ABUNDANT EXACT ANALYTICAL SOLUTIONS AND NOVEL INTERACTION PHENOMENA OF THE GENERALIZED (3+1)-DIMENSIONAL SHALLOW WATER EQUATION

by

Xiaomin WANG, Sudao BILIGE*, and Yueyang FENG

Department of Mathemaitcs, Inner Mongolia University of Technology, Hohhote, China

Original scientific paper
https://doi.org/10.2298/TSCI191123103W

This paper reveals abundant exact analytical solutions to the generalized (3+1)-D shallow water equation. The generalized bilinear method is used in the solution process and the obtained solutions include the high-order lump-type solutions, the three-wave solutions, the breather solutions. The interaction between the high-order lump-type solutions and the soliton solutions is also elucidated. These solutions have greatly enriched the generalized (3+1)-D shallow water equation in open literature.

Key words: *exact analytical solution, high-order lump-type solution, interaction solution, generalized bilinear method, generalized (3+1)-D shallow water equation*

Introduction

It is well-known that non-linear evolution equations (NLEE) have a wide applications in areas of mathematics, physics, fluid mechanics, plasma, optical fiber communication, biologic nerve propagation, atmospheric science, marine science, and thermal science. Therefore, the exact solutions of NLEE play an important role in understanding the non-linear phenomena of non-linear science. To find exact solutions of non-linear systems is a difficult and tedious but very important and meaningful work. So far, several effective methods have been established by mathematicians and physicists to obtain exact solutions of NLEE [1-8]. By using these methods, researchers constructed the exact solutions of NLEE, such as soliton [9], rogue wave [10], breathers [11], periodic wave [12], three-wave solution [13], rational solutions [14], lump solution [15] and interaction solutions [16-18], *etc.*

As one of the three branches of non-linear science, the theory of solitons has become an important research field of non-linear science. It has a wide and important role in the fields of non-linear wave theory and elementary particle theory. Recently, the research of breather waves [11], rational solutions [14] become a hot research topic. In general, breather waves, which have a periodic outline in one direction, can degenerate into the rogue wave solutions in the limiting case. The rational solutions have appeared in many non-linear fields, such as non-linear optic fibers, Bose-Einstein condensates, biophysics and economics. As one of critical exact solutions, rational solutions can be used to describe natural phenomena well. In contrast to soliton solutions, lump solutions are a kind of rational function solutions, localized in all directions in the space. In soliton theory, lump solutions have received increasing attention

* Corresponding author, e-mail: inmathematica@126.com

recently [15]. In particular, collisions between lump solutions and other forms of soliton solutions have been studied [16-18].

Exact analytical solutions of the generalized (3+1)-D shallow water equation

We consider the generalized (3+1)-D shallow water equation [14]

$$u_{xxxy}+3u_{xx}u_y+3u_xu_{xy}-u_{yt}-u_{xz}=0 \tag{1}$$

Researchers studied the rational solutions and the lump solutions (only when $z = x$) of the generalized (3+1)-D shallow water equation. In the following, we will study the general exact analytical solutions of eq. (1).

Step 1. By using the Cole-Hopf transformation:

$$u(x,y,z,t)=2[\ln f(x,y,z,t)]_x \tag{2}$$

Equation (1) becomes the generalized Hirota bilinear equation:

$$GB_{\rm GSW}(f):=(D_{p,x}^3D_{p,y}-D_{p,y}D_{p,t}-D_{p,x}D_{p,z})f\cdot f=0 \tag{3}$$

where p is an arbitrarily natural number and D – the generalized bilinear differential operator [3]. We note that when $p = 2$, the generalized bilinear form is transformed into Hirota bilinear form.

When taking $p = 3$, we can obtain the generalized bilinear shallow water equation:

$$GB_{\rm GSW}(f):=(D_{3,x}^3D_{3,y}-D_{3,y}D_{3,t}-D_{3,x}D_{3,z})f\cdot f=$$

$$=2(3f_{xx}f_{xy}+f_yf_t-f_{yt}f+f_xf_z-f_{xz}f)=0 \tag{4}$$

Step 2. We suppose that the generalized bilinear eq. (4) has the following solution:

$$f=a_0+\sum_{i=1}^{N}\xi_i^{2n_i}+\sum_{j=1}^{M}m_jg_j(\eta_j) \tag{5}$$

$$\xi_i=a_{i0}+a_{i1}x+a_{i2}y+a_{i3}z+a_{i4}t,\quad \eta_j=b_{j0}+b_{j1}x+b_{j2}y+b_{j3}z+b_{j4}t \tag{6}$$

where a_{ik}, m_j,b_{jk} ($i = 1,\ldots,N$, $j = 1,\ldots,M$, and $k = 0, 1, 2, 3, 4$) are arbitrary real constants.

To search for the high-order lump-type solutions, three-wave solutions, breather solutions and interaction solutions between the high-order lump-type solution and other function solutions, we suppose:

$$N=3,\quad M=4,\quad n_1=2,\quad n_2=1,\quad n_3=1,\quad g_1(\eta_1)=e^{\eta_1},\quad g_2(\eta_2)=e^{\eta_2}$$

$$g_3(\eta_3)=\cos\eta_3,\quad g_4(\eta_4)=\cosh\eta_4 \tag{7}$$

The exact analytical solutions of generalized bilinear eq. (4) is written:

$$f=a_0+\xi_1^4+\xi_2^2+\xi_3^2+m_1e^{\eta_1}+m_2e^{\eta_2}+m_3\cos\eta_3+m_4\cosh\eta_4 \tag{8}$$

Step 3. By substituting eq. (8) into eq. (4) and collecting all terms with the same order of x, y, z, t, e^{η_1}, e^{η_2}, $\sin\eta_3$, $\cos\eta_3$, $\sinh\eta_4$, and $\cosh\eta_4$ together, the left-hand side of eq. (4) is converted into another polynomial in x, y, z, t, e^{η_1}, e^{η_2}, $\sin\eta_3$, $\cos\eta_3$, $\sinh\eta_4$, and $\cosh\eta_4$. Equating each coefficient of this different power terms to zero yields a set of non-linear algebraic equations for a_0, a_{ik}, b_{jk}, and m_j. Solving the algebraic equations by symbolic computation Maple, yields the following sets of solutions. According to different parameter values, we can obtain abundant exact analytical solutions of the generalized (3+1)-D shallow water eq. (1).

