}^p[\sin(wt+\theta)] &= w^p\sin\left(wt+\theta+\frac{p\pi}{2}\right) \end{aligned} \tag{4}$$

To investigate the bifurcation behaviours of the 3:1 super-harmonic resonance of system (3), it is assumed that the excitation frequency is close to 1/3 of the natural frequency:

$$9w^2 = {w_0}^2 + \varepsilon\sigma \tag{5}$$

Then, system (3) could be transformed into:

$$\ddot{x}(t) + 9w^2x(t) = \varepsilon\sigma x(t) - \frac{c}{m}\dot{x}(t) - \frac{\alpha_1}{m}x^3(t) - \frac{K_1}{m}\,{}_0^C\mathrm{D}^p[x(t)] + \frac{F_1}{m}\cos(wt) \tag{6}$$

The solution of system (6) is composed of the solution of free vibration and the solution of forced vibration, which can be set as:

$$x = a\cos\varphi + B_1\cos(wt) \tag{7}$$

where $\varphi = 3wt + \theta$, $B_1 = F_1/8mw^2$.

According to the calculation of the averaging method, there are:

$$\begin{aligned} \dot{a} &= -\frac{1}{3w}\left[\varepsilon\sigma x - \frac{c}{m}\dot{x} - \frac{\alpha_1}{m}x^3 - \frac{K_1}{m}\,{}_0^C\mathrm{D}^p x\right]\sin\varphi \\ a\dot{\theta} &= -\frac{1}{3w}\left[\varepsilon\sigma x - \frac{c}{m}\dot{x} - \frac{\alpha_1}{m}x^3 - \frac{K_1}{m}\,{}_0^C\mathrm{D}^p x\right]\cos\varphi \end{aligned} \tag{8}$$

From eq. (4), we obtain:

$$\begin{aligned} &{}_0^C\mathrm{D}^p[x(t)] = {}_0^C\mathrm{D}^p[a\cos(3wt+\theta) + B_1\cos(wt)] = \\ &= a(3w)^p\cos\left(3wt+\theta+\frac{p\pi}{2}\right) + B_1w^p\cos\left(wt+\frac{p\pi}{2}\right) \end{aligned} \tag{9}$$

Combined eq. (8) with eq. (9), it yields:

$$\begin{aligned} \dot{a} = -\frac{1}{3w}&\left\{\varepsilon\sigma\left[a\cos\varphi + B_1\cos\left(\frac{\varphi-\theta}{3}\right)\right]\right\} - \frac{c}{m}\left[-3aw\sin\varphi - B_1w\sin\left(\frac{\varphi-\theta}{3}\right)\right] - \\ &-\frac{\alpha_1}{m}\left[a\cos\varphi + B_1\cos\left(\frac{\varphi-\theta}{3}\right)\right]^3 - \frac{K_1}{m}\left[a(3w)^p\cos\left(\varphi+\frac{p\pi}{2}\right) + \right. \\ &\left. + B_1w^p\cos\left(\frac{\varphi-\theta}{3}+\frac{p\pi}{2}\right)\right]\sin\varphi \end{aligned}$$

$$a\dot{\theta}=-\frac{1}{3w}\left\{\varepsilon\sigma\left[a\cos\varphi+B_1\cos\left(\frac{\varphi-\theta}{3}\right)\right]\right\}-\frac{c}{m}\left[-3aw\sin\varphi-B_1w\sin\left(\frac{\varphi-\theta}{3}\right)\right]-$$
$$-\frac{\alpha_1}{m}\left[a\cos\varphi+B_1\cos\left(\frac{\varphi-\theta}{3}\right)\right]^3-\frac{K_1}{m}\left[a(3w)^p\cos\left(\varphi+\frac{p\pi}{2}\right)+\right.$$
$$\left.+B_1w^p\cos\left(\frac{\varphi-\theta}{3}+\frac{p\pi}{2}\right)\right]\cos\varphi \tag{10}$$

