ANALYTICAL SOLUTIONS TO CONTACT PROBLEM WITH FRACTIONAL DERIVATIVES IN THE SENSE OF CAPUTO

by

Muhammad Aslam NOOR[a], Muhammad RAFIQ[b*], Salah-Ud-Din KHAN[c], Muhammad Amer QURESHI[d], Muhammad KAMRAN[b], Shahab-Ud-Din KHAN[e], Faisal SAEED[b], and Hijaz AHMAD[*f]

[a] Department of Mathematics, COMSATS University Islamabad, Islamabad Campus, Pakistan
[b] Department of Mathematics, COMSATS University Islamabad, Wah Campus, Pakistan
[c] College of Engineering, King Saud University, Riyadh, Saudi Arabia
[d] PYP Mathematics, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia
[e] National Tokamak Fusion Program, Islamabad, Pakistan
[f] Department of Basic Sciences, University of Engineering and Technology Peshawar, Pakistan

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The current study extends the applications of the variational iteration method for the analytical solution of fractional contact problems. The problem involves Caputo sense while calculating the derivative of fractional order, we apply the Penalty function technique to transform it into a system of fractional boundary value problems coupled with a known obstacle. The variational iteration method is employed to find the series solution of fractional boundary value problem. For different values of fractional parameters, residual errors of solutions are plotted to make sure the convergence and accuracy of the solution. The reasonably accurate results show that one of the highly effective and stable methods for the solution of fractional boundary value problem is the method of variational iteration.

Key words: *contact problem, obstacle, variational iteration method, fractional derivative*

Introduction

The generic concept of obstacle problems occur naturally across many fields, not only in mathematics, but also in applications such as electrostatics, control theory, fluid mechanics, physics, relaxation processes, *etc*. Non-etheless, the obstacle problem having the fractional derivative occurs in a wide range, such as the study of aberrant diffusion in fluid dynamics, viscoelastic theory, neurology, theory of electromagnetic acoustics and the pricing of American options [1]. Fractional derivatives are more reliable than integral order derivatives models of realistic problems. They can actually be found to be useful techniques to explain some physical problems. Many researchers make extensive use of such models to understand their convoluted processes and establishing nature issues that are easily understandable for these phenomena without losing their underlying hereditary properties [2]. Such notable developments of physics and financial mathematics have recently made the obstacle problem very interesting. Wang [3] was the first, who utilized the Adomian decomposition method to find the approximate solution of Korteweg-de-Vries Burger non-linear fractional equation. Homotopy analysis method was used to study fractional order algebraic differential equations by Zurigat *et al*. [4]. Iomin [5],

* Corresponding author, e-mail: rafiq@ciitwah.edu.pk; hijaz555@gmail.com

introduced the idea of fraction (in both time and space) for Schroedinger equation, that seems like a strong and proper explanation in the complex inhomogeneous media of diffusive wave transportation. Pade approximations are utilized to get the numerical solution of fractional order PDE by Turut and Guzel in [6]. Liu and Hou [7] implemented the generalized differential transform technique to find the solution of coupled Burger equation involving the fractional derivatives with respect to space and time. A finite difference scheme based on B-spline approximation is used to solve 1-D hyperbolic equation by Abbas *et al*. [8]. Many researchers used various invariants of transform and iterative methods to find the series solution third order fractional PDE, for more details see [9-12].

The spline approximate schemes have many advantages over finite difference techniques as they give continuous, differentiable approximate solution for the spatial co-ordinates with significant accuracy. Bashan *et al.* [13] implemented quantic spline approximation based on quadrature scheme to solve Korteweg-de Vries-Burgers equation numerically. Ramadan *et al*. [14] used a non-polynomial quintic spline method for fourth order boundary value problem (BVP), while third-order BVP along with odd order obstacle problems are solved by Khan and Sultana [15] using non-polynomial quantic spline approximation scheme. Srivastava [16] analysed the numerical solution of differential equations using polynomial spline approximations of various orders. Spline collocation methods based on fifth degree polynomials were carried out by Siddiqi and Arshed [17] to solve PDE. Li and Wong [18] employed a parametric quantic spline technique for the solution of fractional sub-diffusion problem.

