HYPERCOMPLEX SYSTEMS AND NON-GAUSSIAN STOCHASTIC SOLUTIONS OF χ-WICK-TYPE (3+1)-DIMENSIONAL MODIFIED BENJAMIN-BONA-MAHONY EQUATION

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In this paper, we seek non-Gaussian stochastic solutions of χ-Wick-type stochastic (3+1)-dimensional modified Benjamin-Bona-Mahony equations. Using the generalized modified tanh-coth method, the connection between hypercomplex system and transforming white noise theory, χ-Wick product and χ-Hermite transform, we generate a new set of exact travelling non-Gaussian wave solutions for the (3+1)-dimensional modified Benjamin-Bona-Mahony equations. This set contains solutions with non-Gaussian parameters of exponential, hyperbolic, and trigonometric types.

Key words: *modified BBM equations, (3+1)-dimensional equations, white noise, Hermite transform, non-Gaussian white noise, Wick product, travelling wave solutions*

Introduction

Let λ be a measure of non-Gaussian probability on a locally compact space $\mathbb{Q}^*$. Consider the quasinuclear chain

$$D_{-q}^{\chi} \supseteq L_2\left(\mathbb{Q}^*, \mathrm{d}\lambda x\right) \supseteq D_q^{\chi}$$

where $L_2[\mathbb{Q}^*, \mathrm{d}\lambda(x)]$ is the zero space defined on a commutative normal hypercomplex system $L_1[\mathbb{Q}^*, \mathrm{d}m_r(x)]$ with the $\mathbb{Q}^*$ basis and multiplicative mr measure. For more details see [1-4]. The spaces D_{-q}^{χ} and D_q^{χ} are the spaces of generalized and test functions that are constructed by the characters Delsarte $\chi_n \in C(\mathbb{Q}^*)$. This paper has a fundamental interest with-Wick-type stochastic (3+1)-dimensional modified Benjamin-Bona-Mahony (BBM) equations with non-Gaussian parameters:

$$\begin{cases} U_t + F_1(t)\Diamond^{\chi} U_{z_1} + F_2(t)\Diamond^{\chi} U^{\Diamond^{\chi} 2}\Diamond^{\chi} U_{x_1} + F_3(t)\Diamond^{\chi} U_{x_1 y_1 t} = 0 \\ V_t + F_4(t)\Diamond^{\chi} V_{x_1} + F_5(t)\Diamond^{\chi} V^{\Diamond^{\chi} 2}\Diamond^{\chi} V_{y_1} + F_6(t)\Diamond^{\chi} V_{x_1 z_1 t} = 0 \\ W_t + F_7(t)\Diamond^{\chi} W_{y_1} + F_8(t)\Diamond^{\chi} W^{\Diamond^{\chi} 2}\Diamond^{\chi} W_{z_1} + F_9(t)\Diamond^{\chi} W_{x_1 x_1 t} = 0 \end{cases} \tag{1}$$

where $\Diamond^{\chi}$ is the χ-Wick product on D_{-q}^{χ} and $(x_1, y_1, z_1, t) \in \mathbb{R}^3 \times \mathbb{R}^+$, $F_i (i = 1, 2,..., 9)$ are non-Gaussian D_{-q}^{χ}-valued functions from $\mathbb{R}^+$ to generalized distribution space D_{-q}^{χ}. In addition we obtain the perturbation variable coefficients (3+1)-dimensional modified BBM equations when the χ-Wick product $\Diamond^{\chi}$ is replaced by the ordinary product in the system (1):

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$$\begin{cases} u_t + f_1(t)u_{z_1} + f_2(t)u^2u_{x_1} + f_3(t)u_{x_1y_1t} = 0 \\ v_t + f_4(t)v_{x_1} + f_5(t)v^2v_{y_1} + f_6(t)v_{x_1z_1t} = 0 \\ w_t + f_7(t)w_{y_1} + f_8(t)w^2w_{z_1} + f_9(t)w_{x_1x_1t} = 0 \end{cases} \tag{2}$$

where f_i ($i = 1, 2,..., 9$) is a non-zero integrable function on $\mathbb{R}^+$. In many non-linear science branches the (3+1)-dimensional modified BBM equations have various applications. The study of (3+1)-Dimensional non-linear equations is thriving as these equations model the real features in a wide array of fields of science, technology, fluid mechanics, wave propagation, electrodynamics, and engineering [5-9]. Recently, Khater *et al*. [10], Korpinar *et al*. [11, 12], have addressed the applications non-linear wave equations in mathematical physics [13-15]. In this paper, we aim to obtain non-Gaussian solutions for stochastic (3+1)-dimensional modified BBM equations of the χ-Wick-type. If the major problem is regarded in a non-Gaussian distributed environment, we obtain non-Gaussian stochastic (3+1)-dimensional modified BBM equations. To give the exact stochastic solutions of the non-Gussian stochastic (3+1)-dimensional modified BBM equations, we only consider this problem in a non-Gussian white noise environment. That is, we will study stochastic (3+1)-dimensional modified BBM, eq. (1), with variable coefficients. Through white noise functional analysis [16], Ghany [17, 18], Ghany and Hyder [19-22], Ghany *et al*. [23, 24], Ghany and Zakarya [25, 26], Hyder and Zakarya [27], Agarwal *et al*. [28], Zakarya *et al*. [29], Agarwal *et al*. [30] studied this model of white noise functional solutions for some non-linear stochastic partial differential equations (SPDE) more intensively. In addition, Okb El Bab *et al*., [1] and Zakarya [3] explored some important topics related to the construction of non-Gaussian white noise analysis using the hyper-complex systems theory and some applications.

