ANALYTICAL METHODS FOR NON-LINEAR FRACTIONAL KOLMOGOROV-PETROVSKII-PISKUNOV EQUATION
Soliton Solution and Operator Solution

by

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Kolmogorov-Petrovskii-Piskunov equation can be regarded as a generalized form of the Fitzhugh-Nagumo, Fisher and Huxley equations which have many applications in physics, chemistry and biology. In this paper, two fractional extended versions of the non-linear Kolmogorov-Petrovskii-Piskunov equation are solved by analytical methods. Firstly, a new and more general fractional derivative is defined and some properties of it are given. Secondly, a solution in the form of operator representation of the non-linear Kolmogorov-Petrovskii-Piskunov equation with the defined fractional derivative is obtained. Finally, some exact solutions including kink-soliton solution and other solutions of the non-linear Kolmogorov-Petrovskii-Piskunov equation with Khalil et al.'s fractional derivative and variable coefficients are obtained. It is shown that the fractional-order affects the propagation velocity of the obtained kink-soliton solution.

Key words: *fractional derivative, analytical method, exact solution, fractional Kolmogorov-Petrovskii-Piskunov equation, operator solution, kink-soliton solution*

Introduction

Fractional calculus [1], which could be traced back to a letter from Leibniz to L'Hospital dated on September 30, 1695, extends the ordinary differention and integration to arbitrary order. Derivatives and integrals of non-integer order have many important applications in physics, chemistry, engineering, mechanics, materials, *etc.* It is worth mentioning that the super-derivative operator $D=\theta\partial_x+\partial_\theta$ in super-symmetry theory can be regarded as a half-order derivative operator because of $D^2=\partial_x$ [2], here θ is an anti-commuting variable. Compared with the classical integer-order differential models, the fractional systems have sometimes their own advantages, especially in modelling the mechanical and electrical properties of real materials [3]. With the development of the theory of fractals, fractal derivatives – a kind of fractional derivatives provide further perspectives in the simulation of dynamical processes in self-similar and porous structures [3, 4]. He [5] defined a new fractal derivative to easily deal with the discontinuous problems in engineering. He [6] explained the fractal calculus from a geometric point of view. Li *et al.* [7] successfully solved a paradox by using a fractal derivative which modifies the surface coverage model in an electrochemical arsenic sensor.

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Although the fractional calculus has a history of more than 300 years, it has become a hot topic only in recent years owing to its more and more applications in various fields. Brockmann *et al.* [8] derived a space-time fractional diffusion equation by which they studied the scaling laws of human travel. Vosika *et al.* [9] used the fractional calculus to model the electrical properties of biological systems. Zhang *et al.* [10] proposed the variable separation method for non-linear time-fractional biological population model. Wang *et al.* [11] adopted the fractal calculus to reveal the thermal properties of the polar bear hairs. Wang and Deng [12] studied the tsunami traveling in an unsmooth boundary by using the fractal derivative. As pointed by Podlubny [3] that the fractional calculus is very suitable for describing the properties of polymers. Solitons, as one of self-localized non-linear excitations [13], are fundamental and inherent features of quasi-one-dimensional conducting polymers [14]. In this way, polymer could be said to be a bridge connecting soliton theory with fractional calculus. Solitons, together with chaos and fractals, constitute three main research contents of non-linear science. In recent years, the application of solitons in Bose-Einstein condensate [15, 16] and optical microresonators [17], soliton molecules [13, 18], rouge wave solutions [19, 20] and long-time asymptotic of soliton equations via the Riemann-Hilbert approach [21] are some hot spots in the academic frontier of soliton research. It is of great significance to extend fractional calculus to soliton theory, which has attracted the attention of researchers. Fujioka *et al.* [22] studied the fractional optical solitons determined by a fractional Schrodinger equation. Zhang and Zhang [23] first proposed the fractional sub-equation method for solving non-linear fractional PDE. Yang *et al.* [24] modeled the fractal waves by means of a local fractional Korteweg-de Vries equation. Zhang *et al.* [25-27] extended the inverse scattering transform [28], Hirota's bilinear method [29] and the operator method [30] to non-linear fractional evolution equations. On the other hand, more and more attention has been paid to finding conformable fractional derivatives, especially those suitable for application has become one of the research focuses of fractional calculus. It is well known that there are various definitions [1, 2] of the fractional derivatives such as Grunwald-Letnikov fractional derivative, Riemann-Liouville fractional derivative, and Caputo fractional derivative. Kolwankar and Gangal [31] proposed the concept of local fractional derivative which can be used to deal with non-differentiable problems. Khalil *et al.* [32] defined a new and natural fractional derivative. It is very interesting that a non-differentiable function of integer order could exist Khalil *et al.*'s fractional-order derivative.