By the integral average of eq. (10), one could establish the standard equation as:

$$\begin{aligned}\dot{a}&=-\frac{a}{2m}C(p)+\frac{\alpha_1B_1^3}{24mw}\sin\theta\\ a\dot{\theta}&=-\frac{3aw}{2}+\frac{aK(p)}{6mw}+\frac{a\alpha_1(a^2+2B_1^2)}{8mw}+\frac{\alpha_1B_1^3}{24mw}\cos\theta\end{aligned} \tag{11}$$

where

$$\begin{aligned}C(p)&=c+K_1(3w)^{p-1}\sin\left(\frac{p\pi}{2}\right)\\ K(p)&=k+K_1(3w)^p\cos\left(\frac{p\pi}{2}\right)\end{aligned} \tag{12}$$

which are the equivalent linear damping and equivalent linear stiffness of system (3), respectively.

Letting $\dot{a}=0$ and $\dot{\theta}=0$, eq. (11) becomes:

$$\begin{aligned}12\bar{a}wC(p)&=\alpha_1B_1^3\sin\bar{\theta}\\ 36\bar{a}mw^2-4\bar{a}K(p)-3\bar{a}\alpha_1(\bar{a}^2+2B_1^2)&=\alpha_1B_1^3\cos\bar{\theta}\end{aligned} \tag{13}$$

where $\bar{a}$ and $\bar{\theta}$ are the steady-state amplitude and phase, respectively. Eliminating $\bar{\theta}$ from equation (13), one could obtain the amplitude-frequency equation:

$$\bar{a}^2\{[36mw^2-4K(p)-3\alpha_1(\bar{a}^2+2B_1^2)]^2+144w^2C^2(p)\}=\alpha_1^2B_1^6 \tag{14}$$

Influence of fractional derivative on the system response

For the sake of simplicity of calculation, letting $A = a^2$, and eq. (14) could be rewritten as:

$$A\{[36mw^2-4K(p)-3\alpha_1(A+2B_1^2)]^2+144w^2C^2(p)\}=\alpha_1^2B_1^6 \tag{15}$$

where $A = a^2$, $K(p)=k+K_1(3w)^p\cos\frac{p\pi}{2}$, $C(p)=c+K_1(3w)^{p-1}\sin\frac{p\pi}{2}$, $B_1=\dfrac{F_1}{8mw^2}$.

Using eq. (15) as the bifurcation equation, we can analyse the changes of the bifurcation characteristics of the system's amplitude-frequency curves along with the variations of the parameters, and the bifurcation equation should be established as the following form:

$$G(A,w;K_1,p)=A\{[36mw^2-4K(p)-3\alpha_1(A+2B_1^{\;2})]^2+144w^2C^2(p)\}-\alpha_1^{\;2}B_1^{\;6} \tag{16}$$

where A is the state variable, w – the bifurcation parameter, and the coefficient K_1 as well as the order p of the fractional derivative are the unfolding parameters.

According to the singularity theory [21], the transition set of eq. (16) includes the following three types:

– B (Bifurcation set)

$$G(A,w;\quad K_1,p)=0,\quad G_A(A,w;\quad K_1,p)=0,\quad G_w(A,w;\quad K_1,p)=0 \tag{17}$$

– H (Hysteresis set)

$$G(A,w;\quad K_1,p)=0,\quad G_A(A,w;\quad K_1,p)=0,\quad G_{AA}(A,w;\quad K_1,p)=0 \tag{18}$$

– D (Double-limit-point set)

$$G(A_i,w;\quad K_1,p)=0,\quad G_A(A_i,w;\quad K_1,p)=0,\quad i=1,2;\quad A_1\neq A_2 \tag{19}$$