The framework of variational inequality has become a powerful tool for the qualitative analysis of obstacles, boundary, unilateral, contact and optimization problems arising in many fields including financial mathematics, physics and engineering sciences. Obstacle problems involving fractional (non-integer) order derivative are modified form of obstacle problems and are evolved a significant area for last few decades for many researchers. Fractional calculus, as compared to conventional ones, can take into consideration memory and inherent properties of various processes and materials. These types of problems arise in chaotic systems, anomalous diffusion, ecological and biological models, *etc*. Stampacchia [19] was the first who introduced variational inequality theory for the analysis of PDE with applications in mechanics. Generally, to obtain the exact solutions of fractional differential equations is difficult. Hence, computational and analytical techniques are therefore, used extensively. Strong and stable computational and theoretical techniques are implemented for the solution of fractional differential equations of practical importance. Many researchers generalized the integer order differential equation to fractional order and employed numerous techniques to find the solution after appropriate modification. Hu *et al*. [20] used Adomian decomposition method, Rani *et al*. [21] employed homotopy perturbation technique and the homotopy analysis method, respectively, Inc [22] implemented variational iteration method (VIM), Modanli and Akgul [23] used theta method for investigating the fractional differential equations, are some of the examples. Method of variation iteration and Laplace transformation based on algorithm were suggested by Martin [24] to solve a fractional differential equation and also discussed the stability of VIM-defined fractional operator. A scheme based on weak formulation is implemented for fractional differential equations of coupled form by Heidarkhani *et al*. [25]. Because of its broad applications in science and engineering, several researchers have recently discussed obstacle-based fractional contact problems.

The most important point to note here is that all of these approaches are attempted for the solutions of non-linear as well as linear BVP that come with obstacle contact and unilateral problems, but less consideration is given to non-linear fractional BVP. The current work uses the VIM to explore the analytical solution of the BVP correlated with the of fractional obstacles problems.

Preliminaries

Here a simple definition is given with respect to the fractional derivative that are used in the following sections. The fractional derivatives of order $\alpha > 0$, see [5] have various definitions. One of the most widely used fractional derivatives definition is attributed to Caputo, for example see [26].

Definition

We suppose that n is the smallest integer equal to or greater than α, then the Caputo derivative of fractional order $\alpha > 0$ is given:

$$\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^x (x-s)^{(n-\alpha-1)} f^{(n)}(s)\mathrm{d}s, & n-1<\alpha<n \\ \dfrac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}, & \alpha = n \end{cases} \tag{1}$$

whereas the value of α is a real number and Γ stands for the gamma function.

Problem

Here fractional obstacle BVP of the type is given:

$$\begin{gathered} -\frac{\mathrm{d}^\alpha w}{\mathrm{d}x^\alpha} \geq h(w,x), \quad x \in [c,d] \\ w(x) \geq \kappa(x), \quad x \in [c,d] \\ \left[\frac{\mathrm{d}^\alpha w}{\mathrm{d}x^\alpha} - f(w,x)\right]\left[w(x)-\kappa(x)\right] = 0, \quad x \in [c,d], \quad 2<\alpha\leq 3 \quad \text{and} \quad c,d \in \Re \end{gathered} \tag{2}$$

Having boundary conditions:

$$u|_{x=c} = \frac{\mathrm{d}w}{\mathrm{d}x}|_{x=a} = \frac{\mathrm{d}w}{\mathrm{d}x}|_{x=b} = 0 \tag{3}$$

In the aforementioned equation, $h(u, x)$ is the continuous functions, the obstacle function $\kappa(x) \leq 0$, $\kappa(x)$ at the end points of the domain, $\Re$ – given for the set of real numbers and $2 < \alpha \leq 3$. Such systems arise in the investigation of unilateral, obstacle, free, and moving boundary value problems and have significant applications in physical oceanography. Many researchers have discussed this problem for $\alpha = 3$, see [27]. They used the theory of variational inequalities to address the uniqueness and existence of the solution of such problems.

If $w(x) = \kappa(x)$, the problem (2) reduce:

$$-\frac{\mathrm{d}^\alpha w}{\mathrm{d}x^\alpha} = h(w,x) \tag{4}$$

with the boundary conditions given in (3). The problems of such type (4) are arising in the framework of mathematical modelling of the physical phenomena in nuclear science, biophysics, coating and draining flow problems [28].

