DETERMINATION OF SOURCE TERM FOR FRACTIONAL HEAT EQUATION ON THE SPHERE

by

Nguyen Duc PHUONG [a], ***Tran Thanh BINH*** [b], ***Nguyen Hoang LUC*** [b], **and *Nguyen Huu CAN*** [c*]

[a] Faculty of Fundamental Science, Industrial University of Ho Chi Minh City, Ho Chi Minh City, Vietnam
[b] Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Thu Dau Mot City, Vietnam
[c] Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

Original scientific paper
https://doi.org/10.2298/TSCI20S1361P

In this work, we study a truncation method to solve a time fractional diffusion equation on the sphere of an inverse source problem which is ill-posed in the sense of Hadamard. Through some priori assumption, we present the error estimates between the regularized and exact solutions.

Key words: *time fractional diffusion, heat on the sphere, convergence estimates, inverse source problem, ill-posed problem*

Introduction

The PDE on the sphere has many applications in various fields, for example, potential theory, physical geodesy, meteorology, and oceanography. The direct problem for heat equation on the sphere and numerical approximation of it has been considered by many authors, such as Le Gia Quoc Thong [1, 2]. However, in the case inverse problem of PDE on the sphere, to the best of authors's knowledge, there are limited results on this area. In the sense of Hadamard, the inverse problem is ill-posed. According to the definition of Hadamard, a problem is called well-posed if it satisfies three conditions:

Condition 1. Existence: There exists a solution of the problem.

Condition 2. Uniqueness: If there exists a solution, it is unique.

Condition 3. Stability: The solution's behaviour changes continuously with the input conditions.

A problem is called ill-posed if it is not well-posed. In this article, we study the problem:

$$u_t - \frac{\partial}{\partial t}\left[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Delta^* u(s)\,\mathrm{d}s\right] = \psi(t) f(x),\ x \in S^n,\ 0 < t < T \tag{1}$$

with the following conditions

$$u(x,0) = 0,\ x \in S^n,\ \text{and}\ u(x,T) = h(x),\ x \in S^n$$

* Corresponding author, e-mail: nguyenhuucan@tdtu.edu.vn

Here, the Riemann-Liouville fractinal integral of w of order α is given:

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} w(s)\mathrm{d}s$$

In the area *fractional calculus* which has a long history and it keeps an important situation in the physical model or real-life, for example, porous media [3], the predator-prey dynamic [4] and references therein.

A suitable fractional operator which is selected depending on the physical system is really necessary. Most selected models are the fractional diffusion equation using the Riemann-Liouville or Cauputo derivatives. Many applications of these models in fields of science such as [3, 5-18]. In recent times, the mathematical model of time-fractional diffusion equation is a class of important physical phenomena. Such equations describe relaxation phenomena in complex viscoelastic materials, subdiffusion processes diffusion, and some other processes.

In recent years, many regularization methods have been introduced to consider the inverse source problem for time-fractional diffusion equation, for example [19, 20] and we can see more in the references therein. Based on our best search results, there no results on inverse source problem for the time fractional diffusion equations on the sphere. This work may be the first study on this topic. Our main goal in this work is to establish a Fourier regularized method for finding approximate solution.

Preliminaries

We introduce a function of the Mittag-Leffler type:

$$E_{\alpha,\beta}(\omega) = \sum_{n=0}^{\infty} \frac{\omega^n}{\Gamma(\alpha n+\beta)}, \quad \omega \in C \tag{2}$$

which is an important role in the time-fractional equations in PDE. Here, $\beta \in R$ and $\alpha > 0$ are given constants. Some discussion in [21] about properties of the Mittag-Leffler function which can be found.

Lemma 1 Let $0 < \alpha_0 < \alpha_1 < 1$. Then there exits A^-, A^+, B^-, $B^+ > 0$ which are constants depending only on γ satisfies:

$$\frac{A^-}{\alpha}\mathrm{e}^{z^{1/\gamma}} \le E_{\alpha,1}(z) \le \frac{A^+}{\alpha}\mathrm{e}^{z^{1/\alpha}}, \text{ for all } z \ge 0$$

$$\frac{B^-}{\Gamma(1-\alpha)}\frac{1}{1-z} \le E_{\alpha,1}(z) \le \frac{B^+}{\Gamma(1-\alpha)}\frac{1}{1-z}, \text{ for all } z \le 0$$

for all $\alpha \in [\alpha_0, \alpha_1]$.

