NON-LINEAR HEAT CONDUCTION WITH RAMPED SURFACE HEATING
Ramp Surface Heating and Approximate Solution

by

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Non-linear heat conduction with a power-law thermal diffusivity and ramped surface temperature has been solved by the double-integration technique of the integral-balance integral method. The case of a semi-infinite medium and infinite ramp of surface temperature has been considered as example demonstrating the versatility of the solution approach. The thermal penetration depth and solution behaviours with finite speeds have been analyzed.

Key words: *ramped surface temperature, transient heat conduction, integral-balance solution, non-linear thermal diffusivity*

Introduction

Problem statement

Let us consider the heat conduction equation:

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left[a\left(T\right)\frac{\partial T}{\partial x}\right],\quad a=a_0\left(\frac{T}{T_{\text{ref}}}\right)^m,\quad m>0 \tag{1}$$

with a scaled power-law relationship which is related to temperature-dependent thermal conductivity $k = k_0(T/T_{\text{ref}})^m$ assuming the product of the density and heat capacity ρC_p as a temperature-independent value when heat conduction is modelled.

The model eq. (1), in contrast to the linear diffusion equation ($m = 0$), is uniformly parabolic in any region where T is not zero, but degenerates in the vicinity of any point where $T = 0$ [1]. The main performance of this degeneracy is that any disturbance propagates at finite speed giving rise to a front or interface in the solution separating disturbed and undisturbed medium [2].

It is worthy to note that despite the fact that the problem eq. (1) is defined as a transient heat conduction there are many physical process described by eq. (1) such as creeping flows [3], non-linear heat conductivity [4], non-linear diffusion [5], porous media flow [6], *etc*. All cases where $m > 1$ belong to the family of slow diffusion problems [2].

The difficulties inherent in obtaining solutions for this class of equations have motivated a variety of solution methods, both exact and approximate ones such as: waiting-time approach [3], asymptotic methods [7], similarity solutions [1, 2], analytic moment methods [1],

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a linearization through the Kirchhoff transformation [8], Heat-balance integral method (HBIM) [9] with Dirichlet boundary condition of the model eq. (1) and transformation the variables $\varphi = T^m$ and $\tau = t/m$ [10]. Double-integration method (DIM) has been applied in [11].

Ramp surface temperature as boundary condition

The surface heating ($x = 0$) represented by a time-dependent (ramped) temperature can be expressed in general form as $h(t) = T_s = T_0 + b_0t^{p/2}$, where $p > 0$. The boundary condition can be defined:

$$T_s = T_0, \quad t < 0, \; h(t) = T_s = T_0 + b_0 t^{p/2}, \; t > 0 \tag{2}$$

The exact solution of this problem developed in [12] with constant thermal diffusivity a and surface temperature $T_s = b_0t^{p/2}$ is (presented in terms of the original solution):

$$\theta_e = b_0\Gamma\left(\frac{p}{2}+1\right)(4t)^{p/2}\left[i^p\Phi\left(\frac{x}{2\sqrt{at}}\right)\right], \quad i^p\Phi(x) = \int_x^\infty i^{p-1}\Phi^*(\xi)\mathrm{d}\xi, \; p = 2,3,4 \tag{3}$$

where Φ is the error function, $\xi = x/2(\mathrm{at})^{1/2}$ is similarity variable, $\Gamma(\bullet)$ is the Gamma function. As it mentioned in [12] about the solution eq. (3) it may be used with tabulated functions. However, despite the exactness, this approach is not always useful to handle in engineering calculations.

Aim

The present article reports an analysis about integral-balance solutions of the model eq. (1) in case of time-dependent surface temperature as boundary condition and a non-linear (power-law) thermal diffusivity. To our knowledge, no attempts to solve this problem by the integral balance method (without a preliminary linearization by the Kirchhoff transform) have been reported in the literature.

Problem development

Dimensionless governing equation and assumed profile

The scaled diffusivity in case of heat conduction is commonly expressed as $a = a_0(T/T_{\mathrm{ref}})^m$ where T_{ref} is commonly accepted room temperature of about 20 °C and differs from the initial medium temperature. In this context, the case of thermal diffusivity with $T_{\mathrm{ref}} \neq T_0 \neq 0$ can be rescaled as $a_{\mathrm{eff}}u^m = a_0k_T\,(T/T_0)^m$ where $u = (T/T_0)$, $k_T = (T/T_{\mathrm{ref}})^m$ = constant and $a_{\mathrm{eff}} = a_0k_T$. When $T_{\mathrm{ref}} = T_0 \neq 0$, we get $k_T = 1$ and with $u = T/T_0$ we have:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a_0u^m\frac{\partial u}{\partial x}\right) \tag{4}$$

Double integration method

Expressing eq. (6) through the new variable U and integrating from 0 to δ we get the first version of eq. (5). Then application of the Leibniz rule to the integral leads to the second form of eq. (5):

$$\int_0^\delta \frac{\partial U}{\partial t}\mathrm{d}x = -\left(a_0U^m\frac{\partial U}{\partial x}\right)_{x=0} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta U\mathrm{d}x = -\left(a_0U^m\frac{\partial U}{\partial x}\right)_{x=0} \tag{5}$$

This is the principle equation of the HBIM of Goodman [9, 10], where replacing U by U_a we can develop an equation about $\delta(t)$. As mentioned earlier, the main drawback comes from the fact that the gradient $a_0U^m\partial U(0, T)/\partial x$ depends on the type of the function chosen as assumed profile.