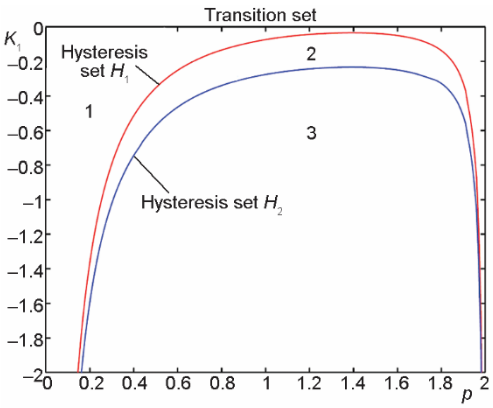


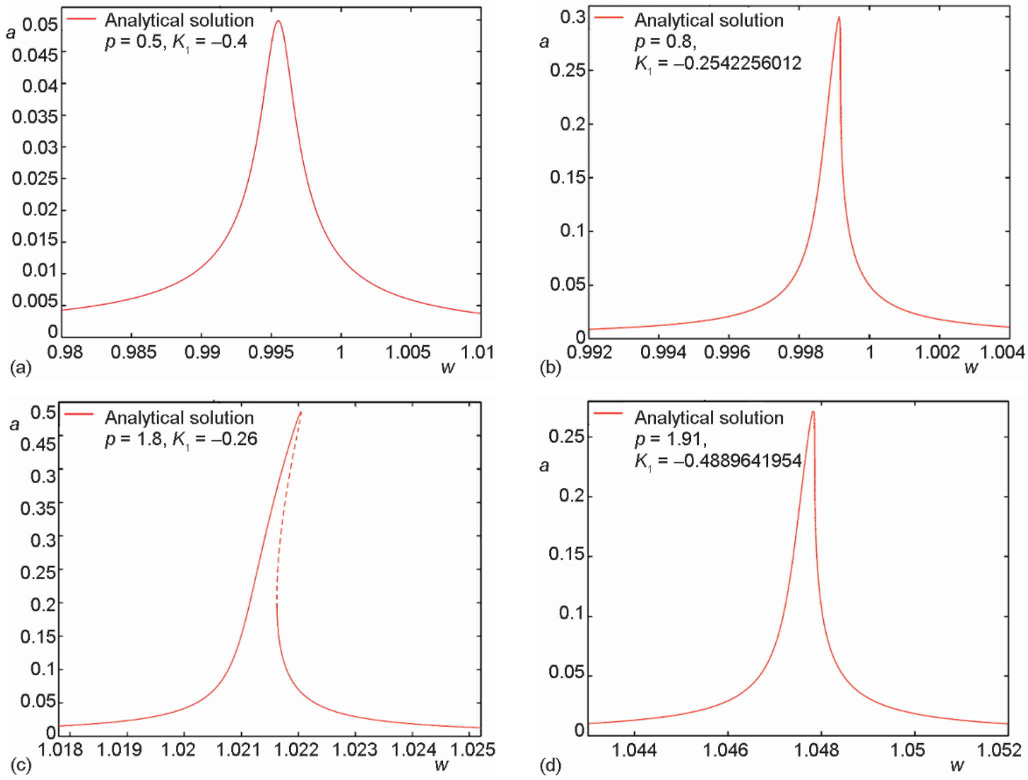
Figure 1. Transition set of system (3)

Taking m = 5, k = 45, c = 0.2, α_1 = 0.4, and F_1 = 15, the calculation results show that both the bifurcation set B and the double-limit-point set DL are empty sets, the hysteresis set H is non-empty set, and the transition set $\Sigma = B \cup H \cup DL = H$, which is shown in fig. 1.

The unfolding parameter p – K_1 plane is divided into three sub-regions by the transition set Σ. According to the singularity theory, the amplitude-frequency response curves of different points (p, K_1) in the same region are qualitatively the same. Taking a point (p, K_1) in each region and on the transition set curve, all varieties of the amplitude-frequency response curves which are qualitatively different could be obtained. For convenience, each region in fig. 1 is marked with a number.

Taking a given point (p, K_1) in the three sub-regions and on the transition set curve of fig. 1, the characteristics of amplitude-frequency responses are analysed, and the corresponding amplitude-frequency response curves are shown in fig. 2.

