TWO-SCALE TRANSFORM FOR 2-D FRACTAL HEAT EQUATION IN A FRACTAL SPACE

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Original scientific paper
https://doi.org/10.2298/TSCI190918124W

A 2-D fractal heat conduction in a fractal space is considered by He's fractal derivative. The two-scale transform is adopted to convert the fractal model to its differential partner. The homotopy perturbation method is used to find the approximate analytical solution.

Key words: *fractal derivative, heat equation, two-scale transform, Homotopy perturbation method*

Introduction

Fractional calculus has been a hot issue. It has been widely adopted to establish a mathematical model for a very complex physical problem. The most commonly used fractional derivatives are defined [1-5]:

– Riemann-Liouville derivative:

$$\mathrm{D}_x^\alpha[u(x)]=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_0^x (x-t)^{n-\alpha-1}u(t)\mathrm{d}t$$

– Caputo's definition:

$$\mathrm{D}_x^\beta[u(x)]=\frac{1}{\Gamma(n-\beta)}\int_0^x (x-t)^{n-\beta-1}\frac{\mathrm{d}^n u(t)}{\mathrm{d}t^n}\mathrm{d}t$$

– Yang's local fractional derivative definition:

$$\mathrm{D}_x^{(\alpha)}u(x_0,t)=u^{(\alpha)}(x_0,t)=\left.\frac{\mathrm{d}^\alpha u(x,t)}{\mathrm{d}x^\alpha}\right|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^\alpha[u(x,t)-f(x_0,t)]}{(x-x_0)^\alpha}$$

where

$$\Delta^\alpha[u(x,t)-u(x_0,t)]\cong\Gamma(1+\alpha)\Delta[u(x,t)-f(x_0,t)]$$

– He's fractal derivative definition:

$$\frac{\mathrm{D}u(x_0)}{\mathrm{D}x^\alpha}=\Gamma(1+\alpha)\lim_{\substack{x-x_0\to\Delta x\\ \Delta x\neq 0}}\frac{u(x)-u(x_0)}{(x-x_0)^\alpha}$$

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where Δx is the smallest scale for measuring a porous structure or an unsmooth medium.

The He's fractal derivative is very similar to the Yang's local fractional derivative, and it is widely adopted to model complex physical phenomena in fractal space. He [6] gave the geometrical explanation of the fractal derivative, which was widely applied to various discontinuous problems [7-16]. Wang *et al.* [17] also explained the physical properties for Yang's local fractional derivation.

In this paper, we will use the two-scale method [18-20] and He's homotopy perturbation method (HHPM) [21] to find an approximate analytical solution of a 2-D fractal heat equation.

Two-scales transform

The two-scale transform is proposed by He [18-20]. It is a powerful tool to converting a discontinuous space into its continuous one.

Consider a fractal differential equation:

$$\frac{Du}{Dt^\beta} + \frac{Du}{Dx^\alpha} + F(u) = 0 \tag{1}$$

where Du/Dt^β and Du/Dx^α are He's fractal derivatives.

We use the two-scale transform;

$$T = t^\beta \tag{2}$$

$$X = x^\alpha \tag{3}$$

Therefore, eq. (1) can be rewritten:

$$\frac{Du}{DT} + \frac{Du}{DX} + F(u) = 0 \tag{4}$$

He's Homotopy perturbation method

He's homotopy perturbation method was proposed in 1998 [21] and it becomes a famous analytical method to various non-linear problems [22-26]. Consider:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{5}$$

with the boundary condition:

$$B\left(u, \frac{\partial u}{n}\right) = 0, \; r \in \Gamma \tag{6}$$

where A is a general differential operator, B – a boundary operator, $f(r)$ – a known analytical function, and Γ – the boundary of the domain Ω.

We can divide operator A into N and L, where N is a non-linear and L is a linear operator.

Therefore eq. (5) can be written:

$$L(u) + N(u) - f(r) = 0 \tag{7}$$

According to the homotopy technique, we can construct a homotopy as $\mu(r, q)$:$\Omega\times[0, 1] \rightarrow R$ which satisfies:

$$T(\mu, q) = (1-q)[L(\mu) - L(u_0)] + q[A(\mu) - f(r)] = 0 \tag{8}$$

or

$$T(\mu,\lambda)=L(\mu)-L(u_0)+\lambda L(u_0)+\lambda[N(\mu)-g(r)]=0 \tag{9}$$

where $\lambda\in[0,1]$ is an embedding parameter and u_0 – an initial approximation of eq. (5), which satisfies the boundary conditions. Using eqs. (8) and (9), we can obtain:

$$T(\mu,0)=L(\mu)-L(u_0)=0 \tag{10}$$

$$T(\mu,1)=A(\mu)-f(r)=0 \tag{11}$$

The changing process of q from zero to unity is just that of μ (r, q) from $u_0(r)$ to $u(r)$. This is called deformation in topology. The $L(\mu)-L(u_0)$ and $A(\mu)-f(r)$ are called homotopy. Using the HPM, we can first apply the embedding parameter q as a small parameter and assume that the solution of eqs. (8) and (9) can be written into a power series in term of q:

$$\mu=\mu_0+\lambda\mu_1+\lambda^2\mu_2+\lambda^3\mu_3+\lambda^4\mu_4+\cdots \tag{12}$$