Lemma 2. Assume that $0 < \alpha < 1$, the Mittag-Leffler function have asymptotics:

$$E_{\alpha,1}(z) = \frac{1}{\alpha}\mathrm{e}^{z^{1/\alpha}} - \frac{1}{z\Gamma(1-\alpha)} + O\left(\frac{1}{z^2}\right), \quad 0 < z \to +\infty$$

$$E_{\alpha,1}(z) = -\frac{1}{z\Gamma(1-\alpha)} + O\left(\frac{1}{z^2}\right), \quad -\infty \leftarrow z < 0$$

$$E_{\alpha,0}(z) = \frac{1}{\alpha}z^{1/\alpha}\mathrm{e}^{z^{1/\alpha}} - \frac{1}{z\Gamma(-\alpha)} + O\left(\frac{1}{z^2}\right), \quad 0 < z \to +\infty$$

$$E_{\alpha,0}(z) = -\frac{1}{z\Gamma(-\alpha)} + O\left(\frac{1}{z^2}\right), \ -\infty \leftarrow z < 0$$

For Δ^* is the Laplacian operator in R^{n+1}. On the surface of the Euclidean sphere S^n, spherical harmonics are in form a polynomial which satisfy $\Delta^* Y(x)$ are restricted.

We have the eigenfunctions are the spherical harmonics $Y_l(x)$ of order l satisfies:

$$\Delta^* Y_l(x) = -\lambda_l Y_l(x)$$

where $l = 0, 1, 2,...$, we have the eigenvalues for $-\Delta^*$:

$$\lambda_l = l(l+n-1)$$

Denote by V_l, the space of all spherical harmonics of degree l on S^n has an orthonormal basis:

$$\{Y_{lk}(x)\}, \text{ for } k = 1,2,3,...,N(n,l)$$

here

$$N(n,l) = \frac{(2l+n-1)\Gamma(l+n-1)}{\Gamma(l+1)\Gamma(n)}, \ l = 1,2,3,...$$

Especially, $N(n, 0) = 1$ for all n. Let:

$$\hat{f}_{lk} \quad \int f\overline{Y}_{lk}$$

and the surface measure of the unit sphere is dS, we can be expanded in terms of spherical harmonics of the function $f \in L^2(S^n)$:

$$f = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} \hat{f}_{lk} Y_{lk}$$

Authentically, we have the fllowing norm stems from an inner product in the Sobolev space $H^\sigma(S^n)$:

$$\langle f, g \rangle_{H^\sigma(S^n)} = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} (1+\lambda_l)^\sigma \hat{f}_{lk} \hat{g}_{lk}$$

where the real parameter σ consists of all distributions f satisfies:

$$\| f \|_{H^\sigma(S^n)}^2 = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} (1+\lambda_l)^\sigma \, | \hat{f}_{lk} |^2 < \infty$$

Regularization and error estimate

Let us recall that Δ^* is the Laplace-Beltrami on the sphere S^n. The inverse problem is for construction the initial value data $u(x, 0) = f(x)$ from the known final data value $u(x, T) = g(x)$. Now, we find an explicit fomula of the mild solution of Problem (1). We assume that dS is the surface measure of the unit sphere and $u \in L^2(S^n)$ which can be expanded in terms of the following spherical harmonics:

$$u(x,t) = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} \hat{u}_{lk}(t) Y_{lk}(x), \ \hat{u}_{lk}(t) = \int_{S^n} u(x,t) \overline{Y}_{lk}(x) \mathrm{d}S$$

let

$$\hat{f}_{lk} = \int_{S^n} f(x)\overline{Y}_{lk}(x)\mathrm{d}S$$

by taking the inner product of Y_{lk}:

$$\frac{\mathrm{d}\hat{u}_{lk}}{\mathrm{d}t} + \lambda_l \frac{\partial}{\partial t}\left[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\hat{u}_{lk}(s)\mathrm{d}s\right] = \psi(t)\hat{f}_{lk},\ \hat{u}_{lk}(T) = \hat{g}_{lk}$$

we get

$$u(x,t) = \sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)} E_{\alpha,1}\left(-\lambda_l t^{\alpha}\right)\hat{u}_{0lk}Y_{lk}(x) + \\ +\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)}\left\{\int_0^t \psi(s)E_{\alpha_1}\left[-\lambda_l(t-s)^{\alpha}\right]\hat{f}_{lk}\mathrm{d}s\right\}Y_{lk}(x) \tag{3}$$