As can be seen from fig. 2, the parametric region where the amplitude-frequency response curve appears multiple solutions is surrounded by the hysteresis set H_1 and H_2. The system's response in Region 2 is triple-value. Two are stable while the other is unstable, and the jump occurs at the two extreme values of the frequency interval of triple-value amplitude as shown in fig. 2(c). On the curves H_1 and H_2, the system's response is in the critical state of occurrence of jump. The type of amplitude-frequency response curve of the system will be from figs. 2(a) to 2(c) on H_1 as shown in fig. 2(b) and from figs. 2(c) to 2(e) on H_2 as shown in fig. 2(d). These results just verify the correctness of the theoretical analysis in this part.



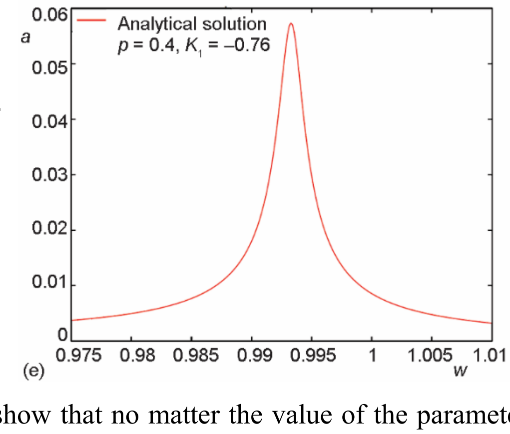


Figure 2. Amplitude-frequency responses of system (3); (a) parameter (p, K_1) in Region 1 of fig. 1, (b) parameter (p, K_1) on H_1 of fig. 1, (c) parameter (p, K_1) in Region 2 of fig. 1, (d) parameter (p, K_1) on H_2 of fig. 1, and (e) parameter (p, K_1) in Region 3 of fig. 1

The results of the previous analysis show that no matter the value of the parameter (p, K_1) crosses any line in fig. 1, the topology structure of the system response will change. So the transition set curve is just the critical parametric condition for qualitative changes of the responses of the system, and the types of dynamic responses of the system could be controlled by selecting the appropriate parameters (p, K_1) of the fractional derivative.

Numerical simulation

From the viewpoint of numerical simulation, we verify the correctness of the theoretical results previously obtained. Since the solution (7) contains two frequency components *w* and 3*w*, we need to do further spectrum analysis of the steady-state response of system (3) in order to obtain the corresponding amplitudes at the frequencies *w* and 3*w*, respectively. Then they are compared with $B_1 = F_1/8mw^2$ and eq. (14), respectively. Taking the parameters as given in fig. 2 to analyse, the corresponding waterfall figures could be obtained as shown in fig. 3.