The construction of non-Gaussian white noise analysis

In the case of Gaussian, if the objects of the differential equation are considered to be $(K)_{-1}$-valid, $(K)_{-1}$ is the Kondratiev space of stochastic distributions based on the Gaussian measure). We often get a more realistic mathematical model of the situation. This model referred to as the Wick-type stochastic differential equation, and for further details, see [16]. A non-Gaussian χ-Wick-type stochastic model can be introduced by replacing $(K)_{-1}$ with D^{χ}_{-q} and the Wick product associated with the Gaussian measure with the χ-Wick product. Wick ‘s product was first introduced by Wick [31] who used a tool to renormalize certain infinite quantities of field theory. Subsequently, Hida and Ikeda [32] considered the Wick product in a stochastic setting. Dobroshin and Minlos [33] had a comprehensive study of this particular topic in both mathematical physics and probability theory. Currently, the Wick product provides a useful concept for various applications, for example in the study of stochastic ordinary and PDE, see [26-30]. In this section, we present a new product and definition of the so-called χ-Wick product and χ-Hermite transform in the space D^{χ}_{-q}, respectively, with regard to the non-Gaussian probability measure λ.

Definition 1. [3]. The χ-Wick product of φ, ψ denoted $\varphi\Diamond^{\chi}\psi$ is defined:

$$\varphi\Diamond^{\chi}\psi = \sum_{m,n=0}^{\infty} \varphi_m \ \ \psi_n \ \ q^{\chi}_{m+n}$$

where

$$\varphi = \sum_{m=0}^{\infty}\varphi_m \ q^{\chi}_m, \ \psi = \sum_{n=0}^{\infty}\psi_n q^{\chi}_n \ \forall \ \varphi, \ \ \psi \in D^{\chi}_{-q} \ \text{ with } \ \varphi_m, \ \psi_n \in \mathbb{C}$$

It is important to show that the spaces D^{χ}_{-q} and D^{χ}_{q} are closed under χ-Wick product.

Remark 1. If ξ_1, η_1, $\in D^{\chi}_{-q}$, and ξ_2, η_2, $\in D^{\chi}_{q}$, the spaces D^{χ}_{-q}, D^{χ}_{q} are closed under the operation χ-Wick product. We have, ξ_1, $\Diamond^{\chi}\eta_1$, $\in D^{\chi}_{-q}$, and ξ_2, $\Diamond^{\chi}\eta_2$, $\in D^{\chi}_{q}$.

Remark 2. For each $\pi_1, \pi_2, \pi_3 \in D^{\chi}_{-q}$,we have the important algebraic properties of the χ-Wick product (Commutative, Associative, and Distributive law), respectively:

$$\pi_1\Diamond^{\chi}\pi_2 = \pi_2\Diamond^{\chi}\pi_1,\ \pi_1\Diamond^{\chi}\left(\pi_2\Diamond^{\chi}\pi_3\right) = \left(\pi_1\Diamond^{\chi}\pi_2\right)\Diamond^{\chi}\pi_3,\ \ \pi_1\Diamond^{\chi}\left(\pi_2+\pi_3\right) = \pi_1\Diamond^{\chi}\pi_2 + \pi_1\Diamond^{\chi}\pi_3$$

Definition 2. [3]. Let

$$\Theta = \sum_{n=0}^{\infty}\Theta_n q_n^{\chi} \in D_{-q}^{\chi} \ \text{ with } \Theta_n \in \mathbb{C}$$

Then, the χ-Hermite transform of Θ, denoted by $H^{\chi}\Theta$ or $\tilde{\Theta}$ is defined:

$$H^{\chi}\Theta\left(s\right) = \tilde{\Theta}\left(s\right) = \sum_{n=0}^{\infty}\Theta_n s^n \in \mathbb{C}$$

Now, we define the neighborhoods $N_r(h)$ of zero in $\mathbb{C}$ for $0 < h$, $r < \infty$:

$$N_r\left(h\right) = \left\{ s \in \mathbb{C}: \sum_{n=0}^{\infty}\left|s^n\right|^2 K^{rn} < h^2 \right\}$$