Applying the penalty function technique, we may write the problem (2) as the following equation given:

$$\frac{\mathrm{d}^\alpha w}{\mathrm{d}x^\alpha} - \mu\{w(x)-\kappa(x)\}\big(w(x)-\kappa(x)\big) = f(u,x) \tag{5}$$

Here the penalty function is denoted by $\nu\{.\}$. The penalty function is given:

$$\mu(x)=\begin{cases}0, & \chi<0\\ 2, & \chi\geq 0\end{cases} \tag{6}$$

We define the $\kappa(x)$ which is obstacle function:

$$\kappa(x)=\begin{cases}-\dfrac{1}{2}, & c\leq x<m_1\\ \dfrac{1}{2}, & m_1\leq x<m_2\\ -\dfrac{1}{2}, & m_2\leq x<d\end{cases} \tag{7}$$

where m_1, m_2 are real numbers in $[c, d]$. Using eqs. (4)-(6), we get the beneath system of BVP:

$$\frac{\mathrm{d}^\alpha v}{\mathrm{d}x^\alpha}=\begin{cases}f(w,x)+2v+1, & c\leq x<m_1\\ f(w,x), & m_1\leq x<m_2\\ f(w,x)+2v+1, & m_2\leq x<d,\ 1<\alpha\leq 2\end{cases} \tag{8}$$

along with the given boundary conditions in (3). Whereas the continuity conditions of $w(x)$, $\mathrm{d}v/\mathrm{d}x$, and $\mathrm{d}^2v/\mathrm{d}x^2$ are given at m_1 and m_2.

Illustration of variational iteration method

The fundamental structure of the procedure is explained by letting the problem of obtaining $z(\chi)$:

$$Lw(\chi)+Nw(\chi)=g(\chi) \tag{9}$$

where L and N are the linear and non-linear operators and $g(\chi)$ is the source term. The approximate solution w_{p+1} of eq. (9), for the given w_0, can be found:

$$w_{p+1}(\chi)=w_p(\chi)+\int_0^\chi \lambda(s,\chi)\Big[Lw_p(s)+Nw_p(s)-g(s)\Big]\mathrm{d}s,\quad p=0,1,2,\ldots \tag{10}$$

where λ is mentioned as the Lagrange multiplier. Which is derived by applying variation δ on either side of the eq. (10) with respect to the variable w_p:

$$\delta w_{p+1}(\chi)=\delta w_p(\chi)+\delta\int_0^\chi \lambda(s,\chi)\Big[Lw_p(s)+N\widetilde{w_p(s)}-g(s)\Big]\mathrm{d}s$$

where $\widetilde{w_p(s)}$ is a term that is restricted which provides $\delta\widetilde{w_p(s)}=0$. The Lagrange multiplier $\lambda(s,\chi)$is discovered by applying the optimal condition, see [29]. This results in a real solution $z(\chi)$, where:

$$w(\chi)=\lim_{p\to\infty} w_p(\chi) \tag{11}$$

This approach for obtaining the approximate solution is referred to as VIM. The VIM was presented by Inokuti *et al.* [30]. Moreover, the approach introduced by He [31] has been applied in numerous fields of applied and pure science for addressing a wider range of problems [32, 33]. For fast convergence of solution, selction of initial approximation is very important in VIM, for details, see [34].

Implementation of VIM

Here we present an example of systems that demonstrate the efficiency and implementation of VIM with fractional boundary values problems of a specific type eq. (8).

Example

Considering, $f(w, x) = x$, $a = 1/4$, $b = 6/4$, $c = 3/4$, and $d = 1$, the problem eq. (6) is re-written:

$$\frac{\mathrm{d}^\alpha v}{\mathrm{d}x^\alpha} = \begin{cases} -2w + x - 2, & \frac{1}{4} \le x \le \frac{3}{4} \\ x, & \frac{3}{4} \le x \le 1 \\ -2w + x - 2, & 1 \le x \le \frac{3}{2}, \quad 2 < \alpha \le 3 \end{cases} \tag{12}$$

with the boundary conditions:

$$w(x)|_{x=1/4} = 0, \; w(x)|_{x=3/2} = 0 \tag{13}$$

and with the given continuity conditions of $w(x)$, dv/dx at 3/4 and 1 by applying VIM, the required functional (12) is developed:

$$w_{n+1}(x) = \begin{cases} w_n(x) + \int\limits_0^x \lambda,(\zeta, x)\left[w_n^{(\alpha)}(\zeta) - 2w_n(\zeta) - \zeta - 1\right]\mathrm{d}\zeta, & \frac{1}{4} \le x \le \frac{3}{4} \\ w_n(x) + \int\limits_0^x \lambda(\zeta, x)[w_n^{(\alpha)}(\zeta) - \zeta]\mathrm{d}\zeta, & \frac{3}{4} \le x \le 1 \\ w_n(x) + \int\limits_0^x \lambda(\zeta, x)\left[w_n^{(\alpha)}(\zeta) - 2w_n(\zeta) - \zeta - 1\right]\mathrm{d}\zeta, & 1 \le x \le \frac{3}{2} \end{cases} \tag{14}$$