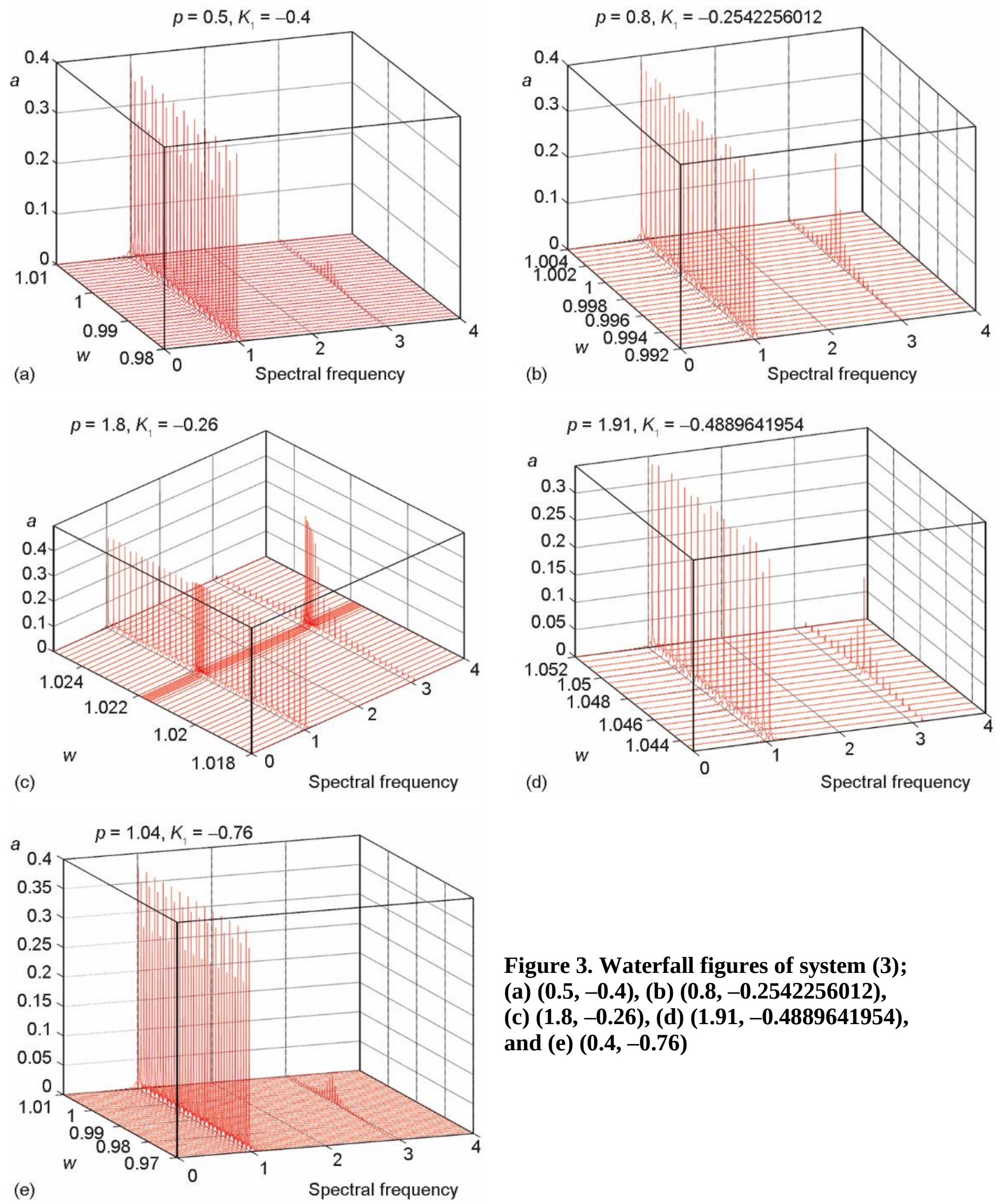
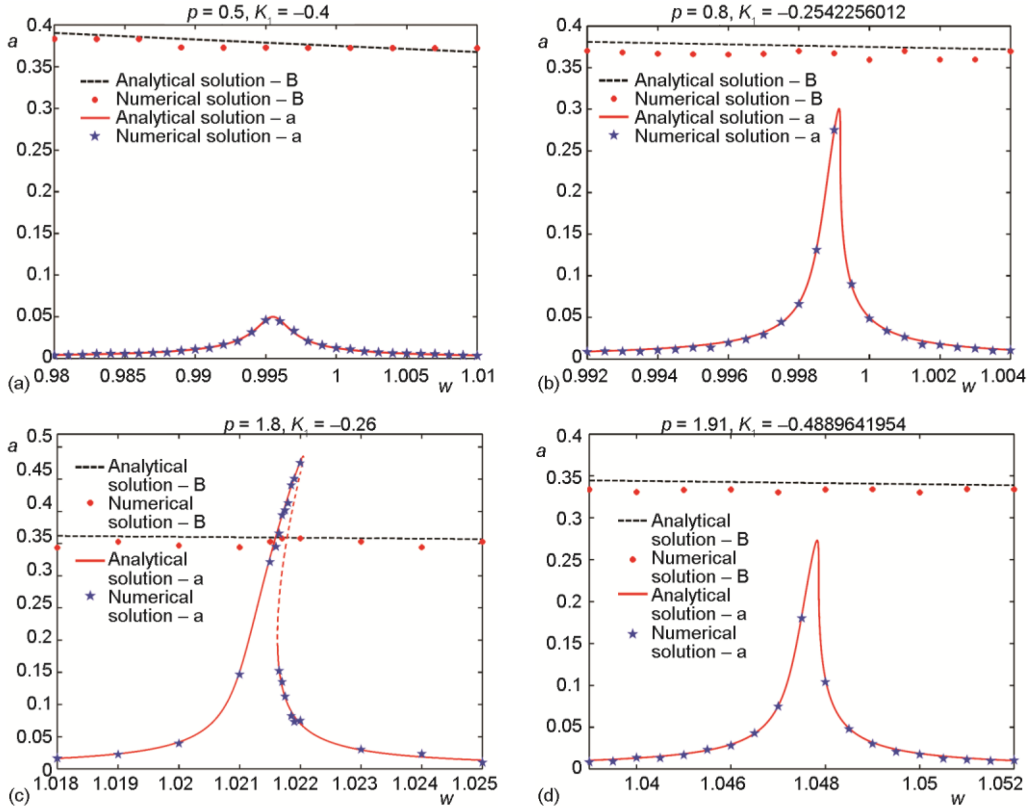