Proposition [2]. If $\varphi \in D^{\chi}_{-q}$ converges for all $s \in N_r(h)$ for all $r > 0$, $h < \infty$. The useful property of the χ-Hermite transform is that it transforms the χ-Wick product into ordinary (complex) product. If φ, $\psi \in D^{\chi}_{-q}$. Then, $H^{\chi}(\varphi\Diamond^{\chi}\psi)(s) = H^{\chi}\varphi(s)H^{\chi}\psi(s)$, for all s such that $H^{\chi}\varphi$ and $H^{\chi}\psi$ exist. For χ-Brownian motion, we hav $W^{\chi}(t) = (\mathrm{d}/\mathrm{d}t)B^{\chi}(t)$ in D^{χ}_{-q}. One advantage of operating in the general space of stochastic distributions D^{χ}_{-q}, is that it involves solutions of several non-Gussian stochastic differential equations, both ordinary and partial in arbitrary dimensions. In addition, if the artifacts of such equations are assumed to be D^{χ}_{-q}-valued, differentiation can be represented in D^{χ}_{-q} in the normal strong sense.

The basic idea of generalized modified tanh – coth method

Consider a non-linear unlit dimensional wave propagation PDE:

$$F(u, u_t, ux_i, ux_ix_j, ux_ix_jx_k,...) = 0$$

where the dependent variable is u, and the independent variables are $t = x_0, x_1, x_2,...x_n$. Apply the transformation of waves: $u = u(\mu)$, $\mu = \sum^{n}_{i=0}a_ix_i$, where, $a_i(i = 0, 1, 2,...,n$: $n \in \mathbb{N})$ constants are unknown. The non-linear PDE can therefore, be transformed into a non-linear ordinary differential equation (NLODE):

$$G\left(u,u',u'',u''',\ldots\right)=0 \tag{3}$$

For simplicity, we integrate the NLODE (3), provide that all terms include derivatives, and take the integration constants to zero. Eventually, eq. (3) which can be solved by the following extension of its general solution in finite series:

$$u(\mu) = \sum_{k=0}^{N} A_k \varPhi^k\left(\mu\right)\sum_{k=1}^{N} B_k \varPhi^{-k}\left(\mu\right) \tag{4}$$

where ϕ solves the first order Riccati equation [34]:

$$\varPhi'(\mu)=\delta_0 + \delta_1\varPhi\left(\mu\right) + \delta_2\varPhi^2\left(\mu\right) \tag{5}$$

where the constants to be calculated are δ_0, δ_1, and δ_2. The positive constant N may be defined by balancing the highest-order linear and non-linear terms in eq. (3). Equations (4) and (5) inserted into eq. (3), in any case, produce an algebraic equation in $\varPhi$ and its powers. Equating

Φ^k to zero coefficients gives an algebraic equation method in A_k and B_k, can be obtained with the aid of the code symbolic program MAPLE. In eq. (5) the Riccati equation has the following special solution [34]:

$$\begin{gathered}\phi(\mu)=e^{\mu}-1,\quad \delta_0=1,\ \delta_1=1,\ \delta_2=0\\ \phi(\mu)=\coth(\mu)\pm\operatorname{csch}(\mu),\quad \tanh(\mu)\pm i\operatorname{sech}(\mu),\quad \delta_0=\frac{1}{2},\ \delta_1=0,\ \delta_2=\frac{-1}{2}\\ \phi(\mu)=\tan(\mu),\ -\cot(\mu),\quad \delta_0=1,\delta_1=0,\ \delta_2=1\\ \phi(\mu)=\frac{1}{2}\cot(2\mu),\ \frac{1}{2}\tan(2\mu),\quad \delta_0=1,\ \delta_1=0,\ \delta_2=4\end{gathered}\tag{6}$$

Travelling wave solutions of (3+1)-dimensional modified BBM equations

In this section, we investigate the model (1) of the stochastic (3+1)-dimensionally modified BBM equations of the χ-wick-type. The application of χ-Hermite transform to eq. (1) results in the deterministic equation:

$$\begin{gathered}\tilde{U}_t(x_1,y_1,z_1,t,s)+\tilde{F}_1(t,s)\tilde{U}_{z_1}(x_1,y_1,z_1,t,s)+\\ +\tilde{F}_2(t,s)U_2(x_1,y_1,z_1,t,s)U_{x_1}(x_1,y_1,z_1,t,s)+\\ +\tilde{F}_3(t,\ s)\tilde{U}_{x_1y_1z_1}(x_1,y_1,z_1,t,s)=0\end{gathered}\tag{7}$$

where $s=(s_1,\ s_2,...)\in(C_n)$. We implement the following transformation to obtain travelling wave solution eq. (7): $\tilde{F}_1(t,\ s):=f_1\ (t,\ s)$, $\tilde{F}_2(t,\ s):=f_2\ (t,\ s)$, $\tilde{F}_3(t,\ s):=f_3(t,\ s)$, and $\tilde{U}(x_1,\ y_1,\ z_1,\ t,\ s):=u(x_1,\ y_1,\ z_1,\ t,\ s)=u\ [\mu(x_1,\ y_1,\ z_1,\ t,\ s)]$ with:

$$\mu(x_1,y_1,z_1,t,s)=a_1x_1+a_2y_1+a_3z_1+b\int_0^t(\tau,s)\,\mathrm{d}\tau+c\tag{8}$$

where $a_i=(i=1,\ 2,\ 3)$, b, and c satisfying a, $b\neq 0$ are arbitrary constants and Ω is a non-zero function that needs to be defined. Accordingly eq. (7) might be converted to NLODE:

$$(b\Omega+a_3f_1)u+\frac{1}{3}a_1f_2u^3+a_1a_2b\Omega f_2u''=0\tag{9}$$

Balancing u^3 with u'' provides $N=1$. Hence, we are putting eq. (7) solution into the form:

$$u(x_1,y_1,z_1,t,s)=A_0(t,s)+A_1(t,s)\Phi(\mu)+\frac{B_1(t,s)}{\Phi(\mu)}\tag{10}$$

where Φ is the solution of eq. (5). Substituting eqs. (10) and (5) into eq. (9), collecting the coefficients of $\Phi^k(k=-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3)$ and attempting to equate them to zero comprises seven algebraic equations system in A_0, A_1, B_1, and Ω:

$$\begin{cases}(b\Omega+a_3f_1)A_0+\frac{1}{3}a_1f_2T_0+a_1a_2b\Omega f_3E_0=0,\ \ \frac{1}{3}a_1f_2T_2+a_1a_2b\Omega f_3E_2=0\\ (b\Omega+a_3f_1)A_1+\frac{1}{3}a_1f_2T_1+a_1a_2b\Omega f_3E_1=0,\ \ \frac{1}{3}a_1f_2T_3+a_1a_2b\Omega f_3E_3=0\\ (b\Omega+a_3f_1)B_1+\frac{1}{3}a_1f_2J_1+a_1a_2b\Omega f_3R_1=0,\ \ \frac{1}{3}a_1f_2J_2+a_1a_2b\Omega f_3R_2=0\\ \frac{1}{3}a_1f_2J_3+a_1a_2b\Omega f_3R_3=0\end{cases}\tag{11}$$

where $T_0 = A_0G_0 + A_1H_1 + B_1G_1$, $T_1 = A_0G_1 + A_1G_0 + B_1G_2$, $T_2 = A_0G_2 + A_1G_1$, $T_3 = A_1G_2$, $J_1 = A_0H_1 + A_1H_2 + B_1G_0$, $J_2 = A_0H_2 + B_1H_1$, $J_3 = B_1H_2$, $G_0 = A_0^2 + 2A_1B_1$, $G_1 = 2A_0A_1$, $G_2 = A_0^2$, $H_1 = 2A_0B_1$, $H_2 = B_1^2$, $E_0 = \delta_0C_1 - \delta_2D_1$, $E_1 = \delta_1C_1 + 2\delta_0C_2$, $E_2 = \delta_2C_1 + 2\delta_1C_2$, $E_3 = 2\delta_2C_2$, $R_1 = -\delta_1D_1 - 2\delta_2D_2$, $R_2 = -\delta_0D_1 - 2\delta_1D_2$, $R_3 = -2\delta_0D_2$, $C_0 = \delta_0A_1 - \delta_2B_1$, $C_1 = \delta_1A_1$, $C_2 = \delta_2A_1$, $D_1 = -\delta_1B_1$, $D_2 = -\delta_0B_1$. Now, we are solving the system (11) for some Riccat equation cases (5):

*Case A***.** Let, $\delta_0 = \delta_1 = 1$ and $\delta_2 = 0$. Using MAPLE, we get a set of solutions:

$$A_0 = \pm i\sqrt{\frac{3a_3 f_1(t,s)}{a_1 f_2(t,s)}},\ A_1 = 0,\quad B_1 = \pm\sqrt{\frac{3a_2a_3 f_1(t,s)}{a_1a_2 f_2(t,s) f_3(t,s) - 2}}\ ,\ \Omega = \frac{2a_3 f_1(t,s)}{b\left(a_1a_2 f_3(t,s) - 2\right)}$$

Substituting by this values in eq. (10). Using eq. (6), eq. (7) travelling wave solution is of an exponential type:

$$u_1(x_1, y_1, z_1, t, s) = \\ = \pm i\frac{\left[\begin{array}{c}\sqrt{3a_3 f_1(t,s)\left[a_1a_2 f_2(t,s) f_3(t,s) - 2\right]}\cdot \\ \cdot\left[\exp\mu_1(x_1, y_1, z_1, t, s)\right] - 1) \pm \sqrt{3a_1a_2a_3 f_1(t,s) f_2(r,s)}\end{array}\right]}{\left[\exp\mu_1(x_1, y_1, z_1, t, s)\right] - 1)\sqrt{a_1 f_2(t,s)\left[a_1a_2 f_2(t,s) f_3(t,s) - 2\right]}} \tag{12}$$

where

$$\mu_1(x_1, y_1, z_1, t, s) = a_1x_1 + a_2y_1 + a_3z_1 + 2a_3\int_0^t \frac{f_1(\tau,s)}{a_1a_2 f_3(\tau,s) - 2}\mathrm{d}\tau$$