This paper has two purposes: one is to define a new and more general fractional derivative – two-parameter fractional derivative denoted by:

$$\mathrm{D}_{\beta_j}^{(\alpha)} \quad (0<\alpha\le 1,\quad 0<\beta_1\le 1,\quad 1\le\beta_2\in\mathbb{Z},\quad j=1,2)$$

two special cases of which are $\mathrm{D}_1^{(\alpha)}$ – the local fractional derivative [33] and $\mathrm{D}_{\beta_1}^{(1)}$ – Khalil *et al.*'s [32] fractional derivative; the other is to construct operator solution and soliton solution of two new fractional extensions:

$$\mathrm{D}_{\beta_2,t}^{(\alpha)}u-\mathrm{D}_{\beta_2,x}^{(2\alpha)}u+f(u)=0 \tag{1}$$

$$\mathrm{D}_{\beta_1,t}^{(1)}u-u_{xx}+\gamma(t)u+\delta(t)u^2+\zeta(t)u^3=0 \tag{2}$$

of the following non-linear Kolmogorov-Petrovskii-Piskunov (KPP) equation [34-36]:

$$u_t - u_{xx} + au + bu^2 + cu^3 = 0\,,\ x \in \mathbb{R}\,,\ t \in [0,\infty) \tag{3}$$

where $f(u)$ is a polynomial of u, $\gamma(t)$, $\delta(t)$, and $\zeta(t)$ are all the differentiable functions of t, while *a, b,* and *c* are all constants. It is Kolmogorov *et al.* [37] who first presented the KPP equation in a genetics model used for a population to spread an advantageous gene. Noting that the KPP equation includes as special cases (see [34] and references therein) the Fitzhugh-Nagumo equation $u_t - u_{xx} + au - (a+1)u^2 + u^3 = 0$ with $0 < a < 1$ has appeared in electronic circuitry, nerve impulses and population genetics, the Fisher equation $u_t - u_{xx} - u + u^2 = 0$ has appeared in chemical kinetics, population dynamics, mutant gene and neurophysiology, and the Huxley equation $u_t - u_{xx} + au - (a+1)u^2 + u^3 = 0$ with $a \neq 0$ has appeared in living muscles. In recent years, researchers have been devoted to the study of the fractional extensions like those [38-42] of the KPP equation. More specifically, Chu *et al.* [38] obtained kink soliton solutions of the space-fractional KPP equation via the fractional sub-equation method. Qin *et al.* [39] obtained vector fields, symmetry reductions, conservation laws and analytical solutions of the time-fractional KPP equation by employing the fractional Lie group analysis method and the power series method. Hashemi *et al.* [40] obtained symmetry properties and exact solutions of the time-fractional derivative KPP equation by means of the Lie symmetry approach and the simplest equation method. Khan and Altaf [41] obtained an approximate solution of the time-fractional KPP equation by introducing the reduced differential transform method. Veeresha *et al.* [42] obtained approximate solutions of the time-fractional KPP equation with three distinct initial conditions by the *q*-homotopy analysis transform method. But, as far as we know these two fractional extended eqs. (1) and (2) are novel due to the new fractional derivatives $\mathrm{D}^{(\alpha)}_{\beta_2,t}$ and $\mathrm{D}^{(2\alpha)}_{\beta_2,x}$ adopted in eq. (1) and the variable coefficients $\gamma(t)$, $\delta(t)$ and $\zeta(t)$ embedded into eq. (2).

Definitions and properties

Definition 1. Combined with the fractal geometry, the local fractional derivative [33] of $\phi(\mu)$ of order α at the point u_0 is defined:

$$\mathrm{D}^{(\alpha)}\phi(\mu_0) = \left.\frac{d^\alpha \phi(\mu)}{d\mu^\alpha}\right|_{\mu=\mu_0} = \lim_{\mu\to\mu_0} \frac{\Delta^\alpha[\phi(\mu) - \phi(\mu_0)]}{(\mu-\mu_0)^\alpha} \tag{4}$$

where $\Delta^\alpha[\phi(\mu) - \phi(\mu_0)] \cong \Gamma(1+\alpha)[\phi(\mu) - \phi(\mu_0)]$.

Definition 2. Khalil *et al.*'s [32] conformable fractional derivative is defined:

$$\mathrm{T}_\alpha(f)(t) = f^{(\alpha)}(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{5}$$

for a given function $f(t)$: $[0,\infty) \to \mathbb{R}$ and the fractional-order $\alpha \in (0,1)$.

