A FRACTAL VARIATIONAL THEORY OF THE BROER-KAUP SYSTEM IN SHALLOW WATER WAVES

by

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The Broer-Kaup equation is one of many equations describing some phenomena of shallow water wave. There are many errors in scientific research because of the existence of the non-smooth boundaries. In this paper, we generalize the Broer-Kaup equation to the fractal space and establish fractal variational formulations through the semi-inverse method. The acquired fractal variational formulation reveals conservation laws in an energy form in the fractal space and suggests possible solution structures of the morphology of the solitary waves.

Key words: *Broer-Kaup equation, He's fractal derivatives, two-scale transform, fractal variational formulation*

Introduction

Recently, many methods have been used to study shallow water waves [1, 2]. The Broer-Kaup (BK) equation, as a model describing the bi-directional propagation of the long waves, plays an important role in the study of shallow water wave. The form can be written:

$$\begin{gathered}\frac{\partial u}{\partial T}=2u\frac{\partial u}{\partial X}+2\frac{\partial v}{\partial X}-\frac{\partial^2 u}{\partial X^2}\\ \frac{\partial v}{\partial T}=\frac{\partial^2 v}{\partial X^2}+2\frac{\partial uv}{\partial X}\end{gathered} \tag{1}$$

in which $u(x,t)$ is bound up with the horizontal velocity, $v(x,t)$ is the height of the water surface above a horizontal bottom. This equation was first proposed by Broer [3] and Kaup [4] and was also derived from the Kadomtsev-Petviashvili equation. There are many works about this equation. Kupershmidt [5] proved that the equation is integrable and has tri-Ha-miltonian structure and infinite conservation laws. Non-Noether symmetries and its applications were studied in [6]. Zhou and Li [7] discussed some exact solutions by Darboux transformation method. Satsuma *et al.* [8] established solitons and presented the fission and fusion phenomena. Svinin [9] further given more soliton solutions. Jacobi elliptic function solutions, soliton-like solutions and trigonometric function solutions were acquired by the generalized F-expansion method in [10]. Chen and Lou [11] proved that the BK system is CTE solvable us-

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ing consistent tanh expansion and some exact interaction solutions were also given. Meng [12] constructed smooth and peaked solitary wave solutions of the eq. (1) through the bifurcation approach of dynamical system. By qualitative analysis method, Jiang and Bi [13] obtained a sufficient condition for the existence of peaked periodic wave solutions of the BK equation and some exact explicit formulas of peaked periodic wave solutions.

As we all know, BK equation is one of many equations describing some phenomena of shallow water wave. In the real world, there are many deviations in scientific research due to the existence of the non-smooth boundaries (for example, coast, *etc.*). So, fractals are a very effective tool for studying discontinuous boundaries. Usually, the smooth space (X, T) should be replaced by the fractal space (X^β, T^α), with α and β are fractal dimensions in time and boundary, respectively.

Under the fractal space, the fractal Broer-Kaup equation can be written:

$$\begin{gathered}\frac{\partial u}{\partial T^\alpha}=2u\frac{\partial u}{\partial X^\beta}+2\frac{\partial v}{\partial X^\beta}-\frac{\partial^2 u}{\partial X^{2\beta}}\\ \frac{\partial v}{\partial T^\alpha}=\frac{\partial^2 v}{\partial X^{2\beta}}+2\frac{\partial uv}{\partial X^\beta}\end{gathered} \tag{2}$$

where $\partial u/\partial T^\alpha$ and $\partial u/\partial X^\beta$ are the fractal derivative defined as [14, 15]:

$$\frac{\partial u}{\partial T^\alpha}(T_0,X)=\Gamma(1+\alpha)\lim_{\substack{T-T_0\to\Delta T\\ \Delta T\neq 0}}\frac{u(T,X)-u(T_0,X)}{(T-T_0)^\alpha} \tag{3}$$

$$\frac{\partial u}{\partial X^\beta}(T,X_0)=\Gamma(1+\alpha)\lim_{\substack{X-X_0\to\Delta X\\ \Delta X\neq 0}}\frac{u(T,X)-u(T,X_0)}{(X-X_0)^\beta} \tag{4}$$

with ΔT and ΔX are the smallest time scale and the smallest spatial scale of the discontinuous boundaries. When $\alpha = \beta = 1$ we have the classical BK equation eq. (1). If the time scale is large than ΔT, the properties of a smooth wave can be investigate, and if the spatial scale is large than ΔX, the non-smooth boundaries become the traditional smooth mechanical problem.