With the double integration method the first step is the integration of eq. (4) from 0 to x:

$$\int_0^x \frac{\partial U}{\partial t}\mathrm{d}x = \int_0^x \frac{\partial}{\partial x}\left(a_0 U^m \frac{\partial U}{\partial x}\right)\mathrm{d}x \tag{6}$$

Representing the integral in the left-side of eq. (5):

$$\int_0^\delta f(\bullet)\mathrm{d}x = \int_0^x f(\bullet)\mathrm{d}x + \int_x^\delta f(\bullet)\mathrm{d}x$$

we get

$$\int_0^x \frac{\partial U}{\partial t}\mathrm{d}x + \int_0^\delta \frac{\partial U}{\partial t}\mathrm{d}x = -\left(a_0 U^m \frac{\partial U}{\partial x}\right)_{x=0} \tag{7}$$

Subtracting eq. (6) from eq. (7) and integrating the resulting equation from 0 to δ:

$$\int_0^\delta\left(\int_x^\delta \frac{\partial U}{\partial t}\mathrm{d}x\right)\mathrm{d}x = -a_0\int_0^\delta U^m \frac{\partial U(x,t)}{\partial x}\mathrm{d}x \tag{8}$$

Now, we may rearrange the term under the integral sign in the right-hand side of eq. (8) as $\partial U/\partial x = 1/(m + 1)\ \partial U^{m+1}/\partial x$ [11]. Then, the integration in eq. (8) yields [11]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta\left(\int_x^\delta U\mathrm{d}x\right)\mathrm{d}x = \frac{a_0}{m+1}U^{m+1}(0,t) \tag{9}$$

Equation (9) is the principle relationship of DIM when the thermal diffusivity is non-linear (power-law). This is not exactly the method used in [30], since the order of integrations is different, but the demonstrated approach allows straightforward solutions of non-linear problems [27, 28].

Assumed profile

The integral-balance solution uses an assumed parabolic profile with unspecified exponent [11]:

$$T_a = T_s\left(1-\frac{x}{\delta}\right)^n \tag{10}$$

The profile eq. (10) satisfies the condition at $x = 0$, *i. e.* $T(0, t) = T_s$ as well as the Goodman's conditions [9] $T(\delta, t) = T_0$ and $k(\partial T/\partial x)_{x=\delta} = 0$. The condition at $x = 0$ defines $T_s = T_0 + b_0t^{p/2}$:

$$T_a = T_0 + b_0 t^{p/2}\left(1-\frac{x}{\delta}\right)^n \tag{11}$$

Then, with $u_a = T_a/T_0$ and $b_0/T_0 = b$ the dimensionless approximate profile:

$$u_a = 1 + u_s\left(1-\frac{x}{\delta}\right)^n \quad \text{or} \quad U_a = (u_a - 1) = bt^{p/2}\left(1-\frac{x}{\delta}\right)^n$$

The introduction of $U = u - 1$ does not change the structure of the dimensionless profile eq. (4) and the surface temperature in terms of the new dimensionless variable is $U_s = b_0 t^{p/2}$. The approximate profile U_a satisfies the Goodman's conditions because at $x = \delta$ we have $u_a = T/T_0 = 1$ and therefore, $U_a(\delta, t) = 0$ and $k(\partial U_a/\partial x)_{x=\delta} = 0$.

Approximate solution

Penetration depth

Without loss of generality we may define the profile as $\theta_a = U_a/b = t^{p/2}(1 - x/\delta)^n$. We start with the principle DIM eq. (9) which with the boundary condition at $x = 0$ an the approximate profile presented as θ_a:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta\int_x^\delta\left[t^{p/2}\left(1-\frac{x}{\delta}\right)^n\right]\mathrm{d}x\mathrm{d}x=\frac{a_0}{m+1}\left(t^{p/2}\right)^{m+1} \tag{12}$$

The integration of eq. (12) yields:

$$\frac{\mathrm{d}\left(\delta^2 t^{p/2}\right)}{\mathrm{d}t}=a_0\frac{(n+1)(n+2)}{m+1}\left(t^{p/2}\right)^{m+1}\Rightarrow\delta=\sqrt{a_0 t^{\frac{pm+2}{2}}}\sqrt{\frac{2(n+1)(n+2)}{(m+1)\left[p(m+1)+2\right]}} \tag{13}$$

or in a compact form:

$$\delta=\sqrt{a_0 t^{\frac{pm+2}{2}}}F_{n,m,p},\quad F_{n,m,p}=\sqrt{\frac{2(n+1)(n+2)}{(m+1)\left[p(m+1)+2\right]}} \tag{14}$$

This expression indicates that only in the case of $m = 0$ (the temperature-independent thermal diffusivity) the thermal front propagates in accordance with the square-root law. For $m = 0$ and $p = 0$, *i. e.* the linear Dirichlet problem:

$$\delta=\sqrt{a_0 t}\sqrt{(n+1)(n+2)}$$

as it was developed in [14].