Case B. Let, $\delta_0 = 1/2$, $\delta_1 = 0$*,* and $\delta_0 = -1/2$. Using MAPLE, we get a set of solutions:

$$A_0 = 0, A_1 = \pm\sqrt{\frac{3a_2a_3 f_1(t,s) f_3(t,s)}{f_2\left[2 - a_1a_2 f_3(t,s)\right]}},\ B_0 = \pm i\sqrt{\frac{3a_1a_2 f_1(t,s)}{2 f_2(t,s) f_3(t,s)}},\ \Omega = -\frac{4a_3 f_1(t,s)}{b\left[4 + a_1a_2 f_3(t,s)\right]}$$

Substituting by this values in eq. (10). Using eq. (6), eq. (7) travelling wave solution is of an hyperbolic type:

$$u_2(x_1, y_1, z_1, t, s) = \pm\sqrt{\frac{3a_2a_3 f_1(t,s) f_3(t,s)}{f_2(t,s)\left[2 - a_1a_2 f_3(t,s)\right]}}\cdot\left[\coth\left(\mu_2(x_1, y_1, z_1, t, s)\right) \pm \operatorname{csch}\left(\mu_2(x_1, y_1, z_1, t, s)\right)\right] \pm \\ \pm i\frac{\sqrt{3ba_2 f_1(t,s)}}{\left[\begin{array}{r}\sqrt{2 f_2(t,s) f_3(t,s)}\ [\coth\left(\mu_2(x_1, y_1, z_1, t, s)\right) \\ \pm\operatorname{csch}\left(\mu_2(x_1, y_1, z_1, t, s)\right)]\end{array}\right]} \tag{13}$$

$$u_3(x_1, y_1, z_1, t, s) = \pm\sqrt{\frac{3a_2a_3 f_1(t,s) f_3(t,s)}{f_2(t,s)\left[2 - a_1a_2 f_3(t,s)\right]}}\left[\tanh\left(\mu_2(x_1, y_1, z, t, s)\right) \pm i\ \operatorname{sech}\left(\mu_2(x_1, y_1, z_1, t, s)\right)\right] \pm \\ \pm i\ \frac{\sqrt{3ba_2 f_1(t,s)}}{\left[\begin{array}{r}\sqrt{2 f_2(t,s) f_3(t,s)}\ \ [\tanh\left(\mu_2(x_1, y_1, z_1, t, s)\right) \\ \pm i\operatorname{sech}\left(\mu_2(x_1, y_1, z_1, t, s)\right)]\end{array}\right]} \tag{14}$$

where

$$\mu_2(x_1,y_1,z_1,t,s)=a_1x_1+a_2y_1+a_3z_1-4a_3\int_0^t \frac{f_1(\tau,s)}{4+a_1a_2f_3(\tau,s)}\mathrm{d}\tau$$

Case C. Let, $\delta_0 = \delta_2 = 1$ and $\delta_1 = 0$. Using MAPLE, we get a set of solutions:

$$A_0=\pm\sqrt{\frac{3a_2a_3f_1(t,s)}{1-a_1a_2f_2(t,s)f_3(t,s)}},\ A_1=B_1=\sqrt{\frac{6a_2a_3f_1(t,s)f_3(t,s)}{1+2a_1a_2f_2(t,s)f_3(t,s)}},\ \Omega=\frac{-a_3f_1(t,s)}{b\left[1+a_1a_2f_3(t,s)\right]}$$

Substituting by this values in eq. (10) using eq. (6), eq. (7) travelling wave solution is of an trigonometric type:

$$u_4(x_1,y_1,z_1,t,s)=u_5(x_1,y_1,z_1,t,s)=\pm\sqrt{\frac{3a_2a_3f_1(t,s)}{1-a_1a_2f_2(t,s)f_3(t,s)}}\pm\sqrt{\frac{6a_2a_3f_1(t,s)f_3(t,s)}{1+2a_1a_2f_2(t,s)f_3(t,s)}}\cdot$$
$$\cdot\left[\sec\left(\mu_3(x_1,y_1,z,t,s)\right)\pm\csc\left(\mu_3(x_1,y_1,z_1,t,s)\right)\right] \qquad (15)$$

where

$$\mu_3(x_1,y_1,z_1,t,s)=a_1x_1+a_2y_1+a_3z_1-a_3\int_0^t \frac{f_1(\tau,s)}{1+2a_1a_2f_3(\tau,s)}\mathrm{d}\tau$$