High-order lump-type solutions, three-wave solutions and Breather solutions

High-order lump-type solutions and lump solutions

When $m_j = 0$ (j = 1, 2, 3, 4) in eq. (8), the solution (8) represents $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2$.

– Case 1.1:

$$a_{11}=0,\quad a_{13}=-\frac{a_{12}a_{34}}{a_{31}},\quad a_{14}=0,\quad a_{21}=0,\quad a_{22}=-\frac{a_{23}a_{31}}{a_{34}},\quad a_{24}=0$$

$$a_{32}=0,\quad a_{33}=0,\quad a_{31}a_{34}\neq 0$$

– Case 1.2:

$$a_{11}=0,\quad a_{12}=\frac{a_{13}a_{32}}{a_{33}},\quad a_{14}=0,\quad a_{22}=-\frac{a_{31}a_{32}}{a_{21}},\quad a_{23}=-\frac{a_{31}a_{33}}{a_{21}},\quad a_{24}=-\frac{a_{21}a_{33}}{a_{32}}$$

$$a_{30}=\frac{a_{20}a_{31}}{a_{21}},\quad a_{34}=-\frac{a_{31}a_{33}}{a_{32}},\quad a_{21}a_{32}a_{33}\neq 0$$

– Case 1.3:

$$a_{11}=0,\quad a_{14}=0,\quad a_{22}=-\frac{a_{31}a_{32}}{a_{21}},\quad a_{23}=-\frac{a_{13}a_{31}a_{32}}{a_{12}a_{21}},\quad a_{24}=-\frac{a_{13}a_{21}}{a_{12}},\quad a_{33}=\frac{a_{13}a_{32}}{a_{12}}$$

$$a_{34}=-\frac{a_{13}a_{31}}{a_{12}},\quad a_{12}a_{21}\neq 0$$

where other parameters are arbitrary real constants. By substituting the parameters in Cases 1.1-1.3 into the solution (8) and using the transformation (2), we can obtain the high-order lump-type solutions of eq. (1).

When $m_j = 0$, $a_{1j} = 0$ (j = 1, 2, 3, 4) in eq. (8), we obtain $f = \bar{a}_0 + \xi_2^2 + \xi_3^2$, $\bar{a}_0 = a_0 + a_{10}^4$.

– Case 1.4:

$$a_{21}=0,\quad a_{23}=-\frac{a_{22}(a_{10}^4a_{24}^2+3a_{31}^3a_{34}+a_0a_{24}^2)}{3a_{31}^4},\quad a_{32}=\frac{a_{22}a_{24}(a_{10}^4+a_0)}{3a_{31}^3}$$

$$a_{33}=-\frac{a_{22}a_{24}(a_{10}^4a_{34}-3a_{31}^3+a_0a_{34})}{3a_{31}^4},\quad a_{31}\neq 0$$

– Case 1.5:

$$a_{22}=\frac{a_{32}a_{34}(a_{10}^4+a_0)}{3a_{21}^3},\quad a_{23}=-\frac{a_{32}a_{34}(a_{10}^4a_{24}-3a_{21}^3+a_0a_{24})}{3a_{21}^4},\quad a_{30}=0,\quad a_{31}=0$$

$$a_{33}=-\frac{a_{32}(a_{10}^4a_{34}^2+3a_{21}^3a_{24}+a_0a_{34}^2)}{3a_{21}^4},\quad a_{21}\neq 0$$

– Case 1.6:

$$a_{23}=\frac{a_{22}a_{33}}{a_{32}}+\frac{3(a_{22}^2+a_{32}^2)(a_{21}^2+a_{31}^2)(a_{21}a_{22}+a_{31}a_{32})}{a_{32}(a_{10}^4+a_0)(a_{21}a_{32}-a_{22}a_{31})}$$

$$a_{24}=-\frac{a_{21}a_{33}}{a_{32}}-\frac{3(a_{22}^2+a_{31}^2)(a_{21}a_{22}+a_{31}a_{32})^2}{a_{32}(a_{10}^4+a_0)(a_{21}a_{32}-a_{22}a_{31})},\quad a_{30}=\frac{a_{20}a_{31}}{a_{21}}$$

$$a_{34}=-\frac{a_{31}a_{33}}{a_{32}}+\frac{3(a_{21}^2+a_{31}^2)(a_{21}a_{22}+a_{31}a_{32})}{a_{32}(a_{10}^4+a_0)},\quad a_{32}(a_{10}^4+a_0)(a_{21}a_{32}-a_{22}a_{31})\neq 0$$

– Case 1.7:

$$a_0=-a_{10}^4+\frac{3a_{21}a_{22}(a_{21}^2+a_{31}^2)}{a_{31}a_{33}},\quad a_{23}=-\frac{a_{21}a_{33}+a_{22}a_{34}}{a_{31}}$$

$$a_{24}=\frac{a_{21}^2a_{33}+a_{21}a_{22}a_{34}+a_{31}^2a_{33}}{a_{22}a_{31}},\quad a_{30}=\frac{a_{20}a_{31}}{a_{21}},\quad a_{32}=0, a_{21}a_{22}a_{31}a_{33}\neq 0$$

– Case 1.8:

$$a_0=-a_{10}^4+\frac{3(a_{21}a_{22}+a_{31}a_{32})(a_{21}^2+a_{31}^2)^2}{(a_{21}a_{34}-a_{24}a_{31})(a_{21}a_{32}-a_{22}a_{31})},\quad (a_{21}a_{34}-a_{24}a_{31})(a_{21}a_{32}-a_{22}a_{31})\neq 0$$

$$a_{23}=-\frac{a_{21}(a_{22}a_{24}-a_{32}a_{34})+a_{31}(a_{22}a_{34}+a_{24}a_{32})}{a_{21}^2+a_{31}^2}$$

$$a_{33}=-\frac{a_{21}(a_{22}a_{34}+a_{24}a_{32})+a_{31}(a_{32}a_{34}-a_{22}a_{24})}{a_{21}^2+a_{31}^2},\quad a_{21}^2+a_{31}^2\neq 0$$

– Case 1.9:

$$a_{22}=-\frac{a_{31}a_{32}}{a_{21}},\quad a_{23}=\frac{a_{24}a_{31}a_{32}}{a_{21}^2},\quad a_{33}=-\frac{a_{24}a_{32}}{a_{21}},\quad a_{34}=\frac{a_{24}a_{32}}{a_{21}},\quad a_{21}\neq 0$$

where other parameters are arbitrary real constants. By substituting the parameters in Cases 1.4-1.9 into the solution (8) and using the transformation (2), we can obtain the lump solutions of eq. (1) which are different from those lump solutions given in [14].