Approximate profile

Therefore, using expression for $\delta(t)$ and the definition eq. (10), the approximate profile can be presented in several forms:

$$\theta_a=\theta_s\left(1-\frac{x}{\sqrt{a_0 t^{\frac{pm+2}{2}}}F_{n,m,p}}\right)^n\Rightarrow\theta_a=t^{\frac{p}{2}}\left(1-\frac{\eta_p}{F_{n,m,p}}\right)^n \tag{15}$$

$$U_a=bt^{\frac{p}{2}}\left(1-\frac{\eta_p}{F_{n,m,p}}\right)^n\Rightarrow u=1+bt^{\frac{p}{2}}\left(1-\frac{\eta_p}{F_{n,m,p}}\right)^n \tag{16}$$

$$u=1+bt^{\frac{p}{2}}\left(1-\frac{\eta_p}{F_{n,m,p}}\right)^n\Rightarrow T=T_0+b_0 t^{\frac{p}{2}}\left(1-\frac{\eta_p}{F_{n,m,p}}\right)^n \tag{17}$$

The approximate profile defines the non-Boltzmann similarity variable η_p.

For $m = 0$, the new variable η_p reduces to the Boltzmann similarity variable $\eta_0 = (a_0 t)^{1/2}$

and relationship between them:

$$\eta_p = \frac{x}{\sqrt{a_0 t^{(pm+2)/2}}} = \frac{\eta_0}{\sqrt{t^{(pm+1)}}}$$

For $p = 2$ (linear ramp condition) this relationship:

$$\eta_{p(p=2)} = \frac{x}{\sqrt{a_0 t^{m+1}}} = \frac{\eta_0}{\sqrt{t^m}}$$

Further, the solution can be presented against the normalized similarity variable $X = \eta_p/F_{n,m,p}$ in the form $U(X, t) = (1 - X)^n$, where $0 < X < 1$. The function $F_{n,m,p}$ is defined by the penetration depth in the form:

$$\delta = \sqrt{a_0 t^{\frac{pm+2}{2}}} F_{n,m,p}$$

Therefore, the boundary conditions at the front of the penetration layer can be expressed in two different forms:

– With η_p as independent variable. In this case the condition $U_x(\delta, t)$:

$$\frac{\partial U(\delta,t)}{\partial x} = \frac{\partial U(\delta,t)}{\partial \eta_p}\frac{\partial \eta_p}{\partial x} = \left(bt^{\frac{p}{2}}\right) n \left(1 - \frac{\eta_p}{F_{n,m,p}}\right)^{n-1} \frac{1}{\sqrt{a_0 t^{\frac{pm+2}{2}}} F_{n,m,p}} = 0 \tag{18}$$

This condition is satisfied for $\eta_p = F_{n,m,p}$. Therefore, the temperature profile should cross the abscissa at different lengths [1] depending of the values of the exponent n as well as the parameters m and p. In general, for stipulated n and p, the increase in value of m reduces the penetration depth.

– With X as independent variable the condition eq. (24) is satisfied at $X = 1$ and can be re-written as $U_X(1, t) = 0$. Therefore, with the normalized independent variable X, all temperature profiles should cross the abscissa at $X = 1$ [1].

Again, with increase in m the penetration depth reduces and this effect is visible when the similarity variable η_p is used as independent variable, but becomes indistinguishable when the profiles are presented against $X = \eta_p/F_{n,m,p}$ as independent variable. To clarify this point, prior to the further development of the solution, we will simulate temperature profiles with exponents satisfying the reciprocal rule $n = 1/m$ established in the approximate analytical [1] and exact solutions [15] of the problem at $p = 0$ and verified in [10, 11]. The plots presented in fig. 1 undoubtedly demonstrate the retardation effect when the value of m increases. The penetration depth becomes shorter when m increase (more obvious in case of slow diffusions for $m > 1$) with the similarity variable η_p as independent variable. Moreover, the increase in m makes the profile more convex and vice versa. On the other hand, when the profiles are plotted against the normalized similarity variable $X = \eta_p/F_{n,m,p}$, the effect of the variation of $n = 1/m$ is the same, but now all curves cross the abscissa at $X = 1$. Therefore, irrespective of the independent variable used, all exponents obeying the reciprocal law provide convex profiles for slow diffusion ($n < 1$, $m > 1$) and concave profiles for fast diffusion ($n > 1$, $m < 1$) . The present article will continue with slow diffusion problems only.