We determine $\lambda(\zeta, x)$ for the value $\alpha = 2$. Utilizing the variational principles:

$$\lambda(\zeta) = \zeta - x \tag{15}$$

The residual errors are displayed at the end of every solution demonstrate the reliability of the Lagrange multiplier for different values in the given domain $1 < \alpha \le 2$. By the use of eqs. (15) and (14), we have:

$$w_{n+1}(x) = \begin{cases} w_n(x) + \int\limits_0^x (\zeta - x)\left[w_n^{(\alpha)}(\zeta) - 2w_n(\zeta) - \zeta - 1\right]\mathrm{d}\zeta, & \frac{1}{4} \le x \le \frac{3}{4} \\ w_n(x) + \int\limits_0^x (\zeta - x)\left[w_n^{(\alpha)}(\zeta) - \zeta\right]\mathrm{d}\zeta, & \frac{3}{4} \le x \le 1 \\ w_n(x) + \int\limits_0^x (\zeta - x)\left[w_n^{(\alpha)}(\zeta) - 2w_n(\zeta) - \zeta - 1\right]\mathrm{d}\zeta, & 1 \le x \le \frac{3}{2} \end{cases} \tag{16}$$

We consider the initial approximations:

$$w_0\left(x\right)=\begin{cases} a_0x+a_1, & \dfrac{1}{4}\le x\le\dfrac{3}{4} \\ a_2x+a_3, & \dfrac{3}{4}\le x\le 1 \\ a_4x+a_5, & 1\le x\le\dfrac{3}{2} \end{cases} \tag{17}$$

– *Case 1*. We consider $\alpha = 2.9$, in this case. We find the approximate solutions, using eqs. (16) and (17):

$$w_1(x)=\begin{cases} 0.000097656a_0+0.00097655a_1+1.0052a_2+0.00472+ \\ +\left(-0.057292+0.98958a_1-0.0625a_2-0.00097655a_0\right)x+ \\ +\left(0.50260a_0+0.031250a_1+0.25a_2+0.23438\right)x^2+ \\ +\left(-0.33333a_2-0.33333\right)x^3+\left(-0.08334a_1+0.04167\right)x^4, & \dfrac{1}{4}\le x\le\dfrac{3}{4} \\ a_5-0.039551+\left(a_4+0.14063\right)x+\left(0.5a_3-0.14062\right)x^2+0.04167x^4, & \dfrac{3}{4}\le x\le 1 \\ 0.1a_6+0.25a_7+1.3333a_8+0.20833+\left(-0.66667+0.33334a_7-1.0a_8-0.25a_6\right)x+ \\ +\left(0.66666a_6+0.5a_7+a_8+0.75000\right)x^2+ \\ +\left(-0.33333a_8-0.33333\right)x^3+\left(-0.08334a_7+0.04167\right)x^4, & 1\le x\le\dfrac{3}{2} \end{cases}$$

$$w_2(x)=\begin{cases} 0.00011952a_0+0.0011608a_1+1.0058a_2+0.0052254+ \\ +(-0.064338+0.98754a_1-0.0011993a_0-0.070566a_2)x+ \\ +(0.27346+0.50322a_0+0.037806a_1+0.29236a_2)x^2+ \\ +(0.66823-0.000032552a_0-0.00032552a_1-0.66840a_2)x^3+ \\ +(0.29356a_2+0.29356)x^{31/10}+ \\ +(0.088111+0.000081379a_0-0.1658a_1+0.0052083a_2)x^4+ \\ +(0.071605a_1-0.035803)x^{41/10}+ \\ +(-0.0078127-0.0010417a_1-0.0083333a_2-0.03341a_0)x^5+\ldots, & \dfrac{1}{4}\le x\le\dfrac{3}{4} \\ a_5-0.043273+(a_4+0.15489)x+(0.5a_3-0.15689)x^2+0.083337x^4-0.035804x^{41/10}, & \dfrac{3}{4}\le x\le 1 \\ 0.19732+0.11290a_6+0.27169a_7+1.3331a_8+(0.25541a_7-1.0323a_8-0.29003a_6- \\ -0.66001)x+(0.71438a_6+0.611162a_7+1.1278a_8+0.822)x^2+(-0.033333a_6- \\ -0.083333a_7-0.7777a_8-0.73611)x^3+(0.29356a_8+0.29356)x^{31/10}+(0.020833a_6- \\ -0.11112a_7+0.083333a_8+0.13889)x^4+(0.071605a_7-0.035803)x^{41/10}+ \\ +(-0.038882a_6-0.016667a_7-0.033333a_8-0.025)x^5+\ldots, & 1\le x\le\dfrac{3}{2} \end{cases}$$

