HE'S FRACTAL CALCULUS AND ITS APPLICATION TO FRACTAL KORTEWEG-de VRIES EQUATION

by

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He's fractal calculus is a powerful and effective tool to dealing with natural phenomena in a fractal space. In this paper, we study the fractal Korteweg-de Vries equation with He's fractal derivative. We first adopt the two-scale transform method to convert the fractal Korteweg-de Vries equation into its traditional partner in a continuous space. Finally, we successfully use He's variational iteration method to obtain its approximate analytical solution.

Key words: *He's fractal derivative, fractal Korteweg-de Vries equation, fractal space, variational iteration method*

Introduction

In the past decades, both engineers and mathematicians have devoted considerable effort to the study of fractal geometry, fractional calculus and fractal calculus [1-14]. The fractal calculus is a very new and powerful tool to describing a lot of natural phenomena. Recently the fractal calculus has been a hot topic in a lot of fields, especially in mathematics and thermodynamics. In this paper, we will research the Korteweg-de Vries (KdV) equation in a fractal space to describe long wave propagation in a discontinuous space.

The fractal KdV equation with time-fractal derivative can be written in the form:

$$u_t^{\alpha} + m(u^p)_x + n(u^q)_{xxx} = 0 \tag{1}$$

where α is the fractal dimension and *m, n, p, and q* are constants. Equation (1) is called $K(m, n, p, q)$ equation.

In this paper, we first adopt two-scale transform method [15-17] to convert the fractal KdV equation in a fractal space into its classical partner in a continuous space. Finally, we successfully use He's variational iteration method (HVIM) [18-22] to obtain its approximate analytical solution.

He's fractal calculus

There are several definitions for fractal derivative, He's fractal derivative is a special local fractional derivative [23, 24].

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Definition 1. Hausdorff fractal derivative on time fractal space is defined:

$$\frac{\mathrm{D}u}{\mathrm{D}t^{\xi}} = \lim_{t_B \to t_A} \frac{u(t_B) - u(t_A)}{(t_B)^{\xi} - (t_A)^{\xi}} \tag{2}$$

where ξ is the fractal dimensions of time.

Definition 2. A general definition of fractal derivative is:

$$\frac{\mathrm{D}^{\theta}u}{\mathrm{D}t^{\theta}} = \lim_{t_B \to t_A} \frac{u^{\theta}(t_B) - u^{\theta}(t_A)}{(t_B)^{\theta} - (t_A)^{\theta}} \tag{3}$$

Definition 3. He's fractal derivative can be defined [25, 26]:

$$\frac{\mathrm{D}u}{\partial x^{\alpha}} = \Gamma(1+\alpha) \lim_{x_A - x_B \to L_0} \frac{u(A) - u(B)}{(x_A - x_B)^{\alpha}} \tag{4}$$

He's variational iteration method

Consider the following differential equation:

$$Lu + Nu = g(x) \tag{5}$$

where N is a non-linear operator, L – a linear operator, and $g(x)$ – a homogeneous term.

According to the variational iteration method, we construct a correct functional for eq. (5):

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda[Lu_n(\xi) + N\hat{u}_n(\xi) - g(\xi)]\,\mathrm{d}\xi \tag{6}$$

where λ is a Lagrange multiplier, which can be identified optimally *via* variational theory.

The second term on the right is called the correction, and $\hat{u}_n$ is considered a restricted variation, *i. e.* $\delta\hat{u}_n = 0$.

Numerical applications

We consider the fractal $K(-3,1,2,1)$ equation:

$$u_t^{\alpha} - 3(u^2)_x + u_{xxx} = 0, \; x \in R, t > 0 \tag{7}$$

with the initial condition:

$$u(x,0) = 6x \tag{8}$$

Firstly, we use the properties of the fractal derivative to convert eq. (7) into its partner. We assume:

$$T = \frac{t^{\alpha}}{\Gamma(1+\alpha)} \qquad (9$$

Equation (7) can easily been written into the follow form:

$$u_T - 3(u^2)_x + u_{xxx} = 0, \quad x \in R, \quad t > 0 \tag{10}$$

with the initial condition:

$$u_0 = 6x \tag{11}$$

Now we use HVIM to solve eqs. (10) and (11). Its correction variational functional in x and T can be expressed:

$$u_{m+1}(x,T)=u_m(x,T)+\int_0^T \lambda\{u_\tau-3(u)_x^2+u_{xxx}\}\mathrm{d}\tau \tag{12}$$

where λ is general Lagrange multiplier.