Surface thermal impedance

With $T_s = T_0{*}t$ at the boundary $x = 0$ we may calculate the surface thermal impedance as $Z_s = T_s/q_s$:

$$Z_s = \frac{z_{0(m=0)}}{t^m} \quad \text{where} \quad z_{0(m=0)} = \left(\frac{\sqrt{a_0} F_{n,m}}{k_0 n}\right)\left(T_0^*\right)^{-m}$$

If, for example, $m = 0$ (the linear case), then the exact surface thermal impedance:

$$Z_{s(m=0)} = z_{0(m=0)}\sqrt{t}, \quad \text{where} \quad z_{0(m=0)} = \frac{\sqrt{a_0 \pi T_0}}{2k_0}$$

That is, the surface thermal impedance decreases in time due the non-linearity in the thermal diffusivity in contrast to the linear case. Similar relationships can be developed in the general case for $p \neq 2$. In this context, with a power-law ramp of the surface temperature the surface flux:

$$q_{s(\text{power-law ramp})} = \left(\frac{n}{\sqrt{a_0} F_{n,m}}\right) t^{\frac{p(m+1)}{4}} \equiv t^{\frac{p(m+1)}{4}}$$

Thus, the thermal impedance can be evaluated in straightforward manner.

Optimization of the approximate solution

The optimization procedure focuses on the determination of the optimal exponent n assuring minimum error of approximation. Since the exponent of the parabolic profile cannot be defined through the boundary conditions we have to apply additional conditions to find optimal one. Since both HBIM and DIM are restricted to the zeroth moment, the accuracy of approximation depends on the values of the exponent n.

Restrictions at the boundaries of the penetration layer

First of all, the approximate profile satisfies the heat-balance integral but not the original heat conduction equation. Therefore, the function $\varphi[u_a(x, t)]$:

$$\varphi\left[u_a(x,t)\right] = \frac{\partial u_a}{\partial t} - \frac{\partial}{\partial x}\left(a_0 u_a^m \frac{\partial u_a}{\partial x}\right) \tag{19}$$

should be zero if u_a matches the exact solution, otherwise it should attain a minimum for a certain value of the exponent n (the only unspecified parameter of the approximate profile). At this point, we have to remember that the assumed profile is designed with respect to the space x-ordinate, especially with respect to the Zener' co-ordinate [16] $\xi = x/\delta$, where $0 < \xi < 1$, and therefore, the normalized approximate profile in the square [1, 1], *i. e.* $0 < \xi < 1$ and is $V = \theta_a/t^{p/2} = (1 - \xi)^n$ with $0 < V < 1$. With $V = (1 - x/\delta)^n$ as well as expressed through the developed penetration depth, the function φ_p^N (the subscript means normalized) at $x = 0$:

$$\varphi_p^N(0,t) = \frac{n[nm+n-1]}{\delta^2} = \frac{n[nm+n-1](m+1)(m+2)}{(n+1)(n+2)} \tag{20}$$

Obviously, the function $\varphi_p^N(0, t)$ satisfies the heat-equation, but minimizing the second term of eq. (20) we get $n = 1/(m + 1)$ and taking into account that $\varphi_p^N(0, t)$ should be positive we need $n > 1/(m + 1)$.

On the other hand, thanking into account that for $x = 0$:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial T(0,t)}{\partial t} = \frac{\partial T^2(0,t)}{\partial x^2}$$

and, therefore, from

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \partial\left[\frac{T^m \partial T(0,t)}{\partial x}\right]\partial x$$

in terms of θ_a

$$\frac{p}{2}t^{\frac{p}{2}-1} = \frac{a_0}{m+1}t^{\frac{mp}{2}}\left[n(m+1)-1\right]\frac{1}{\delta^2} \Rightarrow \delta^2 = t^{\frac{mp}{2}-\frac{p}{2}+1}\frac{a_0}{m+1}\left[n(m+1)-1\right] \tag{21}$$

Using the expression for δ derived by DIM and equating it to eq. (21):

$$t^{\frac{mp}{2}-\frac{p}{2}+1}\frac{2}{p}\frac{a_0}{m+1}\left[n(m+1)-1\right] = a_o t^{\frac{mp}{2}+1}F^2_{n,m,p} \Rightarrow n = \frac{1}{m+1}\left(1+\frac{p}{2}t^{p/2}F^2_{n,m,p}\right) \tag{22}$$

For $p = 0$ we get the condition $n = 1/(m + 1)$ established through the normalized profile. Further, for short times we may neglect the time-dependent term in eq. (22) and accept that n is a constant established at $x = 0$. The expression eq. (22) generally states that n should be time-dependent. Moreover, the last relationship in eq. (22) clearly state that the determination of n needs a non-linear equation be solved since $F_{n,m,p}$ depends on n. However, this case is beyond the scope of this work and we turn on our study to case where the exponent n is constant but ensures minimal mean-squared error of approximation over the entire penetration depth.