The continuity conditions on $w_{25}(x)$ and using the given boundary conditions eq. (13), one can get a system of linear equations. We used MAPLE for solving that system, and get:

$$a_0 = .4370321959073731565, \quad a_1 = -.1092580489768432832, \quad a_2 = 0.0136572561221054098$$

$$a_3 = -.2079641140545480981, \quad a_4 = .1919246082935821929, \quad a_5 = -0.0631955741684549584$$

$$a_6 = 0.0424404142196270447\,9, \quad a_7 = -0.0282242738199136410\,9, \quad a_8 = 0.0342284349719912483\,6$$

In fig. 1, the graph shown, is attained by 15th iteration $w_{15}(x)$ of the problem eq. (16).

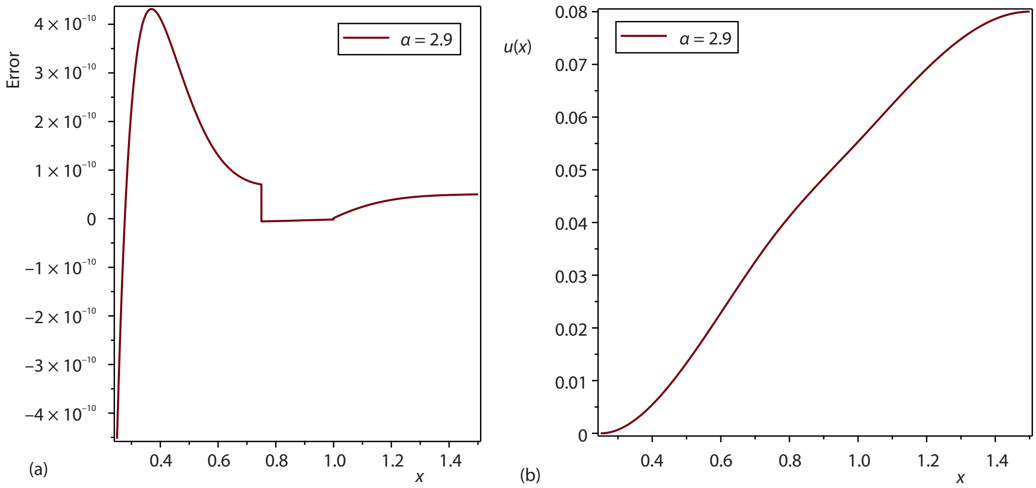

Figure 1. Graphical representation of the (a) residual error $r_{25}(x)$ and (b) approximate solution $w_{25}(x)$

Figure 1 represents the residual error of the problem (12), for $\alpha = 2.9$.

$$r_{25}(x) = \begin{cases} \dfrac{d^{2.9} w_{20}(x)}{dx^{1.95}} - 2w_{25} - x - 1, & \dfrac{1}{4} \le x \le \dfrac{3}{4} \\ \dfrac{d^{2.9} w_{20(x)}}{dx^{1.95}} - x, & \dfrac{3}{4} \le x \le 1 \\ \dfrac{d^{2.9} w_{20}(x)}{dx^{1.9}} - 2w_{25} - x - 1, & 1 \le x \le \dfrac{3}{2} \end{cases} \tag{18}$$

In fig. 1, the graphical representation of residual error (17) has been plotted, which show that the residual error is close to zero and very small having the maximum absolute error -4.5×10^{-14} at $x = 1$.