Three-wave solutions

When $a_{ik}=0$ ($i = 1, 2, 3$ and $k = 0, 1, 2, 3, 4$), $\eta_2=-\eta_1$, $m_2=m_1$ in eq. (8), the solution (8) represents $f=a_0+m_1e^{\eta_1}+m_1e^{-\eta_1}+m_3\cos\eta_3+m_4\cosh\eta_4$.

– Case 2.1:

$$b_{12}=b_{13}=b_{31}=b_{34}=b_{42}=b_{43}=0,\quad b_{32}=-\frac{b_{11}b_{33}}{b_{14}},\quad b_{41}=\frac{b_{11}b_{44}}{b_{14}},\quad b_{14}\neq 0$$

– Case 2.2:

$$b_{12}=b_{13}=b_{31}=b_{34}=b_{41}=b_{44}=0,\quad b_{32}=-\frac{b_{11}b_{33}}{b_{14}},\quad b_{42}=-\frac{b_{11}b_{43}}{b_{14}},\quad b_{14}\neq 0$$

– Case 2.3:

$$b_{11} = b_{14} = b_{32} = b_{33} = b_{42} = b_{43} = 0, \quad b_{34} = -\frac{b_{13}b_{31}}{b_{12}}, \quad b_{44} = -\frac{b_{13}b_{41}}{b_{12}}, \quad b_{12} \neq 0$$

– Case 2.4:

$$b_{11} = b_{14} = b_{32} = b_{33} = b_{41} = b_{44} = 0, \quad b_{34} = -\frac{b_{13}b_{31}}{b_{12}}, \quad b_{42} = \frac{b_{12}b_{43}}{b_{13}}, \quad b_{12}b_{13} \neq 0$$

– Case 2.5:

$$b_{11} = b_{14} = b_{31} = b_{34} = b_{42} = b_{43} = 0, \quad b_{33} = \frac{b_{13}b_{32}}{b_{12}}, \quad b_{44} = -\frac{b_{13}b_{41}}{b_{12}}, \quad b_{12} \neq 0$$

– Case 2.6:

$$b_{12} = b_{13} = b_{32} = b_{33} = b_{41} = b_{44} = 0, \quad b_{34} = \frac{b_{14}b_{31}}{b_{11}}, \quad b_{43} = -\frac{b_{14}b_{42}}{b_{11}}, \quad b_{11} \neq 0$$

– Case 2.7:

$$b_{12} = b_{13} = b_{31} = b_{34} = b_{42} = b_{43} = 0, \quad b_{33} = -\frac{b_{14}b_{32}}{b_{11}}, \quad b_{44} = \frac{b_{14}b_{41}}{b_{11}}, \quad b_{11} \neq 0$$

where other parameters are arbitrary real constants. By substituting the parameters in Cases 2.1-2.7 into the solution (8) and using the transformation (2), we can obtain the three-wave solutions of eq. (1).

Breather solutions and solitary wave solutions

When $a_{ik} = 0$ (i = 1, 2, 3 and k = 0, 1, 2, 3, 4), $m_4 = 0$ or $m_3 = 0$, $\eta_4 = \eta_3$ in eq. (8), the solution (8) represents $f = a_0 + m_1 e^{\eta_1} + m_2 e^{\eta_2} + m_3\cos\eta_3$ and $f = a_0 + m_1 e^{\eta_1} + m_2 e^{\eta_2} + m_4\cosh\eta_3$.

– Case 3.1:

$$b_{11} = b_{14} = b_{21} = b_{24} = b_{32} = b_{33} = 0, \quad b_{12} = -\frac{b_{13}b_{31}}{b_{34}}, \quad b_{22} = -\frac{b_{23}b_{31}}{b_{34}}, \quad b_{34} \neq 0$$

– Case 3.2:

$$b_{11} = b_{14} = b_{22} = b_{23} = b_{32} = b_{33} = 0, \quad b_{13} = -\frac{b_{12}b_{24}}{b_{21}}, \quad b_{34} = \frac{b_{24}b_{31}}{b_{21}}, \quad b_{21} \neq 0$$

– Case 3.3:

$$b_{11} = b_{14} = b_{22} = b_{23} = b_{31} = b_{34} = 0, \quad b_{13} = -\frac{b_{12}b_{24}}{b_{21}}, \quad b_{33} = -\frac{b_{24}b_{32}}{b_{21}}, \quad b_{21} \neq 0$$

– Case 3.4:

$$b_{12} = b_{13} = b_{21} = b_{24} = b_{31} = b_{34} = 0, \quad b_{22} = -\frac{b_{11}b_{23}}{b_{14}}, \quad b_{32} = -\frac{b_{11}b_{33}}{b_{14}}, \quad b_{14} \neq 0$$

– Case 3.5:

$$a_0 = b_{12} = b_{13} = b_{21} = b_{24} = b_{32} = b_{33} = 0, \quad b_{11} = \frac{b_{14}b_{31}}{b_{34}}, \quad b_{22} = -\frac{b_{23}b_{31}}{b_{34}}, \quad b_{34} \neq 0$$

– Case 3.6:

$$a_0 = b_{12} = b_{13} = b_{22} = b_{23} = b_{31} = b_{34} = 0, \quad b_{14} = -\frac{b_{11}b_{33}}{b_{32}}, \quad b_{24} = -\frac{b_{21}b_{33}}{b_{32}}, \quad b_{32} \neq 0$$

where other parameters are arbitrary real constants. By substituting the parameters in Cases 3.1-3.6 into the solution (8) and using the transformation (2), we can obtain the breather solutions and the solitary wave solutions of eq. (1).