In addition, for the Goodman's conditions $u_a(\delta, t) = \partial u_a(\delta, t)/\partial x = 0$, *i. e.* for $x \to \delta$, we get:

$$\varphi_p(\delta,t) = -\frac{n\left[n(m+1)-1\right]}{\delta^2}\left(\frac{1}{t}\right)\lim_{x\to\delta}\left(1-\frac{x}{\delta}\right)^{n(m+1)-2} \Rightarrow n > \frac{2}{(m+1)} \tag{23}$$

In both cases we have the inequality $1/m > 1/(m + 1) > 1/(m + 1/2)$, thus the general reciprocal law $1/m$ is stronger. Following the estimated constrains we may define an area bounded by the lines $1/m$ and $1/(m + 1)$ which tend to converge at large m. These lines form a *Horn-shaped zone* where the exponents of the optimal profile should lie down when the function $n_{\mathrm{opt}} = f(m)$ is plotted. This behaviour was established in [11] when the case for $p = 0$ was solved. Hereafter, we will refer to it as to *the reciprocal law or the H-rule* and will see do the optimal exponents relevant to the problem at issue satisfy it.

Optimal exponents

In the Zener's co-ordinate [16] the profile:

$$\theta_a = t^{p/2}(1-\xi)^n = t^{p/2}V(\xi,t)$$

the heat eq. (1):

$$\frac{p}{2}t^{\frac{p}{2}-1}V - t^{p/2}\frac{\xi}{\delta}\frac{\mathrm{d}\delta}{\mathrm{d}t} = a_0 t^{\frac{mp}{2}}\left[\frac{m}{\delta^2}V^{m-1}\left(\frac{\partial V}{\partial \xi}\right)^2\right] \tag{24}$$

Then, the residual function reads:

$$\varphi_p(\xi,t) = a_0\frac{t^{mp/2}}{\delta^2}\omega(\xi,t)$$

where:

$$\omega(\xi,t)=t^{p/2}\left[pF_{n,m,p}^{2}\left(1-\xi\right)^{n}+\xi\frac{1}{t^{mp/2}}\frac{\delta\mathrm{d}\delta}{\mathrm{d}t}\left(1-\xi\right)^{n-1}\right]-\\-m\left[n\left(m+1\right)-1\right]\left(1-\xi\right)^{\left[n\left(m+1\right)-2\right]} \tag{25}$$

Now, using the expression for δ^2:

$$\varphi_p\left(\xi,t\right)=\frac{1}{t}\frac{\omega\left(\xi,t\right)}{F_{n,m,p}^{2}}$$

inasmuch the group

$$\frac{1}{t^{mp/2}}\frac{\delta\mathrm{d}\delta}{\mathrm{d}t}=M$$

is time-independent. Therefore, the residual function decays in time and the optimization problem refers to a minimum of $[\omega(\xi, t)/F_{n,m,p}^2]$ with respect to n for given m and p. When $p = 0$ (fixed temperature boundary condition), for instance, the function $\omega(\xi, t)$ is time-independent. If we consider, that the maximum value of the residual function occurs at $t = 0$, then setting $t = 0$ in $\omega(\xi, t)$ we get

$$\varphi_p\left(\xi,t=0\right)\approx-2a_0m\left[n\left(m+1\right)-1\right]\left(1-\xi\right)^{\left[n\left(m+1\right)-2\right]}$$

The integration from 0 to 1 yields:

$$\int_0^1\varphi_p\left(\xi,t=0\right)\mathrm{d}\xi\approx\frac{2}{t}\frac{a_0m}{F_{n,m,p}^{2}}$$

a results which becomes zero for $m = 0$, which is unacceptable. Therefore, we have to evaluate the squared-error function defined:

$$E\left(n,m,p\right)=\frac{1}{t^2}\int_0^1\left[\frac{\omega\left(\xi,t\right)}{F_{n,m,p}^{2}}\right]^2\mathrm{d}\xi \tag{26}$$

The integration in eq. (26) yields:

$$E\left(n,m,p\right)=t^{p}\left[\frac{1}{2n+1}p^2F_{n,m,p}^{2}+\frac{M^2}{2n\left(2n-1\right)}+\frac{2pF_{n,m,p}^{2}M}{2n\left(2n+1\right)}\right]+\frac{4m^2\left[n\left(m+1\right)-2\right]^2}{2n\left(m+1\right)-3}-\\-t^{p/2}\left[\frac{4pF_{n,m,p}^{2}m}{n+n\left(m+1\right)-1}+4M\frac{1}{n+n\left(m+1\right)-2}\right]\left[n\left(m+1\right)-2\right] \tag{27}$$

Assessment of E(n, m, p) by asymptotes and optimal exponents

Short times

Only one term of eq. (27) are time-independent and if we will look for the case $t \to 0$ then one obtain $n = 2/(m + 1)$ and varying $m > 1$, all exponents will satisfy the reciprocal law resulting in convex profiles. In fact, this approach coincides with the early technique of Myers [15, 16].