Based on the properties of the fractal derivatives, we introduce the two-scale transform [16, 17]:

$$t=T^\alpha,\quad x=X^\beta \tag{5}$$

Then eq. (1) turns out

$$\begin{gathered}u_t=2uu_x+2v_x-u_{xx}\\ v_t=v_{xx}+2u_x v+2uv_x\end{gathered} \tag{6}$$

In the following part, we mainly discuss the variational formulations of eq. (6).

Variational principle

The variational principle plays an important role in mathematics, mechanics, physics and other fields [18-21]. Wang [22] constructed a fractal space variational principle of the travelling wave through the semi-inverse method which was first proposed by He and Sun [23] and He [24-26]. Wang and He [27] modified the Wang's fractal variational principle to fit into the fractal space.

Accordingly and in view of the use of the semi-inverse method, we convert eq. (4) into the conservative form:

$$u_t + [-(u^2 + 2v - u_x)_x] = 0 \tag{7}$$

$$v_t + [-(v_x + 2uv)_x] = 0 \tag{8}$$

Based on the aforementioned equations, two potential auxiliary functions ψ and ϕ can be introduced:

$$\psi_t = -(u^2 + 2v - u_x) \tag{9}$$

$$\psi_x = -u \tag{10}$$

$$\phi_t = -(v_x + 2uv) \tag{11}$$

$$\phi_x = -v \tag{12}$$

It has become appallingly obvious that the system eq. (4) is equivalent to the eqs. (7), (11), and (12) or eqs. (8)-(10). The aim of our paper is to find a variational principle whose stationary conditions fulfil the eqs. (7), (11), and (12) or eqs. (8)-(10). Here, we use the semi-inverse method to establish a series of variational formulas, the trial-functional can be written:

$$J(u,v,\psi) = \iint L(u,u_t,u_x,u_{xx},v,v_t,v_x,v_{xx},\psi,\psi_t,\psi_x,\psi_{xx})\mathrm{d}x\mathrm{d}t \tag{13}$$

where L is the trial-Lagrange function. Then, we suppose the trial-Lagrange function has the following form:

$$L = u\phi_t - (u^2 + 2v - u_x)\phi_x + F(u,v) \tag{14}$$

where $F(u, v)$ is an trial-undetermined function with respect to u, v, and/or their derivatives.

The stationary conditions with respect to u, v are presented:

$$\phi_t - 2u\phi_x - \phi_{xx} + \frac{\delta F}{\delta u} = 0 \tag{15}$$

$$-2\phi_x + \frac{\delta F}{\delta v} = 0 \tag{16}$$

where $\delta F/\delta u$ and $\delta F/\delta v$ are called He's variational derivative presented:

$$\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial u_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) + \cdots \tag{17}$$

$$\frac{\delta F}{\delta v} = \frac{\partial F}{\partial v} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial v_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) + \cdots \tag{18}$$

According to the eqs. (15) and (16), we will find the specific $F(u, v)$, makes eqs. (15) and (16), respectively result in the field equations of the given equation. Hence, we have:

$$\frac{\delta F}{\delta u} = -(\phi_t - 2u\phi_x - \phi_{xx}) = 0 \tag{19}$$

$$\frac{\delta F}{\delta v} = 2\phi_x = -2v \tag{20}$$

In view of eqs. (19) and (20), $F(u, v)$ can be written:

$$F=-v^2 \tag{21}$$

Finally, we successfully establish the variational formulation:

$$J(u,v,\psi)=\iint[u\phi_t-(u^2+2v-u_x)\phi_x-v^2]\mathrm{d}x\mathrm{d}t \tag{22}$$

Proof: Through the previous variational principle, the stationary condition about ϕ, u, and v, we gain the Euler-Lagrange equations, reads:

$$\phi_t-2u\phi_x-\phi_{xx}=0 \tag{23}$$

$$-2\phi_x-2v=0 \tag{24}$$

$$-u_t-(u^2+2v-u_x)_x=0 \tag{25}$$

It's easy to see that eqs. (24) and (25) are equivalent to eqs. (12) and (7), respectively. Under the constraint eq. (24), eq. (23) leads to eq. (11).