Definition 3. A new and more general fractional derivative is defined:

$$\mathrm{D}^{(\alpha)}_{\beta_j} u(t) = u^{(\alpha)}_{\beta_j}(t) = \Gamma(1+\alpha) \lim_{\varepsilon\to 0} \frac{u(t+\varepsilon t^{1-\beta_j}) - u(t)}{\varepsilon^\alpha}\,,\ 0 < \alpha \le 1 \tag{6}$$

for the given function $u(t)$: $t \in [0,\infty) \to \mathbb{R}$ $(j = 1, 0 < \beta_1 \le 1)$ or $t \in (0,\infty) \to \mathbb{R}$ $(j = 2, 1 \le \beta_2 \in \mathbb{Z})$.

Definition 4. Khalil *et al.*'s [32] conformable fractional integral for $\alpha \in (0,1)$ is defined by means of the usual Riemann improper integral:

$$\mathrm{I}_{\beta_1}^{a} f(t) = \int_a^t x^{\beta_1 - 1} f(x)\,\mathrm{d}x \tag{7}$$

Property 1. The fractional derivatives $\mathrm{D}_{\beta_j}^{(\alpha)} u(t)$ and $\mathrm{D}_1^{(\alpha)} u(x)$ have the relationship $\mathrm{D}_{\beta_j}^{(\alpha)} u(t) = t^{\alpha(1-\beta_j)} \mathrm{D}_1^{(\alpha)} u(t)$.

Proof. We suppose $p = \varepsilon t^{1-\beta_j}$, namely $\varepsilon^\alpha = p^\alpha t^{\alpha(\beta_j - 1)}$. Then using *Definition 3* yields:

$$\mathrm{D}_{\beta_j}^{(\alpha)} u(x) = \Gamma(1+\alpha) \lim_{\varepsilon \to 0} \frac{u(x+p) - u(x)}{p^\alpha t^{\alpha(\beta_j - 1)}} = t^{\alpha(1-\beta_j)} \mathrm{D}_1^{(\alpha)} u(t) \tag{8}$$

Property 2. The new and more general fractional derivative (6) has many useful properties, some of them are listed:

$$\mathrm{D}_{\beta_2}^{(\alpha)} t^{(\beta_2 + k)\alpha} = \frac{\Gamma[1 + (\beta_2 + k)\alpha]}{\Gamma[1 + (\beta_2 + k - 1)\alpha]} t^{k\alpha} \tag{9}$$

$$\mathrm{D}_{\beta_j}^{(\alpha)} [q u(t) + h v(t)] = q \mathrm{D}_{\beta_j}^{(\alpha)} u(t) + h \mathrm{D}_{\beta_j}^{(\alpha)} v(t)$$

$$\mathrm{D}_{\beta_j}^{(\alpha)} [u(t) v(t)] = [\mathrm{D}_{\beta_j}^{(\alpha)} u(t)] v(t) + u(t) [\mathrm{D}_{\beta_j}^{(\alpha)} v(t)] \tag{10}$$

$$\mathrm{D}_{\beta_j}^{(\alpha)} \frac{u(t)}{v(t)} = \frac{[\mathrm{D}_{\beta_j}^{(\alpha)} u(t)] v(t) - u(t) [\mathrm{D}_{\beta_j}^{(\alpha)} v(t)]}{v^2(t)}$$

$$\mathrm{D}_{\beta_j}^{(\alpha)} \left[t^{\beta_2 \alpha} E_{\alpha, 1+\beta_2\alpha}(t^\alpha) \right] = E_{\alpha, 1+(\beta_2 - 1)\alpha}(t^\alpha) \tag{11}$$

$$\mathrm{D}_{\beta_j, t}^{(\alpha)} u[v(t)] = \{\mathrm{D}_{1,v}^{(1)} u[v(t)]\} \mathrm{D}_{\beta_j, t}^{(\alpha)} v(t) \tag{12}$$

where k, q, and h are constants, and $E_{\alpha,\beta}(\cdot)$ is the two-parameter Mittag-Lefeler function [3].