Alternative approach with restrictions imposed of vertical asymptotes

If we set $t = 1$ then, then all terms in eq. (27) will be with equal weights with given p and m, $E(n, m, p)$ is a function only of n. For the optimization of $E(n, m, p)$ with the imposed constraints on the exponent n we have to define the area where the minima and consequently the optimal values of the exponent n should be searcher for. The error function eq. (27) is a rational function $E(n, m, p) = P(n)/Q(n)$ where the nominator $P(n)$ has one order higher than the denominator $Q(n)$ and, therefore, it has a slant asymptote allowing representing it as $E(n, m, p) = kx + b + R(n)/Q(n)$. More over the denominator $Q(n)$ defines the vertical asymptotes (for $n > 0$): $n = 0.5$, $n = 1.5/(m + 1)$, $n = 1/(m + 2)$. We will avoid any cumbersome calculations and will give a particular example. For a linear ramp ($p = 2$) and $m = 1$, we have:

$$E(n,1,2)= \\ =\frac{744n^8-605n^7-4541.1n^6+9937.2n^5-7845.1n^4+2283.8n^3+113.7.17n^2-111n-6}{(n+2)(n+1)(3n+2)(3n-1)(4n-3)n(2n-1)} \tag{28}$$

The vertical asymptotes for $n > 0$ are: $n = 1/3$, $n = 1/2$, and $n = 3/4$. The nominator $P(n)$ is of order 8 while the denominator $Q(n)$ is of order 7 and the slant asymptote is defined by the line $E = –16.178 + 10.33n + R(n)/Q(n)$. With the constraints $n > 1/ (m + 1) = 1/2$ and $n < 1$ the value of n should be searched for the branch of $E(n, 1, 2)$ bounded by the vertical asymptotes $n = 1/2$ from the left and $n = 3/4$ from the right, and beneath by the slant asymptote. The minimization (numerical) of the eq. (28) by help of Maple results in an optimal exponent $n \approx 0.722$ providing $E(n, 1, 2) \approx 0.5625$. This optimal exponent is bounded by the constraints corresponding to the reciprocal law, namely: $0.5 = 1/(m + 1) < 0.722 < 1/m = 1$. In addition, the function $E(n, 1, 2)$ has a local minimum of about $4.373 \cdot 10^{-3}$ for $n \approx 1.391$ in the area bounded from the left by the vertical asymptote $n = 1/3$ and beneath the slant asymptote. However, this exponent provides a concave profile albeit it satisfies the constraints of the reciprocal law. Similarly, for the case of a linear ramp ($p = 2$) and higher values of m the optimal exponents $n_{p,m}$ are: $n_{2,2} \approx 0.656$, $n_{2,3} \approx 0.553$, $n_{2,4} \approx 0.4920$, $n_{2,5} \approx 0.3290$.

Numerical experiments with approximate solutions

Temperature profiles with stipulated exponents

As a first step of demonstration of the developed approximate solution plots with stipulated exponents defined by the rule are shown in fig. 1 . As commented earlier the profiles are convex for $n < 1$ an cross the abscissa in points definition the position of the penetration front. In case when the profiles are presented through the variable $0 < X < 1$ all of them end at $X = 1$ but with increase in the value of m they become with more steeper fronts. This behaviour is characteristic for the degenerate diffusion, especially slow diffusion.

Time evolution of the front

The second interesting question is about the behaviour of the front, especially its time evolution. It is clear that for $m = 0$ (normal, Gaussian Diffusion) we have $\delta \equiv t^{1/2}$. However, for various values of m and different ramping parameter p the behaviour will change. It is possible to see this behaviour from the plots in fig. 2. In all cases for small times, precisely when $0 < t < 1$the plots are below the Gaussian line and therefore, the exhibited behaviour is subdiffusive. However, for $t > 1$ there is superdiffusion in all cases. It is clearly demonstrated