Letting $t = T$ in previous expression and noting that $u_0 = 0$, which yields:

$$h(x) = u(x,T) = \sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)}\left\{\int_0^T \psi(s)E_{\alpha,1}\left[-\lambda_l(T-s)^{\alpha}\right]\mathrm{d}s\right\}\hat{f}_{lk}Y_{lk}(x) \tag{4}$$

Due to the unique of Fourier expansion of the function on $L^2(S^n)$, we deduce:

$$\hat{h}_{lk} = \left\{\int_0^T \psi(s)E_{\alpha,1}\left[-\lambda_l(T-s)^{\alpha}\right]\mathrm{d}s\right\}\hat{f}_{lk} \tag{5}$$

Hence, we obtain:

$$\hat{f}_{lk} = \frac{\hat{h}_{lk}}{\int_0^T \psi(s)E_{\alpha,1}\left[-l(l+n-1)(T-s)^{\alpha}\right]\mathrm{d}s} \tag{6}$$

So, we have the equation:

$$f(x) = \sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)}\frac{\hat{h}_{lk}Y_{lk}(x)}{\int_0^T \psi(s)E_{\alpha,1}\left[-l(l+n-1)(T-s)^{\alpha}\right]\mathrm{d}s} \tag{7}$$

Lemma 3. Let $0 < \alpha < 1$ and $\lambda_j > 0$. Then:

$$\frac{\overline{D}_1}{l(l+n-1)} \le \int_0^T E_{\alpha,1}\left[-l(l+n-1)(T-s)^{\alpha}\right]\mathrm{d}s \le \frac{\overline{D}_2}{l(l+n-1)} \tag{8}$$

where

$$\overline{D}_1 = \frac{D_1 T}{\frac{1}{\lambda_1} + T^{\beta}},\ \overline{D}_2 = \frac{D_2 T^{1-\beta}}{1-\beta}$$

Proof. Since the bound of the function $E_{\alpha 1}$ we derive:

$$\int_0^T E_{\alpha,1}\left[-l\left(l+n-1\right)\left(T-s\right)^{\alpha}\right]\mathrm{d}s \le D^2\int_0^T \frac{\mathrm{d}s}{1+l\left(l+n-1\right)\left(T-s\right)^{\alpha}} \le$$
$$\le \frac{D_2}{l\left(l+n-1\right)}\int_0^T \frac{\mathrm{d}s}{\left(T-s\right)^{\alpha}} = \frac{D_2 T^{1-\alpha}}{\left(1-\alpha\right)l\left(l+n-1\right)} = \frac{\overline{D}_2}{l\left(l+n-1\right)} \tag{9}$$

noting that

$$l\left(l+n-1\right)\left(\frac{1}{l^2}+T^{\alpha}\right) \ge 1+l\left(l+n-1\right)(T-s)^{\alpha}$$

we also get

$$\int_0^T E_{\alpha,1}\left[-l\left(l+n-1\right)\left(T-s\right)^{\alpha}\right]\mathrm{d}s \ge \frac{D_1}{\frac{1}{l^2}+T^{\alpha}}\int_0^T \frac{\mathrm{d}s}{l\left(l+n-1\right)} = \frac{\overline{D}_1}{l\left(l+n-1\right)} \tag{10}$$

where

$$\overline{D}_1 = \frac{D_1 T}{\frac{1}{l^2}+T^{\alpha}}$$

Now, we give the following *Theorem* which show the converence rate between the exact and regularized solutions. In Part 1 of *Theorem 1*, we give the convergence rate in $L^2(S^n)$ and Part 2 of *Theorem 1*, we show the convergence rate in higher space $H^p(S^n)$ for $p > 0$.

Theorem 1. Let $g_\delta \in L^2(S^n)$:

$$\| h_\delta - h \|_{L^2\left(S^n\right)} + \| \psi_\delta - \psi \|_{L^\infty\left(0,T\right)} \le \delta \tag{11}$$

Let us assume that

$$m_0 \le \max\left(\| \psi_\delta \|_{L^\infty\left(0,T\right)}, \| \psi \|_{L^\infty\left(0,T\right)}\right) \le m_1.$$

Let us construct a regularized solution:

$$f_{\mathcal{M},\delta}\left(x\right) = \sum_{l=0}^{l\le\mathcal{M}_\delta}\sum_{k=1}^{N(n,l)} \frac{\hat{h}_{\delta,lk} Y_{lk}\left(x\right)}{\int_0^T \psi_\delta\left(s\right) E_{\alpha,1}\left[-l\left(l+n-1\right)\left(T-s\right)^{\alpha}\right]\mathrm{d}s} \tag{12}$$

– Let us assume that $f \in H^\sigma(S^n)$ for any $\sigma > 0$. Let us choose $\mathcal{M}_\delta = \delta^{(k-1)/2}$ for any $0 < k < 1$:

$$\| f_{\mathcal{M},\delta} - f \|_{L^2\left(S^n\right)} \le C\max\left[\delta^k, \delta^{\left(1-k\right)\sigma}\right] \tag{13}$$

where C depends on $\| f \|_{H^\sigma(S^n)}$ and independent of δ.