In order to make the previous correction functional stationary, we have its stationary conditions:

$$1+\lambda=0$$
$$\lambda'\big|_{\xi=T}=0 \tag{13}$$

which implies $\lambda = -1$. So, we obtain the following iteration formula:

$$u_{m+1}(x,T)=u_m(x,T)-\int_0^T \{u_\tau-3(u)_x^2+u_{xxx}\}\mathrm{d}\tau \tag{14}$$

We start with u_0 as initial approximation:

$$u_0=6x \tag{15}$$

According to eq. (14), we obtain the iteration results:

$$u_1(x,T)=6x(1+36T)$$

$$u_2(x,T)=6x(1+36T+1296T^2+1552T^3)$$

$$\cdots$$

$$u_5(x,T)=6x(1+36T+1296T^2+15552T^3+1679616T^4+60466176T^5)$$

$$\cdots$$

$$u_m(x,T)=6x(1+36T+1296T^2+15552T^3+1679616T^4+60466176T^5+\cdots)$$

So, the approximate analytical solution of eq. (7) is:

$$u_m(x,t)=6x(1+36t^\alpha+1296t^{2\alpha}+15552t^{3\alpha}+1679616t^{4\alpha}+60466176t^{5\alpha}+\cdots) \tag{16}$$

Remark 1. When $\alpha = 1$, the exact solution of eq. (7) is the follow form:

$$u(x,t)=6x(1-36t)^{-1} \tag{17}$$

It is obvious that eq. (16) converges to the eq. (17) when $m\to\infty$.

We consider the fractal $K(1,1,2,2)$ equation:

$$u_t^\alpha+(u^2)_x+(u)^2{}_{xxx}=0,\quad x\in R,\quad t>0 \tag{18}$$

with the initial condition:

$$u(x,0)=x \tag{19}$$

We assume:

$$T = t^{\alpha} \tag{20}$$

Equation (18) can been written into the follow form:

$$u_T + (u^2)_x + (u)^2{}_{xxx} = 0, \quad x \in R, \quad t > 0 \tag{21}$$

with the initial condition:

$$u_0 = x \tag{22}$$

Now we use HVIM to solve eqs. (21) and (22). Its correction variational functional in x and T can be expressed, respectively:

$$u_{m+1}(x,T) = u_m(x,T) + \int_0^T \lambda\{u_\tau + (u)_x^2 + (u)^2{}_{xxx}\} \mathrm{d}\tau \tag{23}$$

where λ is general Lagrange multiplier.

We have its stationary conditions:

$$\begin{aligned} 1 + \lambda &= 0 \\ \lambda'\Big|_{\xi = T} &= 0 \end{aligned} \tag{24}$$

which implies $\lambda = -1$. So, we obtain the following iteration formula:

$$u_{m+1}(x,T) = u_m(x,T) - \int_0^T [u_\tau + (u)_x^2 + (u)^2{}_{xxx}] \,\mathrm{d}\tau \tag{25}$$

We start with u_0 as initial approximation:

$$u_0 = x \tag{26}$$

According to eq. (25), we obtain the iteration results:

$$u_1(x,T) = x(1 - 2T)$$

$$u_2(x,T) = x\left(1 - 2T + 4T^2 - \frac{8}{3}T^3\right)$$

$$\cdots$$

$$u_6(x,T) = x(1 - 2T + 4T^2 - 8T^3 + 16T^4 - 32T^5 + 64T^6)$$

$$\cdots$$

$$u_m(x,T) = x(1 - 2T + 4T^2 - 8T^3 + 16T^4 - 32T^5 + 64T^6 + \ldots) \tag{27}$$

Thus, the approximate analytical solution of eq. (18) is:

$$u_m(x,t) = x(1 - 2t^{\alpha} + 4t^{2\alpha} - 8t^{3\alpha} + 16t^{4\alpha} - 32t^{5\alpha} + 64t^{6\alpha} + \cdots) \tag{28}$$

Remark 2. when $\alpha = 1$, the exact solution of eq. (18) is the follow form:

$$u(x,t) = x(1 + 2t)^{-1} \tag{29}$$

It is also obvious that eq. (28) converges to the eq. (29) when $m \to \infty$.

Conclusion

In this paper, we give a simple introduction to the fractal calculus. The fractal KdV equation is described by fractal derivative. We use HVIM to find the approximate analytical solutions of fractal KdV equations. Our results show that the fractal calculus is a powerful and effective tool to dealing with natural phenomena in a fractal space.

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