Proof. We just need to prove eq. (9) in the case of $k = 0$ and eq. (12), the proof of the others is direct. When $k = 0$ and $\beta_2 = 1$, it is easy to see eq. (9) holds. For any $1 < \beta_2 \in \mathbb{Z}$, we have:

$$\mathrm{D}_{\beta_2}^{(\alpha)} t^{\beta_2 \alpha} = t^{(1-\beta_2)\alpha} \Gamma(1+\alpha) \lim_{\varepsilon \to 0} \frac{(t+\varepsilon)^{\beta_2 \alpha} - t^{\beta_2 \alpha}}{\varepsilon^\alpha} =$$

$$= t^{(1-\beta_2)\alpha} \Gamma(1+\alpha) \lim_{\varepsilon \to 0} \frac{t^{\beta_2 \alpha} + \dfrac{\Gamma(1+\beta_2 \alpha)}{\Gamma(1+\alpha)\Gamma[1+(\beta_2 - 1)\alpha]} t^{(\beta_2 - 1)\alpha} \varepsilon^\alpha + \cdots - t^{\beta_2 \alpha}}{\varepsilon^\alpha} =$$

$$= \frac{\Gamma(1+\beta_2 \alpha)}{\Gamma[1+(\beta_2 - 1)\alpha]} \tag{13}$$

which can be rewritten as eq. (9) in the case of $k = 0$.

We next prove eq. (12). Supposing:

$$g = v(t), \quad \Delta g = v(t+\Delta t) - v(t) \tag{14}$$

$$H(\Delta t) = \begin{cases} \dfrac{u(g+\Delta g)-u(g)}{\Delta g}, & \Delta g \neq 0 \\ \mathrm{D}_{1,g}^{(1)} u(g), & \Delta g = 0 \end{cases} \tag{15}$$

and then using *Definition 1* and *Property 1*, we have:

$$\mathrm{D}_{\beta_j}^{(\alpha)} u[v(t)] = t^{\alpha(\beta_j - 1)} \Gamma(1+\alpha) \lim_{\Delta t \to 0} \frac{u[v(t+\Delta t)] - u[v(t)]}{(\Delta t)^{\alpha}} =$$

$$= t^{\alpha(\beta_j - 1)} \Gamma(1+\alpha) \lim_{\Delta t \to 0} \frac{u(g+\Delta g)-u(g)}{(\Delta t)^{\alpha}} = t^{\alpha(\beta_j - 1)} \Gamma(1+\alpha) \lim_{\Delta t \to 0} \frac{H(\Delta t)\Delta g}{(\Delta t)^{\alpha}} =$$

$$= t^{\alpha(\beta_j - 1)} H(0) \Gamma(1+\alpha) \lim_{\Delta t \to 0} \frac{\Delta v}{(\Delta t)^{\alpha}} = \{\mathrm{D}_{1,v}^{(1)} u[v(t)]\} \mathrm{D}_{\beta_j,t}^{(\alpha)} v(t) \tag{16}$$

Operator solution

Theorem 1. If we take the transformation:

$$\xi = k \frac{\Gamma[1+(2\beta_2 - 1)\alpha]\Gamma[1+(\beta_2 - 1)\alpha]}{\Gamma(1+2\beta_2\alpha)\Gamma(1+\beta_2\alpha)} x^{2\beta_2\alpha} + w \frac{\Gamma[1+(\beta_2 - 1)\alpha]}{\Gamma(1+\beta_2\alpha)} t^{\beta_2\alpha}, \; x,t \in (0,\infty) \tag{17}$$

and suppose that:

$$u(\xi) = \theta, \quad \mathrm{D}_1^{(1)} u(\xi)|_{\xi = v} = \vartheta \tag{18}$$

then the fractional KPP eq. (1) has a solution in the form of operator representation:

$$u = \sum_{k=0}^{\infty} \frac{(\xi - v)^k}{k!} \left\{ \mathrm{D}_{1,v}^{(1)} + \vartheta \mathrm{D}_{1,\theta}^{(1)} + \left[\frac{w}{k^2}\vartheta + \frac{1}{k^2} f(\theta) \right] \mathrm{D}_{1,\vartheta}^{(1)} \right\}^k \theta, \; v \in R \tag{19}$$

Proof. In view of eqs. (17) and (18), we rewrite eq. (1):

$$\mathrm{D}_{1,\xi}^{(2)} = \frac{w}{k^2}\vartheta + \frac{1}{k^2} f(\theta) \tag{20}$$

We finally arrive at eq. (19) by employing eq. (12) in [30] and eq. (20).