Remark 2.1: In addition to Cases 3.1-3.6, we can get the special results of Cases 2.2 and 2.4 when $b_{43} = 0$.

Collisions between lump and soliton solutions

Between lumps and a pair of line soliton solution

When $m_3 = 0$, $m_4 = 0$ in eq. (8), the solution (8) represents $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_1 e^{\eta_1} + m_2 e^{\eta_2}$.

When $m_2 = m_1$, the solution (8) is written $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_1 e^{\eta_1} + m_1 e^{\eta_2}$.

– Case 4.1:

$$a_{11} = 0, a_{13} = -\frac{a_{12}a_{24}}{a_{21}}, \quad a_{14} = a_{22} = a_{23} = a_{31} = 0, \quad a_{33} = -\frac{a_{24}a_{32}}{a_{21}}, \quad a_{34} = b_{12} = b_{13} = 0$$

$$b_{14} = \frac{a_{24}b_{11}}{a_{21}}, \quad b_{22} = b_{23} = 0, \quad b_{24} = \frac{a_{24}b_{21}}{a_{21}}, \quad (a_{21} \neq 0)$$

– Case 4.2:

$$a_{11} = a_{14} = 0, \quad a_{21} = -\frac{a_{12}a_{24}}{a_{13}}, \quad a_{22} = a_{23} = 0, \quad a_{31} = -\frac{a_{12}a_{34}}{a_{13}}, \quad a_{32} = a_{33} = b_{12} = b_{13} = 0$$

$$b_{14} = -\frac{a_{13}b_{11}}{a_{12}}, \quad b_{22} = b_{23} = 0, \quad b_{24} = -\frac{a_{13}b_{21}}{a_{12}}, \quad (a_{12}a_{13} \neq 0)$$

– Case 4.3:

$$a_{10} = a_{11} = 0, \quad a_{12} = -\frac{a_{13}a_{31}}{a_{34}}, \quad a_{14} = a_{21} = a_{22} = a_{23} = a_{24} = a_{32} = a_{33} = b_{12} = b_{13} = 0$$

$$b_{14} = \frac{a_{34}b_{11}}{a_{31}}, \quad b_{22} = b_{23} = 0, \quad b_{24} = \frac{a_{34}b_{21}}{a_{31}}, \quad (a_{31}a_{34} \neq 0)$$

– Case 4.4:

$$a_{11} = a_{14} = a_{21} = a_{22} = a_{23} = a_{24} = a_{31} = 0, \quad a_{33} = \frac{a_{13}a_{32}}{a_{12}}, \quad a_{34} = b_{12} = b_{13} = 0$$

$$b_{14} = -\frac{a_{13}b_{11}}{a_{12}}, \quad b_{22} = b_{23} = 0, \quad b_{24} = -\frac{a_{13}b_{21}}{a_{12}}, \quad (a_{12} \neq 0)$$

where other parameters are arbitrary real constants.

When $m_2 = m_1$, $\eta_2 = -\eta_1$, the solution (8) is written $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_1 e^{\eta_1} + m_1 e^{-\eta_1}$.

– Case 4.5:

$$a_{11} = 0, \quad a_{13} = -\frac{a_{12}a_{34}}{a_{31}}, \quad a_{14} = 0, \quad a_{21} = -\frac{a_{31}a_{32}}{a_{22}}, \quad a_{23} = -\frac{a_{22}a_{34}}{a_{31}}, \quad a_{24} = -\frac{a_{32}a_{34}}{a_{22}}$$

$$a_{33} = -\frac{a_{32}a_{34}}{a_{31}}, \quad b_{12} = b_{13} = 0, \quad b_{14} = \frac{a_{34}b_{11}}{a_{31}}, \quad (a_{22}a_{31} \neq 0)$$

– Case 4.6:

$$a_{11} = a_{14} = a_{21} = 0, \quad a_{23} = \frac{a_{13}a_{22}}{a_{12}}, \quad a_{24} = a_{31} = 0, \quad a_{32} = \frac{a_{12}a_{33}}{a_{13}}, \quad a_{34} = b_{12} = b_{13} = 0$$

$$b_{14} = -\frac{a_{13}b_{11}}{a_{12}}, \quad (a_{12}a_{13} \neq 0)$$

– Case 4.7:

$$a_{11} = 0, \quad a_{13} = -\frac{a_{12}a_{34}}{a_{31}}, \quad a_{14} = a_{21} = 0, \quad a_{22} = -\frac{a_{23}a_{31}}{a_{34}}, \quad a_{24} = a_{32} = a_{33} = b_{11} = 0$$

$$b_{13} = -\frac{a_{34}b_{12}}{a_{31}}, \quad b_{14} = 0, \quad (a_{31}a_{34} \neq 0)$$

– Case 4.8:

$$a_{11} = 0, \quad a_{12} = -\frac{a_{13}a_{31}}{a_{34}}, \quad a_{14} = a_{21} = a_{22} = a_{23} = a_{24} = a_{32} = a_{33} = b_{12} = b_{13} = 0$$

$$b_{14} = \frac{a_{34}b_{11}}{a_{31}}, \quad (a_{31}a_{34} \neq 0)$$

where other parameters are arbitrary real constants. By substituting the parameters in Cases 4.1-4.8 into the solution (8) and using the transformation (2), we can obtain the interaction solutions between the high-order lump-type solution and a pair of line soliton solution of eq. (1).

Remark 2.2: In addition to Cases 4.5-4.8, we can get the special results of Cases 4.1, 4.2, and 4.4 when $b_{21} = b_{11}$.

Between lumps and one line-soliton solution

When $m_3 = 0$, $m_4 = 0$ and $m_2 = 0$ or $m_1 = 0$, $\eta_2 = -\eta_1$ in (8), the solution (8) represents $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_1 e^{\eta_1}$ or $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_2 e^{-\eta_1}$.