– *Case 2*. In this case we consider $\alpha = 2.7$, the approximate solutions are found:

$$w_1(x)=\begin{cases} 1.062a_1+0.01041a_0+0.03646+\left(0.9375a_0-0.2812-0.5000a_1\right)x+ & \\ +\left(a_1+0.5000\right)x^2+\left(0.3334a_0+0.1667\right)x^3, & \frac{1}{4}\le x\le\frac{3}{4} \\ a_3+0.1406+\left(a_2-0.2812\right)x+0.1667x^3, & \frac{3}{4}\le x\le 1 \\ 0.8333+2.0a_5+0.6666a_4+\left(-1.500-2.0a_5\right)x+ & \\ +\left(a_5+0.5000\right)x^2+\left(0.3334a_4+0.1667\right)x^3, & 1\le x\le\frac{3}{2} \end{cases}$$

$$w_2(x)=\begin{cases} 0.01670a_0+1.091a_1+0.05433+\left(-0.4410-0.7790a_1+0.8972a_0\right)x+ & \\ +\left(0.01041a_0+1.036+2.062a_1\right)x^2+\left(-0.3354-0.6708a_1\right)x^{12/5}+ & \\ +\left(-0.1667a_1+0.2396+0.6459a_0\right)x^3+\left(-0.09869-0.1974a_0\right)x^{17/5}+ & \\ +\left(0.08333+0.1667a_1\right)x^4+\left(0.01667+0.03334a_0\right)x^5, & \frac{1}{4}\le x\le\frac{3}{4} \\ a_3+0.1922+\left(a_2-0.3942\right)x+0.3334x^3-0.09868x^{17/5}, & \frac{3}{4}\le x\le 1 \\ 0.9926a_4+2.228a_5+1.110+\left(-2.443-0.8284a_4-3.057a_5\right)x+ & \\ +\left(3.0a_5+1.834+0.6665a_4\right)x^2+\left(-0.3354-0.6708a_5\right)x^{12/5}+ & \\ +\left(0.3334a_4-0.1666-0.6667a_5\right)x^3+\left(-0.09869-0.1974a_4\right)x^{17/5}+ & \\ \left(0.08333+0.1667a_5\right)x^4+\left(0.01667+0.03334a_4\right)x^5, & 1\le x\le\frac{3}{2} \end{cases}$$

On using the boundary conditions given in (13) as well as continuity conditions for $w_{30}(x)$, one can get a linear system of equations. Solving that system, one obtains:

$$a_0=.43703219590737315654,\ a_1=-.10925804897684328318,\ a_2=0.013657256122105409813$$

$$a_3=-.20796411405454809813, a_4=.19192460829358219294, a_5=-0.063195574168454958438$$

$$a_6=0.042440414219627044788,\ a_7=-0.028224273819913641085,$$
$$a_8=0.034228434971991248357.$$

The graph is obtained by VIM and shown in fig. 2.

The maximum height of 2 is -0.3209968346×10^4. The residual error is given:

$$r_{30}\left(x\right)=\begin{cases} \dfrac{\mathrm{d}^{1.6}w_{30}}{\mathrm{d}x^{1.6}}-2w_{30}-x-1, & \frac{1}{4}\le x<\frac{3}{4} \\ \dfrac{\mathrm{d}^{1.6}w_{30}}{\mathrm{d}x^{1.6}}-x, & \frac{3}{4}\le x\le 1 \\ \dfrac{\mathrm{d}^{1.6}w_{30}}{\mathrm{d}x^{1.6}}-2w_{30}-x-1, & 1\le x\le\frac{3}{2} \end{cases} \tag{19}$$

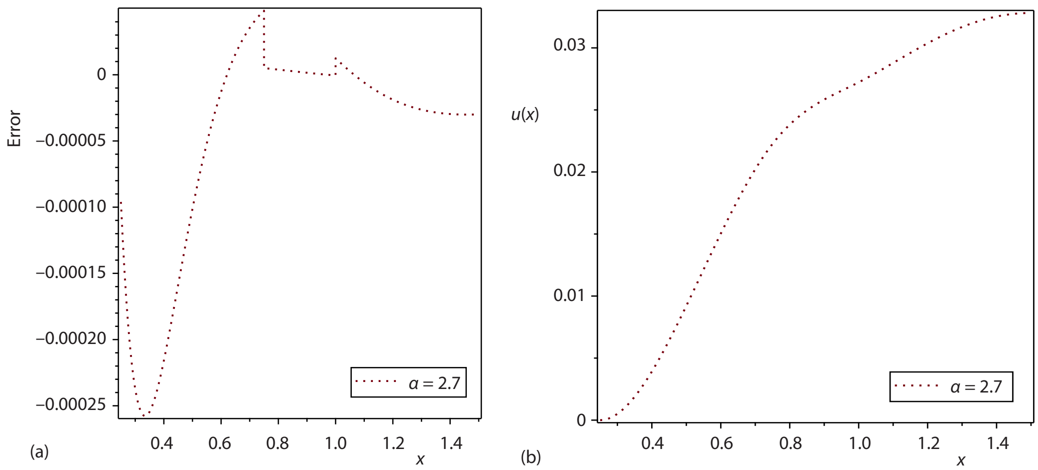

Figure 2. Graphical representation of the (a) residual rrror $r_{15}(x)$ and (b) approximate solution $w_{15}(x)$

The graphical representation of residual error eq. (19) is plotted, which can be seen in fig. 2. From this figure, it is clear that value of residual error is very small near to zero. The maximum error is $-3.209968346 \times 10^{-5}$ at $x = 0.26646049779$. From the analysis of figs. 1 and 2, it can be seen that as the derivative order α decreases, the value of the maximum point of the solution is also decreases.