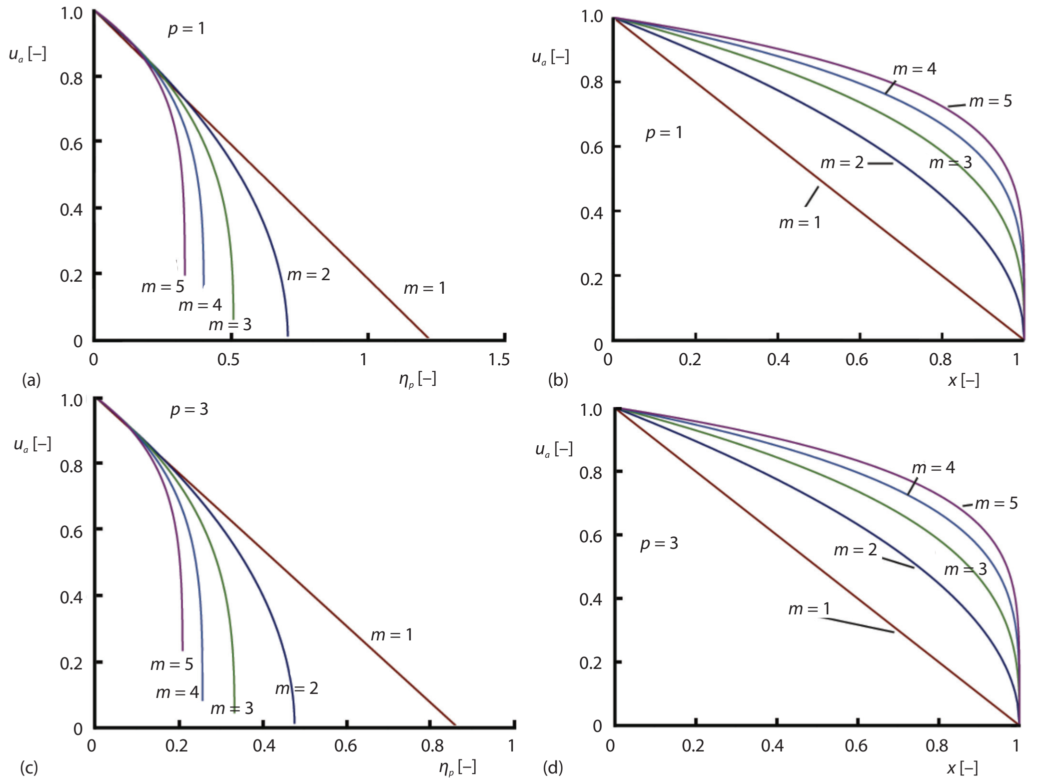

Figure 1. The DIM solutions (normalized profiles $u_a = \theta a/\theta s$ with stipulated exponents following the rule 1/(m + 1) for various values of the ramping parameter p; the upper plots are presented as functions of the similarity variable η as argument, while the lower plots are developed with X as independent variable

that the increase in the values of m make the growth of the penetration depths slower, as it follows from previous results where it was shown that the increase in m reduces the penetration depth.

To clarify this behavior let us see what is the rate of the front. From the functional relation about $\delta(t)$:

$$\delta \equiv t^{\frac{pm+2}{4}} \Rightarrow \frac{\mathrm{d}\delta}{\mathrm{d}t} \equiv \left(\frac{pm+2}{4}\right) t^{\frac{pm-2}{4}} \tag{29}$$

To have a positive growing rate dδ/dt > 0 we need $pm - 2 > 0$ which requires $p > m/2$. This condition is satisfied, since at the very beginning the condition for slow diffusion is $m > 1$. Thus, for $1 < m < 2$ this requirement is satisfied,but when $m > 2$, the general condition imposed by eq. (3) cannot be obeyed. Despite this, we may consider the conditions in eq. (3) about the values of p as non-mandatory. In this case, since the condition $m > 1$ comes from the model physics, then the ramp process is of secondary importance and different front rates can be obtained depending on the value of m and the value of the ramping parameter p. In general, higher values of the non-linear parameter m, the higher value of p in order to obtain positive rate of the front are required.

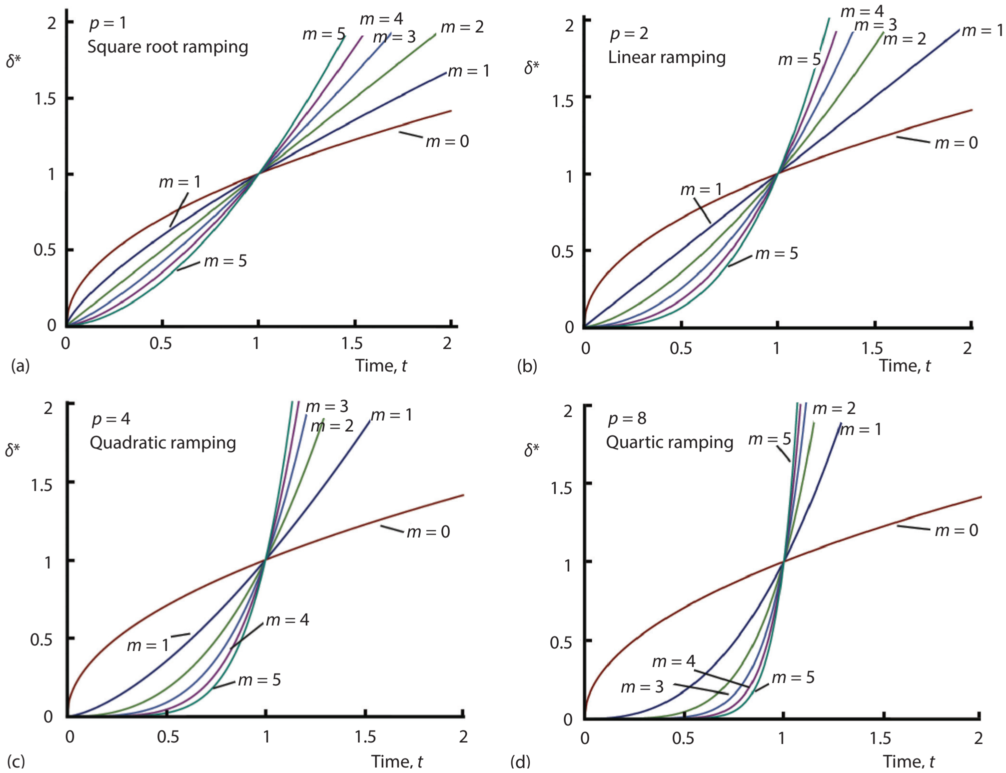

Figure 2. Time evolution of the front $\delta(t)$ represented only by its time-dependent part for different values $\delta^* = [t^{(pm+2)/2}]^{1/2}$ of the non-linear exponent m and the ramping rate constant p