– Let us assume that $f \in H^\sigma(S^n)$ for any $\sigma > 0$. Let us choose $\mathcal{M}_\delta = \delta^{(r-1)/(2+p)}$ for any $0 < r < 1$:

$$\| f_{\mathcal{M},\delta} - f \|_{H^p\left(S^n\right)} \le \overline{C}\max\left[\delta^r, \delta^{\frac{(1-r)(\sigma-p)}{p+2}}\right] \tag{14}$$

for any $0 < p < \sigma$, where $\overline{C}$ is independent of δ and depends on $\| f \|_{H^\sigma(S^n)}$.

Proof. Let us show Part I of *Theorem 1*:

$$b_{1,\delta}(x)=\sum_{l=0}^{l=\mathcal{M}_\delta}\sum_{k=1}^{N(n,l)}\frac{\hat{h}_{lk}Y_{lk}(x)}{\int_0^T\psi_\delta(s)E_{\alpha,1}\left[-l(l+n-1)(T-s)^\alpha\right]\mathrm{d}s} \tag{15}$$

and

$$b_{2,\delta}(x)=\sum_{l=0}^{l=\mathcal{M}_\delta}\sum_{k=1}^{N(n,l)}\frac{\hat{h}_{lk}Y_{lk}(x)}{\int_0^T\psi(s)E_{\alpha,1}\left[-l(l+n-1)(T-s)^\alpha\right]\mathrm{d}s} \tag{16}$$

Step 1. Estimate $\| f_{\mathcal{M},\delta}-b_{1,\delta}\|_{L^2(S^n)}$. In order to treat this term, we first note:

$$\int_0^T\psi_\delta(s)E_{\alpha,1}\left[-l(l+n-1)(T-s)^\alpha\right]\mathrm{d}s\geq m_0\int_0^T E_{\alpha,1}\left[-l(l+n-1)(T-s)^\alpha\right]\mathrm{d}s\geq$$

$$\geq\frac{m_0D_1T}{\left(\frac{1}{l^2}+T^\alpha\right)l(l+n-1)} \tag{17}$$

Therefore, we find:

$$\| f_{\mathcal{M},\delta}-b_{1,\delta}\|^2_{L^2(S^n)}=\sum_{l=0}^{l\leq\mathcal{M}_\delta}\sum_{k=1}^{N(n,l)}\left(\frac{\hat{h}_{\delta,lk}-\hat{h}_{lk}}{\int_0^T\psi(s)E_{\alpha,1}\left[-l(l+n-1)(T-s)^\alpha\right]\mathrm{d}s}\right)^2\leq$$

$$\leq\sum_{l=0}^{l\leq\mathcal{M}_\delta}\sum_{k=1}^{N(n,l)}\frac{\left(\frac{1}{l^2}+T^\alpha\right)^2l^2(l+n-1)^2}{m_0^2D_1^2T^2}\left(\hat{h}_{\delta,lk}-\hat{h}_{lk}\right)^2\leq\frac{\left(1+T^\alpha\right)^2}{m_0^2D_1^2T^2}\left(|\mathcal{M}_\delta|^2\,(\mathcal{M}_\delta+n-1)^2\right)\cdot$$

$$\cdot\sum_{l=0}^{\infty}\sum_{k=1}^{N(n,l)}\left(\hat{h}_{\delta,lk}-\hat{h}_{lk}\right)^2\leq\frac{\left(1+T^\alpha\right)^2}{m_0^2D_1^2T^2}\left[|\mathcal{M}_\delta|^2\,(\mathcal{M}_\delta+n-1)^2\right]\| h_\delta-h\|^2_{L^2(S^n)}\leq$$

$$\leq\frac{\left(1+T^\alpha\right)^2}{m_0^2D_1^2T^2}\left[|\mathcal{M}_\delta|^2\,(\mathcal{M}_\delta+n-1)^2\right]\delta^2 \tag{18}$$

therefore

$$\| f_{\mathcal{M},\delta}-b_{1,\delta}\|_{L^2(S^n)}\leq\frac{\left(1+T^\alpha\right)}{m_0D_1T}\left[|\mathcal{M}_\delta|(\mathcal{M}_\delta+n-1)\right]\delta \tag{19}$$