– Case 5.1:

$$a_{11} = a_{14} = 0, \quad a_{22} = -\frac{a_{31}a_{32}}{a_{21}}, \quad a_{23} = -\frac{a_{13}a_{31}a_{32}}{a_{12}a_{21}}, \quad a_{24} = -\frac{a_{13}a_{21}}{a_{12}}, \quad a_{33} = \frac{a_{13}a_{32}}{a_{12}}$$

$$a_{34} = -\frac{a_{13}a_{31}}{a_{12}}, \quad b_{11} = 0, \quad b_{13} = \frac{a_{13}b_{12}}{a_{12}}, \quad b_{14} = 0, \quad (a_{12}a_{21} \neq 0)$$

where other parameters are arbitrary real constants. By substituting the parameters in Case 5.1 into the solution (8) and using the transformation (2), we can obtain the interaction solutions between the high-order lump-type solution and one line-soliton solution of eq. (1).

Remark 2.3: In addition to Case 5.1, we can get the same results as Cases 4.7, 4.8 and the special results of Cases 4.1, 4.2, and 4.4 when $b_{21} = b_{11}$.

Between lumps and three-wave solution

When $m_i = m_1$ (i = 2, 3, 4), $m_1 \neq 0$, $\eta_2 = -\eta_1$ in eq. (8), the solution (8) represents $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_1 e^{\eta_1} + m_1 e^{-\eta_1} + m_1\cos\eta_3 + m_1\cosh\eta_4$.

– Case 6.1:

$$a_{11} = 0, \quad a_{13} = -\frac{a_{12}a_{24}}{a_{21}}, \quad a_{14} = a_{22} = a_{23} = a_{31} = 0, \quad a_{33} = -\frac{a_{24}a_{32}}{a_{21}}, \quad a_{34} = b_{12} = b_{13} = 0$$

$$b_{14} = \frac{a_{24}b_{11}}{a_{21}}, \quad b_{32} = b_{33} = 0, \quad b_{34} = \frac{a_{24}b_{31}}{a_{21}}, \quad b_{42} = b_{43} = 0, \quad b_{44} = \frac{a_{24}b_{41}}{a_{21}}, \quad (a_{21} \neq 0)$$

– Case 6.2:

$$a_{10} = a_{11} = a_{14} = 0, \quad a_{21} = -\frac{a_{12}a_{24}}{a_{13}}, \quad a_{22} = a_{23} = 0, \quad a_{31} = -\frac{a_{12}a_{34}}{a_{13}}, \quad a_{32} = a_{33} = b_{12} = 0$$

$$b_{13} = 0, \quad b_{14} = -\frac{a_{13}b_{11}}{a_{12}}, \quad b_{32} = b_{33} = 0, \quad b_{34} = -\frac{a_{13}b_{31}}{a_{12}}, \quad b_{42} = b_{43} = 0$$

$$b_{44} = -\frac{a_{13}b_{41}}{a_{12}}, \quad (a_{12}a_{13} \neq 0)$$

– Case 6.3:

$$a_{11} = a_{14} = a_{21} = a_{22} = a_{23} = a_{24} = a_{31} = 0, \quad a_{33} = \frac{a_{13}a_{32}}{a_{12}}, \quad a_{34} = b_{12} = b_{13} = 0$$

$$b_{14} = -\frac{a_{13}b_{11}}{a_{12}}, \quad b_{32} = b_{33} = 0, \quad b_{34} = -\frac{a_{13}b_{31}}{a_{12}}, \quad b_{42} = b_{43} = 0, \quad b_{44} = -\frac{a_{13}b_{41}}{a_{12}}, \quad (a_{12} \neq 0)$$

– Case 6.4:

$$a_{11} = 0, \quad a_{12} = -\frac{a_{13}a_{31}}{a_{34}}, \quad a_{14} = a_{21} = a_{22} = a_{23} = a_{24} = a_{32} = a_{33} = b_{12} = b_{13} = 0$$

$$b_{14} = \frac{a_{34}b_{11}}{a_{31}}, \quad b_{31} = b_{11}, \quad b_{32} = b_{33} = 0, \quad b_{34} = \frac{a_{34}b_{11}}{a_{31}}$$

$$b_{42} = b_{43} = 0, \quad b_{44} = \frac{a_{34}b_{41}}{a_{31}}, \quad (a_{31}a_{34} \neq 0)$$

– Case 6.5:

$$a_{11} = 0, \quad a_{13} = -\frac{a_{12}a_{34}}{a_{31}}, \quad a_{14} = a_{21} = 0, \quad a_{22} = -\frac{a_{23}a_{31}}{a_{34}}, \quad a_{24} = a_{32} = a_{33} = b_{11} = 0$$

$$b_{13} = -\frac{a_{34}b_{12}}{a_{31}}, \quad b_{14} = b_{31} = 0, \quad b_{33} = -\frac{a_{34}b_{32}}{a_{31}}, \quad b_{34} = b_{41} = 0$$

$$b_{43} = -\frac{a_{34}b_{42}}{a_{31}}, \quad b_{44} = 0, \quad (a_{31}a_{34} \neq 0)$$

– Case 6.6:

$$a_{11}=a_{14}=0,\quad a_{21}=\frac{a_{12}a_{32}a_{34}}{a_{13}a_{22}},\quad a_{23}=\frac{a_{13}a_{22}}{a_{12}},\quad a_{24}=-\frac{a_{32}a_{34}}{a_{22}},\quad a_{31}=-\frac{a_{12}a_{34}}{a_{13}}$$

$$a_{33}=\frac{a_{13}a_{32}}{a_{12}},\quad b_{12}=b_{13}=0,\quad b_{14}=-\frac{a_{13}b_{11}}{a_{12}},\quad b_{31}=b_{11},\quad b_{32}=b_{33}=0$$

$$b_{34}=-\frac{a_{13}b_{31}}{a_{12}},\quad b_{42}=b_{43}=0,\quad b_{44}=-\frac{a_{13}b_{41}}{a_{12}},\quad (a_{12}a_{13}a_{22}\neq 0)$$