Setting $\lambda=1$ in eq. (12), we obtain:

$$u=\lim_{\lambda\to 1}\mu=\mu_0+\mu_1+\mu_2+\mu_3+\mu_4+\cdots \tag{13}$$

Application

Consider the 2-D inhomogeneous fractional heat equation:

$$\frac{\mathrm{D}u}{\mathrm{D}^{\alpha}t}-\frac{\mathrm{D}u}{\mathrm{D}^{2\beta}x}-\frac{\mathrm{D}u}{\mathrm{D}^{2\gamma}y}=\sin x\sin ye^{-t}-4 \tag{14}$$

with the initial condition:

$$u(x,y,0)=\sin x\sin y+x^2+y^2,\quad x,y\in(0,1) \tag{15}$$

where $\mathrm{D}u/\mathrm{D}^{\alpha}t$ is He's fractal derivative [27-37].

Step one: We adopt two-scale transform:

$$T=t^{\alpha} \tag{16}$$

$$X=x^{\beta} \tag{17}$$

$$Y=y^{\gamma} \tag{18}$$

The eq. (14) can be written:

$$\frac{\mathrm{D}u}{\mathrm{D}T}-\frac{\mathrm{D}u}{\mathrm{D}^{2}X}-\frac{\mathrm{D}u}{\mathrm{D}^{2}Y}=\sin X\sin Ye^{-T}-4$$

with the initial condition:

$$u(X,Y,0)=\sin X\sin Y+X^2+Y^2,\quad X,Y\in(0,1)$$

Step two: We use the HHPM, and construct the following equation:

$$\frac{\mathrm{D}u}{\mathrm{D}T}-\frac{\mathrm{D}u_0}{\mathrm{D}T}-\lambda\left[\frac{\mathrm{D}u}{\mathrm{D}^{2}X}+\frac{\mathrm{D}u}{\mathrm{D}^{2}Y}-\sin X\sin Y.\exp(-\lambda T)-\frac{\mathrm{D}u_0}{\mathrm{D}T}\right]=0 \tag{19}$$

So, we have the following form:

$$\lambda_0 : \frac{\mathrm{D}u_0}{\mathrm{D}T} - \frac{\mathrm{D}u_0}{\mathrm{D}T} = 0 \tag{20}$$

$$\lambda_1 : \frac{\mathrm{D}u_1}{\mathrm{D}T} - \frac{\mathrm{D}u_0}{\mathrm{D}^2 X} - \frac{\mathrm{D}u_0}{\mathrm{D}^2 Y} - \sin X \sin Y + 4 = 0 \tag{21}$$

$$\lambda_2 : \frac{\mathrm{D}u_2}{\mathrm{D}T} - \frac{\mathrm{D}u_1}{\mathrm{D}^2 X} - \frac{\mathrm{D}u_1}{\mathrm{D}^2 Y} + \sin X \sin YT = 0 \tag{22}$$

$$\lambda_3 : \frac{\mathrm{D}u_3}{\mathrm{D}T} - \frac{\mathrm{D}u_2}{\mathrm{D}^2 X} - \frac{\mathrm{D}u_2}{\mathrm{D}^2 Y} - \sin X \sin Y \frac{T^2}{2!} = 0 \tag{23}$$

$$\lambda_4 : \frac{\mathrm{D}u_4}{\mathrm{D}T} - \frac{\mathrm{D}u_3}{\mathrm{D}^2 X} - \frac{\mathrm{D}u_3}{\mathrm{D}^2 Y} - \sin X \sin Y \frac{T^3}{3!} = 0 \tag{24}$$

$$\lambda_5 : \frac{\mathrm{D}u_5}{\mathrm{D}T} - \frac{\mathrm{D}u_4}{\mathrm{D}^2 X} - \frac{\mathrm{D}u_4}{\mathrm{D}^2 Y} - \sin X \sin Y \frac{T^4}{4!} = 0 \tag{25}$$

$$\vdots$$

$$\lambda_9 : \frac{\mathrm{D}u_9}{\mathrm{D}T} - \frac{\mathrm{D}u_8}{\mathrm{D}^2 X} - \frac{\mathrm{D}u_8}{\mathrm{D}^2 Y} - \sin X \sin Y \frac{T^8}{8!} = 0 \tag{26}$$