Case D. Let, $\delta_0 = 1$, $\delta_1 = 0$, and $\delta_2 = 4$. Using MAPLE, we get a set of solutions:

$$A_0=\pm3\sqrt{\frac{3a_2a_3f_1(t,s)}{9a_1a_2f_2(t,s)f_3(t,s)-2}},\quad A_1=\pm8\sqrt{\frac{3a_2a_3f_1(t,s)f_3(t,s)}{2+15a_1a_2f_2(t,s)f_3(t,s)}}$$

$$B_1=\pm i\sqrt{\frac{3a_3f_1(t,s)}{a_1f_2(t,s)f_3(t,s)}},\quad \Omega=\frac{2a_3f_1(t,s)}{b\left(9a_1a_2f_2(t,s)f_3(t,s)-2\right)}$$

Substituting by this values in eq. (10). Using eq. (6), eq. (7) travelling wave solution is of an trigonometric type:

$$u_6(x_1,y_1,z_1,t,s)=\pm3\sqrt{\frac{3a_2a_3f_1(t,s)}{9a_1a_2f_2(t,s)f_3(t,s)-2}}\pm$$
$$\pm4\sqrt{\frac{3a_2a_3f_1(t,s)f_3(t,s)}{2+15a_1a_2f_2(t,s)f_3(t,s)}}\cot\left[2\mu_4(x_1,y_1,z_1,t,s)\right]\pm$$
$$\pm i\frac{\sqrt{3a_3f_1(t,s)}}{2\cot\left(2\mu_4(x_1,y_1,z_1,t,s)\right)\sqrt{a_1f_2(t,s)f_3(t,s)}} \qquad (16)$$

$$u_7(x_1,y_1,z_1,t,s)=\pm3\frac{\sqrt{3a_2a_3f_1(t,s)}}{\sqrt{a_1a_2f_2(t,s)f_3(t,s)-2}}\pm4\sqrt{\frac{3a_2a_3f_1(t,s)f_3(t,s)}{2+15a_1a_2f_2(t,s)f_3(t,s)}}\times$$
$$\times\tan\left(2\mu_4(x_1,y_1,z_1,t,s)\right)\pm\frac{i\sqrt{3a_3f_1(t,s)}}{2\left(\tan\left(2\mu_4(x_1,y_1,z_1,t,s)\right)\right)\sqrt{a_1f_2(t,s)f_3(t,s)}} \qquad (17)$$

where

$$\mu_4(x_1,y_1,z_1,t,s)=a_1x_1+a_2y_1+a_3z_1+2a_3\int_0^t\frac{f_1(\tau,s)}{9a_1a_2f_2(\tau,s)-2}$$

Note that, there exist a number of specific solutions for the system (11) with the Riccati eq. (5) coming from a variety of different cases. In the aforementioned cases, we have explained the extent to which our approach is applicable.

Non-Gaussian stochastic travelling wave solutions

In this section, we use the results of Section *Travelling wave solutions of (3+1)-dimensional modified BBM equations*, adopting χ-Hermite transform to get exact non-Gaussian white noise functional solutions for χ-Wick-type (3+1)-Dimensional modified BBM equations (1). Assume, $u(x_1, y_1, z_1, t, s)$ is an equation solution of $\tilde{A}(x_1, y_1, z_1, t, D_t, D_x, u, s)$ [3]. The properties of exponential, hyperbolic and trigonometric functions give rise to the existence of a bounded open set, $X\subset\mathbb{R}^3\times\mathbb{R}^+$, $r>0$, $h<\infty$. That is the solution $u(x_1, y_1, z_1, t, s)$ and all of its partial derivatives, which are actively involved, are uniformly bound for $(x_1, y_1, z_1, t, s)\in X\times N_r(h)$, continuous with respect to $(x_1, y_1, z_1, t)\in X$ for all $s\in N_r(h)$ and analytic with respect $s\in N_r(h)$ for all $(x_1, y_1, z_1, t)\in X$. From *Theorems* (4.1.1., 2.1) in [16, 23], there exists $U(x_1, y_1, z_1, t)\in D^x{}_{-q}$ such that $u(x_1, y_1, z_1, t, s) = H^\chi U(x_1, y_1, z_1, t, s) = \tilde{U}(x_1, y_1, z_1)(s)$ for all $(x_1, y_1, z_1, t, s)\in X\times N_r(h)$ and $U(x_1, y_1, z_1, t)$ solves eq. (1) in $D^x{}_{-q}$. Thus, applying the χ-Hermite transform inverse $(H^\chi)^{-1}u(x_1, y_1, z_1, t, s) = U(x_1, y_1, z_1, t)$, to eqs. (12)-(17), we get non-Gaussian solutions from eq. (1) as the following:

Non-Gaussian stochastic travelling wave s olution of the exponential type

$$U_1(x_1,y_1,z_1,t)=\frac{\left[\begin{gathered}\pm i\sqrt{3a_3F_1(t)\Diamond^\chi\left(a_1a_2F_2(t)-2\right)}\Diamond^\chi\left(\exp^{\Diamond^\chi}\left(\Xi_1(x_1,y_1,z_1,t)\right)-1\right)\\ \pm\sqrt{3a_1a_2a_3F_1(t)\Diamond^\chi F_2(t)}\end{gathered}\right]}{\sqrt{a_1F_2(t)\Diamond^\chi(a_1a_2F_2(t)\Diamond^\chi F_3(t)}\Diamond^\chi\left(exp^{\Diamond^\chi}\left(\Xi_1(x_1,y_1,z_1,t)\right)-1\right)}\tag{18}$$

$$V_1(x_1,y_1,z_1,t)=\frac{\left[\begin{gathered}\pm i\sqrt{3b_3F_4(t)\Diamond^\chi(b_1b_2F_5(t)\Diamond^\chi F_6(t)-2)}\\ \Diamond^\chi(\exp^{\Diamond^\chi}\left(\Xi_2(x_1,y_1,z_1,t)-1\right)\pm\sqrt{3b_1b_2b_3F_4(t)\Diamond^\chi F_5(t)}\end{gathered}\right]}{\sqrt{b_1F_5(t)\Diamond^\chi\left((b_1b_2F_5(t)\Diamond^\chi F_6(t)-2\right)}(exp^{\Diamond^\chi}\left(\Xi_2(x_1,y_1,z_1,t)-1\right)}\tag{19}$$

$$W_1(x_1,y_1,z_1,t)=\frac{\left[\begin{gathered}\pm i\sqrt{3c_1F_7(t)\Diamond^\chi\left(c_1c_2F_8(t)\Diamond^\chi F_9(t)-2\right)})\\ \Diamond^\chi\left(exp^{\Diamond^\chi}\Xi_3(x_1,y_1,z_1,t)\right)-1\pm\sqrt{3c_1^2c_2F_7(t)\Diamond^\chi F_8(t)}\end{gathered}\right]}{\sqrt{c_1F_8(t)\Diamond^\chi\left(c_1c_2F_8(t)\Diamond^\chi F_9(t)-2\right)}\left(\exp^{\Diamond^\chi}\Xi_3(x_1,y_1,z_1,t)\right)-1)}\tag{20}$$

where

$$\Xi_1(x_1,y_1,z_1,t)=a_1x_1+a_2y_1+a_3z_1+2a_3\int_0^t \frac{F_1(\tau)}{a_1a_2F_3(\tau)-2}\mathrm{d}\tau$$

$$\Xi_2(x_1,y_1,z_1,t)=b_1x_1+b_2y_1+b_3z_1+2b_3\int_0^t \frac{F_4(\tau)}{b_1b_2F_6(\tau)-2}\mathrm{d}\tau$$

$$\Xi_3(x_1,y_1,z_1,t)=c_1x_1+c_2y_1+c_3z_1+2c_1\int_0^t \frac{F_7(\tau)}{c_1c_2F_9(\tau)-2}\mathrm{d}\tau$$

Non-Gaussian stochastic travelling wave solutions of the hyperbolic type

$$U_2(x_1,y_1,z_1,t)=\pm\sqrt{\frac{3a_2a_3F_1(t)\lozenge^{\chi}F_3(t)}{2F_2(t)\lozenge^{\chi}\left(2-a_1a_2F_3(t)\right)}}\cdot\lozenge^{\chi}\coth^{\lozenge^{\chi}}\left(\Xi_4(x_1,y_1,z_1,t)\pm\coth^{\lozenge^{\chi}}\left(\Xi_4(x_1,y_1,z_1,t)\right)\right)\pm$$
$$\pm i\left[\frac{\sqrt{3ba_2F_1(t)}}{\sqrt{2F_2(t)\lozenge^{\chi}F_3(t)}\lozenge^{\chi}(\coth^{\lozenge^{\chi}}\left(\Xi_4(x_1,y_1,z_1,t)\pm\operatorname{csch}^{\lozenge^{\chi}}\left(\Xi_4(x_1,y_1,z_1,t)\right)\right)}\right] \tag{21}$$

$$U_3(x_1,y_1,z_1,t)=\pm\sqrt{\frac{3a_2a_3F_1(t)\lozenge^{\chi}F_3(t)}{F_2(t)\lozenge^{\chi}\left(2-a_1a_2F_3(t)\right)}}\lozenge^{\chi}\cdot(\tanh^{\lozenge^{\chi}}\left(\Xi_4(x_1,y_1,z_1,t)\pm i\tanh^{\lozenge^{\chi}}\left(\Xi_4(x_1,y_1,z_1,t)\right)\right)\pm$$
$$\pm i\left[\frac{\sqrt{3ba_2F_1(t)}}{\sqrt{2F_2(t)\lozenge^{\chi}F_3(t)}\lozenge^{\chi}(\tanh^{\lozenge^{\chi}}\left(\Xi_4(x_1,y_1,z_1,t)\pm i\tanh^{\lozenge^{\chi}}\left(\Xi_4(x_1,y_1,z_1,t)\right)\right)}\right] \tag{22}$$