Exact solutions

Theorem 2. Under the conditions:

$$\zeta(t) = \frac{2k^2}{A_1^2} \exp\left\{ -2 \int_0^t \tau^{1-\beta_1} [2k^2\sigma - \gamma(\tau)] \, \mathrm{d}\tau \right\} \tag{21}$$

$$\delta(t) = \pm \frac{3k^2\sqrt{-\sigma}}{\sqrt{A_1^2 \exp\left[2\int_0^t \tau^{1-\beta_1}\gamma(\tau)\mathrm{d}\tau\right] - A_2}} \exp\left\{-\int_0^t \tau^{1-\beta_1}[k^2\sigma - \gamma(\tau)]\mathrm{d}\tau\right\} \tag{22}$$

$$\xi = kx + A_1\int_0^t \tau^{1-\beta_1}\delta(\tau)\exp\left\{\int_0^\tau s^{1-\beta_1}[2k^2\sigma - \gamma(s)]\mathrm{d}s\right\}\mathrm{d}\tau, \quad x,t \in [0,\infty) \tag{23}$$

The KPP equation (2) has the following exact solutions:

$$u = A_1 \exp\left\{\int_0^t t^{1-\beta_1}[2k^2\sigma - \gamma(t)]\mathrm{d}t\right\}\phi(\xi) - \frac{A_1^2\delta(t)}{3k^2}\exp\left\{2\int_0^t \tau^{1-\beta_1}[2k^2\sigma - \gamma(\tau)]\mathrm{d}\tau\right\} \tag{24}$$

where A_1, A_2, and k are constants, and $\phi(\xi)$ can be selected:

$$-\sqrt{-\sigma}\tanh(-\sqrt{-\sigma}\xi) \quad \text{and} \quad -\sqrt{-\sigma}\coth(-\sqrt{-\sigma}\xi) \quad \text{for } \sigma < 0$$

$$\sqrt{\sigma}\tan(\sqrt{\sigma}\xi) \quad \text{and} \quad -\sqrt{\sigma}\cot(\sqrt{\sigma}\xi) \quad \text{for } \sigma > 0$$

$$\text{and } -1/\xi \quad \text{for } \sigma = 0, \text{ respectively}$$

Proof. Firstly, we take the transformation:

$$u = a_1(t)\phi(\xi) + a_0(t), \quad \xi = kx + w(t), \quad x,t \in [0,\infty) \tag{25}$$

where $a_1(t)$, $a_0(t)$, and $w(t)$ are undetermined functions, $\phi(\xi)$ satisfies the Riccati equation $\mathrm{D}_{1,\xi}^{(1)}\phi(\xi) = \sigma + \phi^2(\xi)$ which has five solutions [23]. Secondly, substituting eq. (25) into eq. (2) together with the Riccati equation $\mathrm{D}_{1,\xi}^{(1)}\phi(\xi) = \sigma + \phi^2(\xi)$ and then setting $\phi^i(\xi)$ $(i = 0,1,2,3)$ to zero yields a system of fractional differential equations. Solving the fractional differential equations, we determine $a_1(t)$, $a_0(t)$, and $w(t)$ and hence obtain solution (24) under the conditions (21)-(23).

Let ξ be zero, then we have:

$$\mathrm{D}_{1,t}^{(1)}x(t) = -\frac{A_1}{k}t^{1-\beta_1}\delta(t)\exp\left\{\int_0^t s^{1-\beta_1}[2k^2\sigma - \gamma(s)]\mathrm{d}s\right\} \tag{26}$$

which shows that the fractional order β_1 and the coefficient functions $\gamma(t)$ and $\delta(t)$ influence the propagation velocity $\mathrm{D}_{1,t}^{(1)}x(t)$ of the obtained kink-soliton solution:

$$u = -A_1\sqrt{-\sigma}\exp\left\{\int_0^t t^{1-\beta_1}[2k^2\sigma - \gamma(t)]\mathrm{d}t\right\}\tanh(-\sqrt{-\sigma}\xi) -$$

$$-\frac{A_1^2\delta(t)}{3k^2}\exp\left\{2\int_0^t \tau^{1-\beta_1}[2k^2\sigma - \gamma(\tau)]\mathrm{d}\tau\right\} \tag{27}$$

Conclusion

In summary, we have defined a new fractional derivative which is more general than the local fractional derivative and Khalil *et al.*'s [32] fractional derivative. It is because when $\alpha = 1$ our fractional derivative $D_{\beta_1}^{(1)}$ degenerate into the local fractional derivative [33]. At the same time, our fractional derivative $D_1^{(\alpha)}$ degenerate into Khalil *et al.*'s [32] fractional derivative in the case $\beta_j = 1$. Considering the application of the defined fractional derivative, we analytically solved two new fractional extensions of the known KPP equation and obtained kink soliton solution, operator solution and some other exact solutions. To the best of our knowledge, the obtained solutions are new. In addition, we showed from the mathematical point of view that the propagation velocity of the obtained kink-soliton solution is not only affected by the coefficient functions, but also by the fractional-order.

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