Results and discussions

Though several techniques have been proposed to solve obstacle problems efficiently, for both accuracy and performance, most of the algorithms discussed in the literature still need to be improved. The main objective of this research is to use efficient numeric-analytic methods that depend on the utilization of the theory of VIM for the solution of fractional order BVP associated with obstacle in the Caputo sense. To demonstrate the approximate solution of a fractional BVP, a computational analysis has been conducted. From figs. 1, and 2, we conclude that the order α of fractional derivative depend directly on the height h of the maximum point of the solution. It means height of obstacle decreases accordingly. From the results, we analyse that proposed procedure is high accurate, simple and effective for the solution of fractional order obstacle BVP. The solution behavior is shown quantitatively and graphically to approximate certain values of the fractional order α. Non-etheless, the results obtained clearly show the full regularity and reliability of the proposed methods, which is a well-suited technique for the fractional BVP where order of derivative must lies between 2 and 3. It is evident that as order of derivative gets closer to 3 we see a strong convergence means error reduces significantly.

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References

[1] Podlubny, I., Fractional Differential Equations: An Introduction, in: *Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Elsevier, New York, USA, 1998
[2] Freihet, A., *et al*., Construction of Fractional Power Series Solutions to Fractional Stiff System Using Residual Functions Algorithm, *Adv. Differ Equations*, *2019* (2019), 1, 95
[3] Wang, Q., Numerical Solutions for Fractional kdv-Burgers Equation by Adomian Decomposition Method, *Appl. Math. Comput*., *182* (2006), 2, pp. 1048-1055
[4] Zurigat, M., *et al*., Analytical Approximate Solutions of Systems of Fractional Algebraic-Differential Equations by Homotopy Analysis Method, *Comput. Math. with Appl*., *59* (2010), 3, pp. 1227-1235
[5] Iomin, A., Fractional Evolution in Quantum Mechanics. Chaos, *Solitons Fractals*, *X* (2019), 1, 100001
[6] Turut, V., Guzel, N., Multivariate Pade Approximation for Solving Non-Linear Partial Differential Equations of Fractional Order, *In Abstr. Appl. Anal*., *2013* (2013), ID746401
[7] Liu, J., Hou, G., Numerical Solutions of the Space-and Time-Fractional Coupled Burgers Equations by Generalized Differential Transform Method, Appl. Math. Comput., *217* (2011), 16, pp. 7001-7008
[8] Abbas, M., *et al*., The Application of Cubic Trigonometric b-Spline to the Numerical Solution of the Hyperbolic Problems, *Appl Math. Comput*., *239* (2014), July, pp. 74-88
[9] Inc, M., *et al.*, Modified Variational Iteration Method for Straight Fins with Temperature Dependent Thermal Conductivity, *Thermal Science*, *22* (2018), Suppl. 1, pp. S229-S236
[10] Partohaghighi, M., *et al.*, Ficitious Time Integration Method for Solving the Time Fractional Gas Dynamics Equation, *Thermal Science*, *23* (2019), Suppl. 6, pp. S2009-S2016
[11] Kilicman, A., *et al*., Analytic Approximate Solutions for Fluid-Flow in the Presence of Heat and Mass Transfer, *Thermal Science*, *22* (2018), Suppl. 1, pp. S259-S264
[12] Jassim, H. K., Baleanu, D., A Novel Approach for Korteweg-de Vries Equation of Fractional Order, *Journal Appl. Comput. Mech.*, *5* (2019), 2, pp. 192-198
[13] Baashan, A., *et al*., Approximation of the kdvb Equation by the Quintic b-Spline Differential Quadrature Method, *Kuwait J. Sci*., *42* (2015), 2, pp. 67-92
[14] Ramadan, M., *et al*., Quintic Non-Polynomial Spline Solutions for Fourth Order Two-Point Boundary Value Problem, *Commun Non-Linear Sci. Numer. Simul*., *14* (2009), 4, pp. 1105-1114