– Case 6.7:

$$a_{11}=0,\quad a_{13}=-\frac{a_{12}a_{34}}{a_{31}},\quad a_{14}=0,\quad a_{21}=-\frac{a_{31}a_{32}}{a_{22}},\quad a_{23}=-\frac{a_{22}a_{34}}{a_{31}},\quad a_{24}=-\frac{a_{32}a_{34}}{a_{22}}$$

$$a_{30}=\frac{a_{20}a_{32}}{a_{22}},\quad a_{33}=-\frac{a_{32}a_{34}}{a_{31}},\quad b_{12}=b_{13}=0,\quad b_{14}=\frac{a_{34}b_{11}}{a_{31}},\quad b_{31}=b_{11},\quad b_{32}=b_{33}=0$$

$$b_{34}=\frac{a_{34}b_{31}}{a_{31}},\quad b_{42}=b_{43}=0,\quad b_{44}=\frac{a_{34}b_{41}}{a_{31}},\quad (a_{22}a_{31}\neq 0)$$

– Case 6.8:

$$a_{11}=a_{14}=a_{21}=0,\quad a_{23}=\frac{a_{13}a_{22}}{a_{12}},\quad a_{24}=0,\quad a_{30}=\frac{a_{12}a_{20}a_{33}}{a_{13}a_{22}},\quad a_{31}=0,\quad a_{32}=\frac{a_{12}a_{33}}{a_{13}}$$

$$a_{34}=b_{12}=b_{13}=0,\quad b_{14}=-\frac{a_{13}b_{11}}{a_{12}},\quad b_{31}=b_{11},\quad b_{32}=b_{33}=0,\quad b_{34}=-\frac{a_{13}b_{31}}{a_{12}}$$

$$b_{42}=b_{43}=0,\quad b_{44}=-\frac{a_{13}b_{41}}{a_{12}},\quad (a_{12}a_{13}a_{22}\neq 0)$$

where other parameters are arbitrary real constants. By substituting the parameters in Cases 6.1-6.8 into the solution (8) and using the transformation (2), we can obtain the interaction solutions between the high-order lump-type solution and three-wave solution of eq. (1).

Between lumps and breather solution (or solitary wave solution)

When $\eta_2 = -\eta_1$, $m_i = m_1$ (i = 2, 3), $m_4 = 0$ [or $m_i = m_1$ (i = 2, 4), $m_3 = 0$, $\eta_4 = \eta_3$] in eq. (8), the solution (8) represents $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_1 e^{\eta_1} + m_1 e^{-\eta_1} + m_1\cos\eta_3$ or $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_1 e^{\eta_1} + m_1 e^{-\eta_1} + m_1\cosh\eta_3$.

– Case 7.1:

$$a_{11}=a_{14}=0,\quad a_{21}=-\frac{a_{12}a_{24}}{a_{13}},\quad a_{22}=a_{23}=0,\quad a_{31}=-\frac{a_{12}a_{34}}{a_{13}},\quad a_{32}=0$$

$$a_{33}=b_{12}=b_{13}=0,\quad b_{14}=-\frac{a_{13}b_{11}}{a_{12}},\quad b_{32}=b_{33}=0,\quad b_{34}=-\frac{a_{13}b_{31}}{a_{12}},\quad (a_{12}a_{13}\neq 0)$$

– Case 7.2:

$$a_{11}=0,\quad a_{12}=-\frac{a_{13}a_{31}}{a_{34}},\quad a_{14}=a_{21}=a_{22}=a_{23}=a_{24}=a_{32}=a_{33}=b_{12}=b_{13}=0$$

$$b_{14}=\frac{a_{34}b_{11}}{a_{31}},\quad b_{32}=b_{33}=0,\quad b_{34}=\frac{a_{34}b_{31}}{a_{31}},\quad (a_{31}a_{34}\neq 0)$$

where other parameters are arbitrary real constants. By substituting the parameters in Cases 7.1 and 7.2 into the solution (8) and using the transformation (2), we can obtain the interaction solutions between the high-order lump-type solution and breather solution (or the solitary wave solution) of eq. (1).

Remark 2.4: In addition to Cases 7.1 and 7.2, we can get the special results of Cases 6.1, 6.3, and 6.5-6.8 as long as $b_{41} = 0$.

Between lumps and cos-cosh periodic wave solution

When $m_i = 0$ ($i = 1, 2$), $m_3m_4 \neq 0$ in eq. (8), the solution (8) represents $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_3\cos\eta_3 + m_4\cosh\eta_4$.

– Case 8.1:

$$a_{10}=a_{11}=0,\quad a_{13}=\frac{a_{12}a_{33}}{a_{32}},\quad a_{14}=0,\quad a_{21}=-\frac{a_{24}a_{32}}{a_{33}},\quad a_{22}=a_{23}=a_{31}=a_{34}=b_{32}=b_{33}=0$$

$$b_{34}=-\frac{a_{33}b_{31}}{a_{32}},\quad b_{42}=b_{43}=0,\quad b_{44}=-\frac{a_{33}b_{41}}{a_{32}},\quad (a_{32}a_{33}\neq 0)$$

– Case 8.2:

$$a_{10}=a_{11}=a_{14}=0,\quad a_{21}=\frac{a_{12}a_{32}a_{34}}{a_{13}a_{22}},\quad a_{23}=\frac{a_{13}a_{22}}{a_{12}},\quad a_{24}=-\frac{a_{32}a_{34}}{a_{22}},\quad a_{31}=-\frac{a_{12}a_{34}}{a_{13}}$$

$$a_{33}=\frac{a_{13}a_{32}}{a_{12}},\quad b_{32}=b_{33}=0,\quad b_{34}=-\frac{a_{13}b_{31}}{a_{12}},\quad b_{42}=b_{43}=0,\quad b_{44}=-\frac{a_{13}b_{41}}{a_{12}},\quad (a_{12}a_{13}a_{22}\neq 0)$$

– Case 8.3:

$$a_{10}=a_{11}=0,\quad a_{12}=-\frac{a_{13}a_{31}}{a_{34}},\quad a_{14}=0,\quad a_{21}=-\frac{a_{31}a_{32}}{a_{22}},\quad a_{23}=-\frac{a_{22}a_{34}}{a_{31}},\quad a_{24}=-\frac{a_{32}a_{34}}{a_{22}}$$

$$a_{30}=\frac{a_{20}a_{32}}{a_{22}},\quad a_{33}=-\frac{a_{32}a_{34}}{a_{31}},\quad b_{32}=b_{33}=0,\quad b_{34}=\frac{a_{34}b_{31}}{a_{31}}$$

$$b_{42}=b_{43}=0,\quad b_{44}=\frac{a_{34}b_{41}}{a_{31}},\quad (a_{22}a_{31}a_{34}\neq 0)$$

where other parameters are arbitrary real constants. By substituting the parameters in Cases 8.1-8.3 into the solution (8) and using the transformation (2), we can obtain the interaction solutions between the high-order lump-type solution and cos-cosh periodic wave solution of eq. (1).