Step 2. Estimate $\| b_{1,\delta}-b_{2,\delta}\|_{L^2(S^n)}$:

$$\| b_{1,\delta}-b_{2,\delta}\|^2_{L^2(S^n)}=\sum_{l=0}^{l\leq\mathcal{M}_\delta}\sum_{k=1}^{N(n,l)}|\hat{h}_{lk}|^2\cdot$$

$$\cdot\frac{1}{\int_0^T\psi(s)E_{\alpha,1}\left[-l(l+n-1)(T-s)^\alpha\right]\mathrm{d}s}-\frac{1}{\int_0^T\psi_\delta(s)E_{\alpha,1}\left[-l(l+n-1)(T-s)\alpha\right]\mathrm{d}s} \tag{20}$$

it is obvious

$$\int_0^T \left[\psi(s)-\psi_\delta(s)\right] E_{\alpha,1}\left[-l(l+n-1)(T-s)^\alpha\right]\mathrm{d}s \le$$
$$\le \|\psi_\delta-\psi\|_{L^\infty(0,T)} \int_0^T E_{\alpha,1}\left[-l(l+n-1)(T-s)^\alpha\right]\mathrm{d}s \le \frac{\bar{D}_2\delta}{l(l+n-1)} \tag{21}$$

This latter inequality together with eq. (17) implies:

$$\left| \frac{1}{\int_0^T \psi(s) E_{\alpha,1}\left[-l(l+n-1)(T-s)^\alpha\right]\mathrm{d}s} - \frac{1}{\int_0^T \psi_\delta(s) E_{\alpha,1}\left[-l(l+n-1)(T-s)^\alpha\right]\mathrm{d}s} \right| \le$$
$$\le \frac{\dfrac{\bar{D}_2\delta}{l(l+n-1)}}{\left[\dfrac{m_0 D_1 T}{\left(\dfrac{1}{l^2}+T^\alpha\right)l(l+n-1)}\right]^2} = \frac{\left(1+T^\alpha\right)\bar{D}_2 l(l+n-1)}{m_0^2 T^2 D_1^2}\delta \tag{22}$$

Combining eqs. (20) and (22), we find:

$$\|b_{1,\delta}-b_{2,\delta}\|^2_{L^2(S^n)} \le \left[\frac{\left(1+T^\alpha\right)\bar{D}_2}{m_0^2 T^2 D_1^2}\right]^2 \left[|\mathcal{M}_\delta|^2 \left(\mathcal{M}_\delta+n-1\right)^2\right]\delta^2 \tag{23}$$

which allows:

$$\|b_{1,\delta}-b_{2,\delta}\|_{L^2(S^n)} \le \left[\frac{\left(1+T^\alpha\right)\bar{D}_2}{m_0^2 T^2 D_1^2}\right] |\mathcal{M}_\delta| \left(\mathcal{M}_\delta+n-1\right)\delta \tag{24}$$

Step 3. Estimate $\|f-b_{2,\delta}\|_{L^2(S^n)}$ Indeed, since $\lambda_l = l\,(l+n-1)$, we get:

$$\|f-b_{2,\delta}\|^2_{L^2(S^n)} = \sum_{l>\mathcal{M}_\delta}\sum_{k=1}^{N(n,l)} \left(\frac{\hat{h}_{\delta,lk}}{\int_0^T \psi_\delta(s) E_{\alpha,1}\left[-l(l+n-1)(T-s)^\alpha\right]\mathrm{d}s}\right)^2 =$$
$$= \sum_{l>\mathcal{M}_\delta}\sum_{k=1}^{N(n,l)} (1+\lambda_l)^{-\sigma}(1+\lambda_l)^{\sigma} |\hat{f}_{lk}|^2 \le \left[1+l(l+n-1)\right]^{-\sigma} \|f\|^2_{H^\sigma(S^n)} \le$$
$$\le \frac{1}{\left[1+\mathcal{M}_\delta\left(\mathcal{M}_\delta+n-1\right)\right]^\sigma} \|f\|^2_{H^\sigma(S^n)} \tag{25}$$