[15] Khan, A., Sultana, T., Non-Polynomial Ouintic Spline Solution for the System of Third Order Boundary-Value Problems, *Numer. Algorithms*, *59* (2012), 4, pp. 541-559
[16] Srivastava, P. K., Study of Differential Equations with Their Polynomial and Non-Polynomial Spline Based Approximation, *Acta Tech. Corviniensis-Bulletin Eng*., *7* (2014), 3, 139
[17] Siddiqi, S. S., Arshed, S., Numerical Solution of Time-Fractional Fourth Order Partial Differential Equations, *Int. J. Comput. Math*., *92* (2015), 7, pp. 1496-1518
[18] Li, X., Wong, P. J., An Efficient Non-Polynomial Spline Method for Distributed Order Fractional Subdiffusion Equations, *Math. Methods Appl. Sci*., *41* (2018), 13, pp. 4906-4922
[19] Stampacchia, G., Formes bilinéaires coercitives Sur les ensembles convexes (in Franch), *Comptes Rendus Hebdomadaires Des Seances De L' Academie Des Sciences*, *258* (1964), 18, 4413
[20] Hu, Y., *et al*., Analytical Solution of the Linear Fractional Differential Equation by Adomian Decomposition Method, *Journal Comput. Appl. Math*., *215* (2008), 1, pp. 220-229
[21] Rani, A., *et al*., Solving System of Differential Equations of Fractional Order by Homotopy Analysis Method, *Journal Sci. Arts*, *17* (2017), 3, pp. 457-468
[22] Inc, M., The Approximate and Exact Solutions of the Space-and Time-Fractional Burgers Equations with Initial Conditions by Variational Iteration Method, *Journal Math. Anal. Appl*., *345* (2008), 1, pp. 476-484
[23] Modanli, M., Akgül, A., Numerical Solution of Fractional Telegraph Differential Equations by Theta-Method, *Eur. Phys. J. Spec. Top*., *226* (2017), 16-18, pp. 3693-3703
[24] Martin, O., Stability Approach to the Fractional Variational Iteration Method Used for the Dynamic Analysis of Viscoelastic Beams, *Journal Comput. Appl. Math*., *346* (2019), Jan., pp. 261-276
[25] Heidarkhani, S., *et al*., A Variational Approach to Perturbed Impulsive Fractional Differential Equations, *Journal Comput. Appl. Math*., *341* (2018), Oct., pp. 42-60
[26] Kubica, A., Ryszewska, K., Decay of Solutions to Parabolic-Type Problem with Distributed Order Caputo Derivative, *Journal Math. Anal. Appl*., *465* (2018), 1, pp. 75-99
[27] Noor, M. A., *et al*., Homotopy Perturbation Method for Solving a System of Third-Order Boundary Value Problems, *Int. J. Phys. Sci*., *6* (2011), 16, pp. 4128-4133
[28] Tuck, E., Schwartz, L., A Numerical and Asymptotic Study of Some Third-Order Ordinary Differential Equations Relevant to Draining and Coating Flows, *SIAM Rev*., *32* (1990), 3, pp. 453-469

[29] Ahmad, H., Variational Iteration Method with an Auxiliary Parameter for Solving Differential Equations of the Fifth Order, *Non-Linear Sci. Lett. A*, *9* (2018), Mar., pp. 27-35
[30] Inokuti, M., *et al*., General Use of the Lagrange Multiplier in Non-Linear Mathematical Physics, *Var. Methods Mech. Solids*, *33* (1978), 5, pp. 156-162
[31] He, J.-H., Variational Iteration Method – A Kind of Non-Linear Analytical Technique: Some Examples, *Int. J. Non-Linear Mech*., *34* (1999), 4, pp. 699-708
[32] Ahmad, H., *et al.*, Variational Iteration Algorithm-i with an Auxiliary Parameter for Solving Boundary Value Problems, *Earthline J. Math. Sci*., *3* (2020), 2, pp. 229-247
[33] Ahmad, H., Khan, T. A., Variational Iteration Algorithm-i with an Auxiliary Parameter for Wave-Like Vibration Equations, *Journal Low Freq. Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1113-1124
[34] Ahmad, H., Khan, T. A., Variational Iteration Algorithm i with an Auxiliary Parameter for the Solution of Differential Equations of Motion for Simple and Damped Mass-Spring Systems, *Noise Vib World*, *51* (2020), 1-2, pp. 12-20

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