Remark 2.5: In addition to Cases 8.1-8.3, we can get the special results of Case 4.3 ($\eta_3 = \eta_1$, $\eta_4 = \eta_2$), Case 5.1 ($\eta_3 = \eta_4 = \eta_1$, $a_{10} = 0$), Case 6.2 ($b_{11} = 0$), Case 6.5 ($b_{12} = 0$), and Cases 9.1-9.2 ($\eta_4 = \eta_3$).

Between lumps and cos periodic wave solution (or hyperbolic function solution)

When $m_i = 0$ ($i = 1, 2, 4$), $m_3 \neq 0$, or $m_i = 0$ ($i = 1, 2, 3$), $m_4 \neq 0$, $\eta_4 = \eta_3$), in eq. (8), the solution (8) represents $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_3\cos\eta_3$ or $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_4\cosh\eta_3$.

– Case 9.1:

$$a_{10} = a_{11} = 0, \quad a_{13} = \frac{a_{12}a_{33}}{a_{32}}, \quad a_{14} = 0, \quad a_{22} = -\frac{a_{31}a_{32}}{a_{21}}, \quad a_{23} = -\frac{a_{31}a_{33}}{a_{21}}, \quad a_{24} = -\frac{a_{21}a_{33}}{a_{32}}$$

$$a_{30} = \frac{a_{20}a_{31}}{a_{21}}, \quad a_{34} = -\frac{a_{31}a_{33}}{a_{32}}, \quad b_{31} = 0, \quad b_{33} = \frac{a_{33}b_{32}}{a_{32}}, \quad b_{34} = 0, \quad (a_{21}a_{32} \neq 0)$$

– Case 9.2:

$$a_{10} = a_{11} = a_{14} = a_{22} = a_{23} = 0, \quad a_{24} = -\frac{a_{13}a_{21}}{a_{12}}, \quad a_{30} = -\frac{a_{12}a_{20}a_{34}}{a_{13}a_{21}}, \quad a_{31} = -\frac{a_{12}a_{34}}{a_{13}}$$

$$a_{32} = a_{33} = b_{31} = 0, \quad b_{33} = \frac{a_{13}b_{32}}{a_{12}}, \quad b_{34} = 0, \quad (a_{12}a_{13}a_{21} \neq 0)$$

where other parameters are arbitrary real constants. By substituting the parameters in Cases 9.1 and 9.2 into the solution (8) and using the transformation (2), we can obtain the interaction solutions between the high-order lump-type solution and cos periodic wave solution (or solitary wave solution) of eq. (1).

Remark 2.6: In addition to Cases 9.1-9.2, we can get the special results of Case 4.3 ($\eta_3 = \eta_1$, $b_{21} = 0$), Case 5.1 ($\eta_3 = \eta_1$, $a_{10} = 0$), Case 6.2 ($b_{11} = b_{41} = 0$), Case 6.5 ($a_{10} = b_{11} = b_{41} = 0$), and Cases 8.1-8.3 ($b_{41} = 0$).

Step 4. By substituting the parameters a_0, a_{ik}, b_{jk}, and m_j in Cases 1.1- 9.2 into eq. (8) and using eq. (2), we can obtain abundant exact analytical solutions and novel interaction phenomena of the generalized (3+1)-D shallow water eq. (1).

As the example, substituting the results of Case 6.1 to eq. (8), we can get the exact solution f of the generalized bilinear shallow water eq. (4):

$$f_2 = a_0 + \left(a_{12}y - \frac{a_{12}a_{24}}{a_{21}}z + a_{10}\right)^4 + (a_{21}x + a_{24}t + a_{20})^2 + \left(a_{32}y - \frac{a_{24}a_{32}}{a_{21}}z + a_{30}\right)^2 +$$
$$+ m_1\exp\left(b_{11}x + \frac{a_{24}b_{11}}{a_{21}}t + b_{10}\right) + m_1\exp\left(-b_{11}x - \frac{a_{24}b_{11}}{a_{21}}t - b_{10}\right) +$$
$$+ m_1\cos\left(b_{31}x + \frac{a_{24}b_{31}}{a_{21}}t + b_{30}\right) + m_1\cosh\left(b_{41}x + \frac{a_{24}b_{41}}{a_{21}}t + b_{40}\right) \quad (9)$$

By using the transformation (2), we get the following interaction solution between the high-order lump-type solution and three-wave solution of the generalize shallow water eq. (1):

$$u(x,y,z,t)=\frac{2f_x(x,y,z,t)}{f(x,y,z,t)} \tag{10}$$

where $f(x, y, z, t)$ is given in eq. (9).

In order to exhibit the dynamical characteristics of these waves, we can plot various 3-D, contour and density plots. These Figures can show the physical properties, structures and the energy distribution for the exact solutions (10). The phenomenon of the interaction solution is very strange and analogous to rogue wave. The process of interaction changes the amplitudes, shapes and velocities of both waves. This type of interaction solutions provide a method to forecast the appearance of rogue waves, such as financial rogue wave, optical rogue wave and plasma rogue wave, through analyzing the relations between lump wave part and soliton wave part.