$$\vdots$$

We start with the initial condition:

$$u(X,Y,0) = \sin X \sin Y + X^2 + Y^2 \tag{27}$$

We have the following results:

$$u_1(X,Y,T) = -T \sin X \sin Y \tag{28}$$

$$u_2(X,Y,T) = \frac{T^2}{2!} \sin X \sin Y \tag{29}$$

$$u_3(X,Y,T) = -\frac{T^3}{3!} \sin X \sin Y \tag{30}$$

$$u_4(X,Y,T) = \frac{T^4}{4!} \sin X \sin Y \tag{31}$$

$$u_5(X,Y,T) = -\frac{T^5}{5!} \sin X \sin Y \tag{32}$$

$$\vdots$$

Therefore, the approximate analytical solution of eq. (17) is:

$$u(X,Y,T) = \sin X \sin Y + X^2 + Y^2 - T \sin X \sin Y + \frac{T^2}{2!} \sin X \sin Y - \frac{T^3}{3!} \sin X \sin Y +$$

$$+ \frac{T^4}{4!} \sin X \sin Y + \cdots + (-1)^{n-1} \frac{T^n}{n!} + \cdots \tag{33}$$

Using the eqs. (16)-(18), we obtain the approximate analytical solution of eq. (14):

$$u(x,y,t)=\sin x^{\beta}\sin y^{\gamma}+x^{2\beta}+y^{2\gamma}-t^{\alpha}\sin x^{\beta}\sin y^{\gamma}+\frac{t^{2\alpha}}{2!}\sin x^{\beta}\sin y^{\gamma}-$$

$$-\frac{t^{3\alpha}}{3!}\sin x^{\beta}\sin y^{\gamma}+\frac{t^{4\alpha}}{4!}\sin x^{\beta}\sin y^{\gamma}+\cdots+(-1)^{n-1}\frac{t^{n\alpha}}{n!}+\cdots \tag{34}$$

Remark 1: When $\alpha=\beta=\gamma=1$, the eq. (14) is the classical heat equation, and its exact solution is [14]:

$$u(x,y,t)=x^2+y^2+\mathrm{e}^{-t}\sin x\sin y$$

In figs. 1 and 2, we show the 11-order approximate analytical solution and its exact solution of eq. (14) when $\alpha=\beta=\gamma=1$ at $t=6$. We can see clearly that our approach is accurate and efficient.

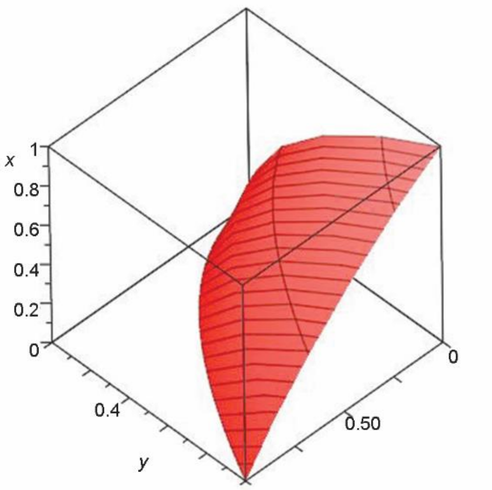

Figure 1. The $\alpha=\beta=\gamma=1$, the 11-order approximate solution of eq. (14) at $t=6$

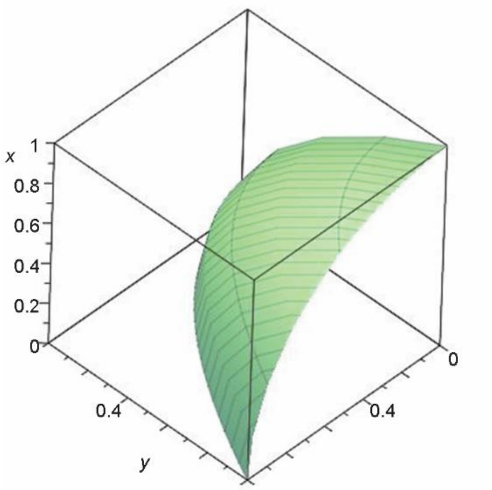

Figure 2. The $\alpha=\beta=\gamma=1$, the exact solution of eq. (14)

Conclusion

In this work, the 2-D fractal heat equation is described by He's fractal derivative in a fractal space. The two-scale transform is adopted to converted the heat transform into its classical partner, and He's homotopy perturbation method is use to find the approximate analytical solution of 2-D fractal heat equation.

Acknowledgment

This paper, was supported by the Fundamental Research Funds for the Universities of Henan Province in China under Grant No. NSFRF210345.

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Paper submitted: September 18, 2019
Paper revised: June 14, 2020
Paper accepted: June 15, 2020