In the fractal space (X^β, T^α), the variational formulation can be given by:

$$J(u,v,\psi)=\iint\left[u\frac{\partial\phi}{\partial T^\alpha}-\left(u^2+2v-\frac{\partial u}{\partial X^\beta}\right)\frac{\partial\phi}{\partial X^\beta}-v^2\right]\mathrm{d}X^\beta\mathrm{d}T^\alpha \tag{26}$$

On the other hand, on account of the other trial-Lagrange function, we can construct a new variational formulation. The new trial-Lagrange function reads:

$$L^*=v\psi_t-(v_x+2uv)\psi_x+F^*(u,v) \tag{27}$$

where $F^*(u, v)$ is the new trial-Lagrange function. The Euler-Lagrange equation is:

$$-2v\psi_x+\frac{\delta F^*}{\delta u}=0 \tag{28}$$

$$\psi_t-2u\psi_x+\psi_{xx}+\frac{\delta F^*}{\delta v}=0 \tag{29}$$

In view of eqs. (28) and (29), we get:

$$\frac{\delta F^*}{\delta u}=2v\psi_x=-2uv \tag{30}$$

$$\frac{\delta F^*}{\delta v}=-(\psi_t-2u\psi_x+\psi_{xx})=-u^2+2v \tag{31}$$

The trial-Lagrange function can be determined by:

$$F^*=-u^2v+v^2 \tag{32}$$

Therefore, the other variational formulation reads:

$$J^*(u,v,\phi)=\iint[v\psi_t-(v_x+2uv)\psi_x-u^2v+v^2]\mathrm{d}x\mathrm{d}t \tag{33}$$

In order to proof the correctness of the variational formulation, we reduce the Euler-Lagrange equations through the previous functional:

$$\psi_x = -u \tag{34}$$

$$\psi_t = -2u^2 - 2v + u_x \tag{35}$$

$$v_t - (v_x + 2uv)_x = 0 \tag{36}$$

It is clear that the eqs. (34) and (36) are equivalent to eqs. (10) and (8), respectively. Under the constraint eqs. (34) and (35) leads to eq. (9). Similarly, the fractal variational formulation can be written:

$$J^*(u,v,\psi) = \iint \left[v\frac{\partial \psi}{\partial T^\alpha} - \left(\frac{\partial v}{\partial X^\beta} + 2uv \right) \frac{\partial \psi}{\partial X^\beta} - u^2 v + v^2 \right] \mathrm{d}X^\beta \mathrm{d}T^\alpha \tag{37}$$

In addition, in view of the generalized variational formulations eqs. (22) and (33), some constrained variational principles can be constructed. For example, we substitute eq. (10) into eq. (33), the constrained functional reads:

$$J^{**}(v,\psi) = \iint (v\psi_t + uv_x + u^2 v + v^2)\mathrm{d}x\mathrm{d}t \tag{38}$$

with the constraint eq. (10). Then, substituting eq. (9) into eq. (38), we have:

$$J^{***}(\psi) = \iint (u_x v + uv_x - v^2)\mathrm{d}x\mathrm{d}t \tag{39}$$

with the constraint eq. (9). Further integrating by parts and ignoring its boundary items of eq. (39), we get:

$$J^{***}(\psi) = \iint (u_x v + uv_x - v^2)\mathrm{d}x\mathrm{d}t = -\frac{1}{4}\iint [(\psi_{xx} + \psi_t + \psi_x^2)]\mathrm{d}x\mathrm{d}t \tag{40}$$

with the constraints $\psi_t = -(u^2 + 2v - u_x)$ and $\psi_x = -u$.

Finally, under the fractal space (X^β, T^α), the constrained fractal variational formulation is given by:

$$J^{***}(\psi) = -\frac{1}{4}\iint \left[\frac{\partial^2 \psi}{\partial X^{2\beta}} + \frac{\partial \psi}{\partial T^\alpha} + \left(\frac{\partial \psi}{\partial X^\beta} \right)^2 \right] \mathrm{d}X^\beta \mathrm{d}T^\alpha \tag{41}$$

with the constraints $\partial\psi/\partial T^\alpha = -(u^2 + 2v - \partial u/\partial X^\beta)$ and $\partial\psi/\partial X^\beta = -u$.

Conclusion

Fractal derivatives are widely used to investigate discontinuous media. This paper, based on the fractal space (X^β, T^α), we construct the fractal variational formulations of the fractal BK equation through the semi-inverse method. The variational principle can be used to construct conservation laws and suggest solution structures of the solitary waves.

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