Conclusions

In this paper, we gave a novel form of exact analytical solution to the generalized (3+1)-D shallow water equation. To search for various kinds of exact analytical solutions, we are free to choose the values of N, M, and the basis function $g(\eta)$ in eq. (5), such as: lump solution ($N = 2$, $n_1 = n_2 = 1$, $m_i = 0$), lump-type solution ($N = 3$, $n_1 = n_2 = n_3 = 1$, $m_i = 0$), high-order lump solution ($N = 2$, $n_1 = n_2 = 2$, $m_i = 0$), high-order lump-type solution ($N = 3$, $n_1 = 2$, $n_2 = n_3 = 1$, $m_i = 0$), lump-kink solution ($N = 2$, $n_1 = n_2 = 1$; $M = 1$, $g(\eta) = e^{\eta}$), and lump-soliton solution ($N = 2$, $n_1 = n_2 = 1$; $M = 1$, $g(\eta) = \cosh\eta$), *etc.*

As the example, by choosing the basic functions $g_1(\eta_1) = e^{\eta_1}$, $g_2(\eta_2) = e^{\eta_2}$, $g_3(\eta_3) = \cos\eta_3$, $g_4(\eta_4) = \cosh\eta_4$, we successfully constructed abundant exact analytical solutions of the generalized (3+1)-D shallow water equation based on the generalized bilinear method, and these solutions contained the high-order lump-type solutions, the three-wave solutions, the breather solutions, the interaction solution between high-order lump-type solutions and soliton solutions. These solutions will greatly expand the exact solutions of the generalized (3+1)-D shallow water equation on the existing [14]. These new solutions are significant to understand the propagation processes for non-linear waves in fluid mechanics and the explanation of some special physical problems.

By using eq. (5), we can construct other rational solution and their interaction solution to the generalized (3+1)-D shallow water equation. Such as, when $g_1(\eta_1) = e^{\eta_1}$, $g_2(\eta_2) = e^{\eta_2}$, $g_3(\eta_3)= \sin\eta_3$, $g_4(\eta_4) = \sinh\eta_4$ or $g_3(\eta_3) = \tan\eta_3$, $g_4(\eta_4) = \tanh\eta_4$, we can obtain the interaction solution between rational solution and soliton solutions, periodic wave solution. But due to the lack of space, we will discuss these solutions in another paper. The method can be used for many other NLEE in mathematical physics.

Acknowledgment

This work is supported by the National Natural Science Foundation of China (12061054, 11661060), Program for Young Talents of Science and Technology in Universities of Inner Mongolia Autonomous Region (NJYT-20-A06) and the Natural Science Foundation of Inner Mongolia Autonomous Region of China (2018LH01013).

References

[1] Ablowitz, M. J., Clarkson, P. A., *Solitons, Non-linear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, Mass., USA, 1991
[2] Hirota, R., Satsuma, J., Soliton Solution of a Coupled KdV Equation, *Phys. Lett. A*, *85* (1981), 8-9, pp. 407-408
[3] Ma, W. X., Generalized Bilinear Differential Equations, *Stud. Non-linear Sci.*, *2* (2011), 4, pp. 140-144
[4] Fan, E. G., Extended Tanh-Function Method and Its Applications to Non-linear Equations, *Phys. Lett. A*, *277* (2000), 4-5, pp. 212-218
[5] He, J. H., Wu, X. H., Exp-Function Method for Non-linear Wave Equations, *Chaos, Solitons and Fractals*, *30* (2006), 3, pp. 700-708
[6] Wang, M. L., *et al.*, Applications of a Homogeneous Balance Method to Exact Solutions of Non-linear Equations in Mathematical Physics, *Phys. Lett. A*, *216* (1996), 1-5, pp. 67-75
[7] Wang, M. L., *et al.*, The (G'/G)-Expansion Method and Travelling Wave Solutions of Non-linear Evolution Equations in Mathematical Physics, *Phys. Lett. A*, *372* (2008), 4, pp. 417-423
[8] Zhang, R. F., Bilige, S. D., Bilinear Neural Network Method to Obtain the Exact Analytical Solutions of Non-linear Partial Differential Equations and its Application to p-gBKP Equation, *Non-Linear Dyn.*, *95* (2019), 4, pp. 3041-3048
[9] Lu, X., *et al.*, Solitary Waves with the Madelung Fluid Description: A Generalized Derivative Non-linear Schrodinger Equation, *Commun. Non-linear Sci. Numer. Simul*., *31* (2016), 1-3, pp. 40-46
[10] Zhang, X. E., Chen, Y., General High-Order Rogue Waves to Non-linear Schrodinger-Boussinesq Equation with the Dynamical Analysis, *Non-Linear Dyn.*, *93* (2018),4, pp. 2169-2184
[11] Xu, H. X., *et al.*, Breathers and Solitons on two Different Backgrounds in a Generalized Coupled Hirota System with Four Wave Mixing, *Phys. Lett. A*, *382* (2018), 26, pp. 1738-1744
[12] Zhang, R. F., *et al.*, New Periodic Wave, Cross-Kink Wave and the Interaction Phenomenon for the Jimbo-Miwa-Like Equation, *Comput. Math. Appl.*, *78* (2019), 3, pp. 754-764
[13] Liu, J. G., *et al.*, New Three-Wave Solutions for the (3+1)-Dimensional Boiti-Leon-Manna-Pempinelli Equation, *Non-Linear Dyn.*, *88* (2017), 1, pp. 655-661
[14] Zhang, Y., *et al.*, Rational Solutions and Lump Solutions to the Generalized (3+1)-Dimensional Shallow Water-Like Equation, *Comput. Math. Appl.*, *73* (2017), 2, pp. 246-252
[15] Ma, W. X., Lump Solutions to the Kadomtsev-Petviashvili Equation, *Phys. Lett. A*, *379* (2015), 36, pp. 1975-1978
[16] Liu, J. G., Lump-Type Solutions and Interaction Solutions for the (2+1)-Dimensional Generalized Fifth-Order KdV Equation, *Appl. Math. Lett.*, *86* (2018), Dec., pp. 36-41
[17] Lü, J. Q., *et al.*, The Study of Lump Solution and Interaction Phenomenon to (2+1)-Dimensional Generalized Fifth-Order Kdv Equation, *Non-Linear Dyn.*, *91* (2018), 3, pp. 1669-1676
[18] Gao, X. Q., *et al.*, Abundant Lump Solutions and Interaction Solutions of the (3+1)-Dimensional Kadomtsev-Petviashvili Equation, *Therm. Sci.*, *23* (2019), 4, pp. 2437-2445

Paper submitted: November 23, 2019
Paper revised: July 3, 2020
Paper accepted: July 5, 2020