From three aforementioned steps:

$$\| f_{\mathcal{M},\delta} - f \|_{L^2(S^n)} \leq \| f_{\mathcal{M},\delta} - b_{1,\delta} \|_{L^2(S^n)} + \| b_{1,\delta} - b_{2,\delta} \|_{L^2(S^n)} + \| f - b_{2,\delta} \|_{L^2(S^n)} \leq$$

$$\leq \sqrt{\frac{1}{\left[1+\mathcal{M}_\delta\left(\mathcal{M}_\delta+n-1\right)\right]^\sigma}} \| f \|_{H^\sigma(S^n)} + \left[\frac{\left(1+T^\alpha\right)\bar{D}_2}{m_0^2 T^2 D_1^2}\right] |\mathcal{M}_\delta| \left(\mathcal{M}_\delta+n-1\right)\delta +$$

$$+\frac{\left(1+T^\alpha\right)}{m_0 D_1 T} |\mathcal{M}_\delta| \left(\mathcal{M}_\delta+n-1\right)\delta \tag{26}$$

Next, we continue to show Part II of *Theorem 1*.

Step 4. Estimate $\| f_{\mathcal{M},\delta} - b_{1,\delta} \|_{H^p(S^n)}$. By a similar argument as *Step 1*:

$$\| f_{\mathcal{M},\delta} - b_{1,\delta} \|^2_{H^p(S^n)} = \sum_{l=0}^{l\leq\mathcal{M}_\delta} \sum_{k=1}^{N(n,l)} \left[1+l(l+n-1)\right]^p \left[\frac{\hat{h}_{\delta,lk} - \hat{h}_{lk}}{\int_0^T \psi(s) E_{\alpha,1}\left[-l(l+n-1)^p (T-s)^\alpha\right] \mathrm{d}s}\right]^2 \leq$$

$$\leq \sum_{l=0}^{l\leq\mathcal{M}_\delta} \sum_{k=1}^{N(n,l)} \frac{\left(\frac{1}{l^2}+T^\alpha\right)^2 l^2 (l+n-1)^2}{m_0^2 D_1^2 T^2} \left[1+l(l+n-1)\right]^p \left(\hat{h}_{\delta,lk} - \hat{h}_{lk}\right)^2 \leq$$

$$\leq \frac{\left(1+T^\alpha\right)^2}{m_0^2 D_1^2 T^2} \left[|\mathcal{M}_\delta|^2 \left(\mathcal{M}_\delta+n-1\right)^2\right] \left[1+\mathcal{M}_\delta\left(\mathcal{M}_\delta+n-1\right)\right]^p \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} \left(\hat{h}_{\delta,lk} - \hat{h}_{lk}\right)^2 \leq$$

$$\leq \frac{\left(1+T^\alpha\right)^2}{m_0^2 D_1^2 T^2} \left[|\mathcal{M}_\delta|^2 \left(\mathcal{M}_\delta+n-1\right)^2\right] \left[1+\mathcal{M}_\delta\left(\mathcal{M}_\delta+n-1\right)\right]^p \| h_\delta - h \|^2_{L^2(S^n)} \leq$$

$$\leq \frac{\left(1+T^\alpha\right)^2}{m_0^2 D_1^2 T^2} \left[|\mathcal{M}_\delta|^2 \left(\mathcal{M}_\delta+n-1\right)^2\right] \left[1+\mathcal{M}_\delta\left(\mathcal{M}_\delta+n-1\right)\right]^p \delta^2 \tag{27}$$

therefore

$$\| f_{\mathcal{M},\delta} - b_{1,\delta} \|_{L^2(S^n)} \leq \frac{\left(1+T^\alpha\right)^2}{m_0 D_1 T} \left[|\mathcal{M}_\delta| \left(\mathcal{M}_\delta+n-1\right)\right] \left[1+\mathcal{M}_\delta\left(\mathcal{M}_\delta+n-1\right)\right]^{p/2} \delta \tag{28}$$

Step 5. Estimate $\| b_{1,\delta} - b_{2,\delta} \|_{H^p(S^n)}$. Indeed, we get:

$$| b_{1,\delta} - b_{2,\delta} \|^2_{H^p(S^n)} = \sum_{l=0}^{l\leq\mathcal{M}_\delta} \sum_{k=1}^{N(n,l)} \left[1+l(l+n-1)\right]^p |\hat{h}_{lk}|^2$$

$$\left[\frac{1}{\int_0^T \psi(s) E_{\alpha,1}\left[-l(l+n-1)^p (T-s)^\alpha\right] \mathrm{d}s} - \frac{1}{\int_0^T \psi_\delta(s) E_{\alpha,1}\left[-l(l+n-1)^p (T-s)^\alpha\right] \mathrm{d}s}\right]^2 \tag{29}$$