Figure 3. Waterfall figures of system (3); (a) (0.5, –0.4), (b) (0.8, –0.2542256012), (c) (1.8, –0.26), (d) (1.91, –0.4889641954), and (e) (0.4, –0.76)



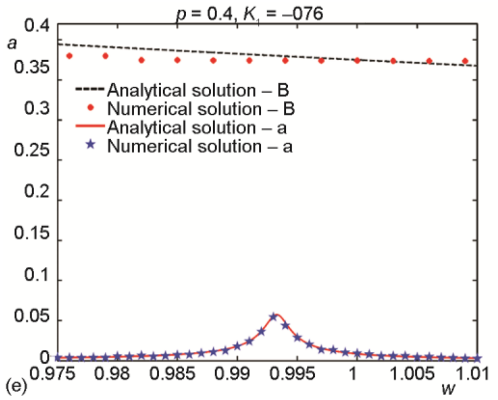


Figure 4. The comparison of the approximately analytical solutions with that by numerical simulation; (a) (0.5, –0.4), (b) (0.8, –0.2542256012), (c) (1.8, –0.26), (d) (1.91, –0.4889641954), and (e) (0.4, –0.76)

In each waterfall figure of fig. 3, for the different excitation frequency *w*, the spectrum analysis amplitudes at the corresponding frequencies *w* and 3*w* are obtained, respectively. Then, they are compared with $B_1 = F_1/8mw^2$ in eq. (7) and the amplitude-frequency response curves in fig. 2, respectively. The results of numerical simulation and the approximately analytical solutions are shown in fig. 4.

The previous results show that the numerical solutions are in good agreements with the approximately analytical solutions, which just verify the correctness of the theoretical analysis and show that it is feasible to use the method in this paper to analyse the bifurcation behaviour of fractional order system.

Conclusion

In this paper, we study the influence of fractional derivative on the super-harmonic resonance response of the system. According to the properties of the Caputo fractional derivative and the periodic characteristics of the response, the original system is transformed into an equivalent integer-order system, and the amplitude-frequency response equation of the system is obtained by using the deterministic averaging method. Also the critical parametric condition for the system's bifurcation is obtained by using the singularity theory, which can provide a theoretical guidance for system design in practical engineering. The results of spectrum analysis of the original system verify the correctness of the theoretical results obtained in this paper. It is concluded that the fractional order p and coefficient K_1 can both cause the mutation phenomenon of the system.

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