Temperature profiles with optimal exponents

The temperature profiles of the optimized solutions are shown in fig. 3 . It is clear that the generative behaviour of the model equation dominates over the form of the ramping functions. In general, the increase in non-linearity exponent m results in shorter penetration depth and steeper fronts of the solutions. Actually, it is hard to detect any effect of the parameter p of the ramping surface departure, and the profiles are practically indistinguishable,with respect to the values of the ramping rate parameter p, especially when they are presented by the dimensionless variable $0 < X < 1$. It is noteworthy that both parameters p and m form a correction factor $f_{(m,\,p)}$ that sharply increases and reduces the penetration depth with increase either in p or in the model degeneracy through m. This effect can be simply explained when numerical factor $F_{n,m,p}$ can be presented as $F_{n,m,p}(\text{DIM})/f_{(m,\,p)}$:

$$F_{n,mp}(DIM) = \sqrt{(n+1)(n+2)}$$

as in the numerical factor of the integral-balance (DIM) solution for $m = 0$ and p = 0. And, $f_{(m,\,p)}$, from eq. (14):

$$f_{(m,p)} = \frac{\{(m+1)[p(m+1)+2]\}}{2}$$

the plot is shown in fig. 3(d).

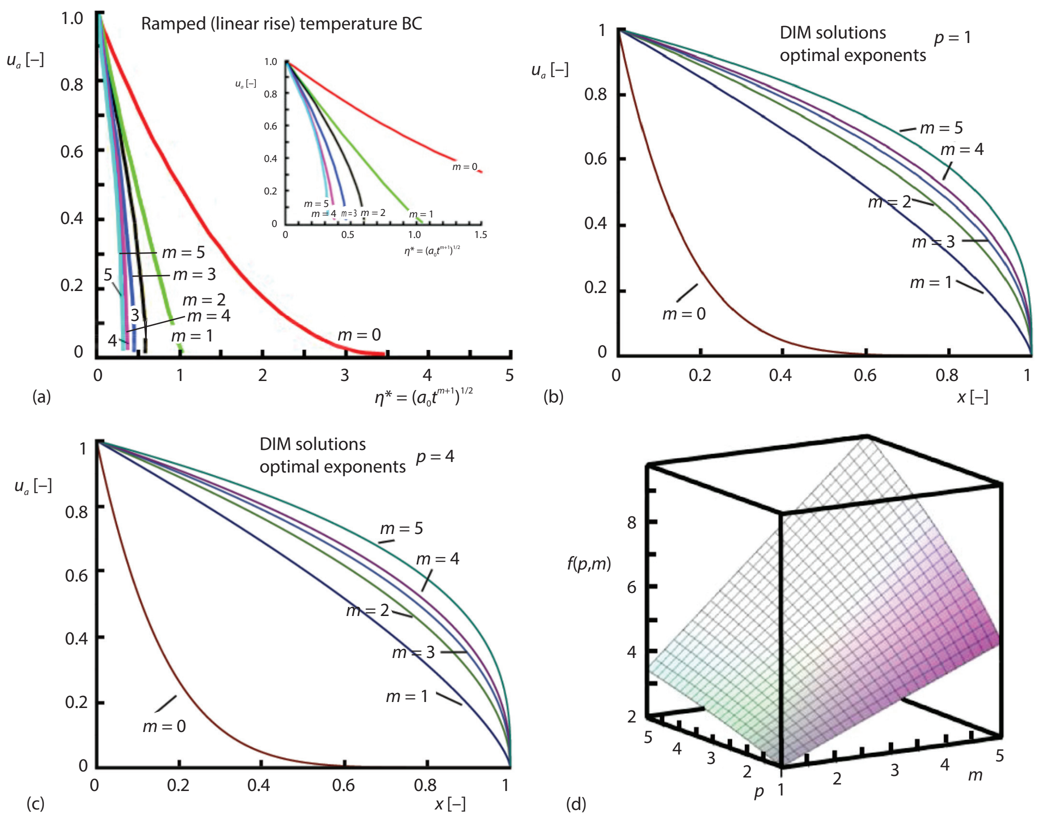

Figure 3. Temperature profiles with optimal exponents (DIM solutions) for various non-linear parameters *m* and different ramping programs of surface heating

Conclusion

The work presented analytical solution of a transient heat conduction with a power-law temperature dependence of the thermal diffusivity with difference ramping surface heating (surface temperature rise). The solution developed by the approximate integral-balance method (double-integration techniques) reveal adequate physical behaviour with sound results which can be attributed to the slow diffusion behaviour of the modelling degenerate parabolic equation. The increase in the non-linearity of the thermal diffusivity leads to steep fronts of the solutions and shorter heated layer in depth of the medium. In general, the rate of ramp rise, has no strong effect on the temperature profiles since the non-linearity of the modelling equations dominates.

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