This latter inequality together with (21) and (22) implies:

$$\| b_{1,\delta} - b_{2},_{\delta} \|^2_{H^p(S^n)} \le \left[\frac{\left(1+T^\alpha\right)\bar{D}_2}{m_0^2 T^2 D_1^2} \right]^2 \left[| \mathcal{M}_\delta |^2 \left(\mathcal{M}_\delta + n - 1\right)^2 \right] \left[1 + \mathcal{M}_\delta \left(\mathcal{M}_\delta + n - 1\right) \right]^p \delta^2 \tag{30}$$

which allows:

$$\| b_{1,\delta} - b_{2},_{\delta} \|_{H^p(S^n)} \le \left[\frac{\left(1+T^\alpha\right)\bar{D}_2}{m_0^2 T^2 D_1^2} \right] \left[| \mathcal{M}_\delta | \left(\mathcal{M}_\delta + n - 1\right) \right] \left[1 + \mathcal{M}_\delta \left(\mathcal{M}_\delta + n - 1\right) \right]^{p/2} \delta \tag{31}$$

Step 6. Estimate $\| f - b_{2,\delta} \|_{H^p(Sn)}$. Indeed, since $\lambda_l = l\,(l + n - 1)$, we get:

$$\| f - b_{2,\delta} \|^2_{H^p(S^n)} = \sum_{l > \mathcal{M}_\delta} \sum_{k=1}^{N(n,l)} \left[1 + l(l+n-1)\right]^p \left[\frac{\hat{h}_{\delta,lk}}{\int_0^T \psi_\delta(s) E_{\alpha,1}\left[-l(l+n-1)(T-s)^\alpha\right] \mathrm{d}s} \right]^2 =$$

$$= \sum_{l > \mathcal{M}_\delta} \sum_{k=1}^{N(n,l)} \left[1 + l(l+n-1)\right]^p \left[1 + l(l+n-1)\right]^{-\sigma} \left[1 + l(l+n-1)\right]^{\sigma} | \hat{h}_{lk} |^2 \le$$

$$\le \frac{1}{\left[1 + \mathcal{M}_\delta \left(\mathcal{M}_\delta + n - 1\right)^{\sigma - p}\right]} \| f \|^2_{H^\sigma(S^n)} \tag{32}$$

for $p < \sigma$. From *Steps 4-6*, previously mentioned, we deduce:

$$\| f_{\mathcal{M},\delta} - f \|_{H^p\left(S^n\right)} \le \| f_{\mathcal{M},\delta} - b_{1,\delta} \|_{H^p\left(S^n\right)} + \| b_{1,\delta} - b_{2,\delta} \|_{H^p\left(S^n\right)} + \| f - b_{2,\delta} \|_{H^p\left(S^n\right)} \le$$

$$\le \frac{1}{\left[1 + \mathcal{M}_\delta \left(\mathcal{M}_\delta + n - 1\right)\right]^{\frac{\sigma - p}{2}}} \| f \|_{H^p\left(S^n\right)} + \frac{\left(1+T^\alpha\right)}{m_0 D_1 T} \left[| \mathcal{M}_\delta | \left(\mathcal{M}_\delta + n - 1\right) \right]$$

$$\left[1 + \mathcal{M}_\delta \left(\mathcal{M}_\delta + n - 1\right)\right]^{p/2} \delta + \frac{\left(1+T^\alpha\right)\bar{D}_2}{m_0^2 T^2 D_1^2} \left[| \mathcal{M}_\delta |^2 \left(\mathcal{M}_\delta + n - 1\right) \right] \left[1 + \mathcal{M}_\delta \left(\mathcal{M}_\delta + n - 1\right)\right]^{p/2} \delta \tag{33}$$

By choose$\mathcal{M}_\delta = \delta^{(r-1)/(2+p)}$ and after some simple caculation, we follow from (33):

$$\| f_{\mathcal{M},\delta} - f \|_{H^p\left(S^n\right)} \le \bar{C} \max\left(\delta^r, \delta^{\frac{(1-r)(\sigma - p)}{p+2}} \right) \tag{34}$$

where $\bar{C}$ is independent of δ and depends on $\| f \|_{H^p(S^n)}$.

Acknowledgment

This research is supported by Industrial University of Ho Chi Minh City (IUH) under grant number 66/HĐĐHCN. The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improve the quality of this paper.

References

[1] Le Gia, Q. T., Approximation of Parabolic PDE on Spheres Using Collocation Method, *Adv. Comput. Math.*, *22* (2005), May, pp. 377-397

[2] Gia, Q. T., L., Approximation of Parabolic PDE on Spheres Using Collocation Method, *Adv. Comput. Math*., *22* (2005), Jan., pp. 377-397
[3] Atangana, A., *Derivative with a New Parameter: Theory*, *Methods and Applications*, Academic Press, New York, USA, 2016
[4] Rihan, A. R., *et al*., Fractional-Order Delayed Predator-Prey Systems with Holling Type-II Functional Response, *Non-Linear Dynam*., *80* (2015), 1-2, pp. 777-789
[5] Kumar, D., *et al*., New Analysis for Fractional Model of Regularized Long-Wave Equation Arising in Ion Acoustic Plasma Waves, *Mathematical Methods in the Applied Sciences*, *40* (2017), 15, pp. 5642-5653
[6] Tuan, N. H., *et al*., Tikhonov Regularization Method for a Backward Problem for the Inhomogeneous Time-Fdiffusion Equation, *Appl. Anal*., *97* (2018), 5, pp. 842-863
[7] Trong, D. D., Tuan, N. H., A Non-Homogeneous Backward Heat Problem: Regularization and Error Estimates, *Electronic Journal of Differential Equations*, (2008), 33, pp. 1-14
[8] Trong, D. D., *et al*., A Quasi-Boundary Value Method for Regularizing Non-Linear Ill-Posed Problems, *Electronic Journal of Differential Equations*, (2009), 109, pp. 1-16
[9] Sakamoto, K., Yamamoto, M., Initial Value/Boundary Value Problems for Fractional Diffusion-Wave Equations and Applications to Some Inverse Problems, *Journal Math. Anal. Appl*., *382* (2011), 1, pp. 426-447
[10] McLean, W., Regularity of Solutions to a Time-Fractional Diffusion Equation, *ANZIAM J.*, *52* (2010), 2, pp. 123-138
[11] Tuan, N. H., Trong, D. D.,On a Backward Parabolic Problem with Local Lipschitz Source, *Journal Math. Anal. Appl*., *414* (2014), 2, pp. 678-692
[12] Tuan, N. H., *et al*., Identification of the Population Density of a Species Model with Non-Local Diffusion and Non-Linear Reaction, *Inverse Problems*, *33* (2017), 5, 055019, 40
[13] Tuan, N. H., *et al*., Analysis of a Quasi-Reversibility Method for a Terminal Value Quasi-Linear Parabolic Problem with Measurements, *SIAM J. Math. Anal*., *51* (2019), 1, pp. 60-85
[14] Fernandez, A., *et al*., On Fractional Calculus with General Analytic Kernels, *Appl. Math. Comput*., *354* (2019), Aug., pp. 248-265
[15] Fernandez, A., *et al*., Series Representations for Fractional-Calculus Operators Involving Generalised Mittag-Leffler Functions, *Commun. Non-linear Sci. Numer. Simul*., *67* (2019), Feb., pp. 517-527
[16] Srivastava, H. M., Some New Fractional-Calculus Connections between Mittag-Leffler Functions, Math*ematics*, 7 (2019), 6, 485
[17] Korpinar, Z., *et al*., Theory and Application for the System of Fractional Burger Equations with Mittag--Leffler Kernel, *Applied Mathematics and Computation*, *367* (2020), Feb., 124781
[18] Akgul, A., *et al*., A New Approach for 1-D Sine-Gordon Equation, *Adv. Difference Equ*., *2016* (2016), Dec., 8
[19] Zhang, Y., Xu, X., Inverse Source Problem for a Fractional Diffusion Equation, *Inverse Probl*., 27, (2011), 3, 035010
[20] Wang, J. G., *et al*., Two Regularization Methods to Identify a Space-Dependent Source for the Time-Fractional Diffusion Equation, *Appl. Numer. Math*., *68* (2013), June, pp. 39-57
[21] Podlubny, I., *Fractional Differential Equations*, Academic Press, London, UK, 1999

Paper submitted: June 5, 2020
Paper revised: June 30, 2020
Paper accepted: July 15, 2020