$$V_2(x_1,y_1,z_1,t)=\pm\sqrt{\frac{3b_2b_3F_4(t)\lozenge^{\chi}F_6(t)}{F_5(t)\lozenge^{\chi}\left(2-b_1b_2F_6(t)\right)}}\cdot$$
$$\cdot\lozenge^{\chi}\left(\coth^{\lozenge^{\chi}}\left(\Xi_5(x_1,y_1,z_1,t)\right)\pm\operatorname{csch}^{\lozenge^{\chi}}\left(\Xi_5(x_1,y_1,z_1,t)\right)\right)\pm$$
$$\pm i\left[\frac{\sqrt{3b_1b_3F_4(t)}}{\sqrt{2F_5(t)\lozenge^{\chi}F_6(t)}\lozenge^{\chi}\left(\coth^{\lozenge^{\chi}}\left(\Xi_5(x_1,y_1,z_1,t)\right)\pm\operatorname{csch}^{\lozenge^{\chi}}\left(\Xi_5(x_1,y_1,z_1,t)\right)\right)}\right] \tag{23}$$

$$V_3(x_1,y_1,z_1,t)=\pm\sqrt{\frac{3b_2b_3F_4(t)\lozenge^{\chi}F_6(t)}{F_5(t)\lozenge^{\chi}F_6(t)}}\lozenge^{\chi}\cdot\left(\tanh^{\lozenge^{\chi}}\left(\Xi_5(x_1,y_1,z_1,t)\right)\pm i\operatorname{sech}^{\lozenge^{\chi}}\left(\Xi_5(x_1,y_1,z_1,t)\right)\right)\pm$$
$$\pm i\left[\frac{\sqrt{3b_1b_2F_4(t)}}{\sqrt{2F_5(t)\lozenge^{\chi}F_6(t)}\lozenge^{\chi}\left(\tanh^{\lozenge^{\chi}}\left(\Xi_5(x_1,y_1,z_1,t)\right)\pm i\operatorname{sech}^{\lozenge^{\chi}}\left(\Xi_5(x_1,y_1,z_1,t)\right)\right)}\right] \tag{24}$$

$$W_2(x_1,y_1,z_1,t)=\pm\sqrt{\frac{3c_1c_2F_7(t)\Diamond^\chi F_9(t)}{F_8(t)\Diamond^\chi\left(2-c_1c_2F_9(t)\right)}}\cdot$$

$$\cdot\Diamond^\chi(\coth^{\Diamond^\chi}\left(\Xi_6(x_1,y_1,z_1,t)\pm\operatorname{csch}^{\Diamond^\chi}\left(\Xi_6(x_1,y_1,z_1,t)\right)\right)\pm$$

$$\pm i\left[\frac{\sqrt{3c_2c_3F_7(t)}}{\sqrt{2F_8(t)\Diamond^\chi F_9(t)}}\Diamond^\chi\left(\tanh^{\Diamond^\chi}\left(\Xi_6(x_1,y_1,z_1,t)\right)\pm i\operatorname{csch}^{\Diamond^\chi}\left(\Xi_6(x_1,y_1,z_1,t)\right)\right)\right] \tag{25}$$

$$W_3(x_1,y_1,z_1,t)=$$

$$=\pm\sqrt{\frac{3c_1c_2F_7(t)\Diamond^\chi F_9(t)}{F_8(t)\Diamond^\chi\left(2-c_1c_2F_9(t)\right)}\Diamond^\chi(\tanh^{\Diamond^\chi}\left(\Xi_6(x_1,y_1,z_1,t)\pm\operatorname{sech}^{\Diamond^\chi}\left(\Xi_6(x_1,y_1,z_1,t)\right)\right)}\pm$$

$$\pm i\frac{\sqrt{3c_1c_2F_7(t)}}{\sqrt{2F_8(t)\Diamond^\chi F_9(t)}\Diamond^\chi\left(\tanh^{\Diamond^\chi}\left(\Xi_6(x_1,y,z,t)\right)\pm i\operatorname{sech}^{\Diamond^\chi}\left(\Xi_6(x_1,y,z,t)\right)\right)} \tag{26}$$

where

$$\Xi_4(x_1,y_1,z_1,t)=a_1x_1+a_2y_1+a_3z_1-4a_3\int_0^t\frac{F_1(\tau)}{4+a_1a_2F_3(\tau)}\mathrm{d}\tau$$

$$\Xi_5(x_1,y_1,z_1,t)=b_1x_1+b_2y_1+b_3z_1-4b_3\int_0^t\frac{F_4(\tau)}{4+b_1b_2F_6(\tau)}\mathrm{d}\tau$$

$$\Xi_6(x_1,y_1,z_1,t)=c_1x_1+c_2y_1+c_3z_1-4c_1\int_0^t\frac{F_7(\tau)}{4+c_1c_2F_9(\tau)}\mathrm{d}\tau$$

Non-Gaussian stochastic travelling wave solutions of the trigonometric type

$$U_4(x_1,y_1,z_1,t)=U_5(x_1,y_1,z_1,t)=$$

$$=\pm\sqrt{\frac{3a_2a_3F_1(t)}{1-a_1a_2F_2(t)\Diamond^\chi F_3(t)}}\pm\sqrt{\frac{6a_2a_3F_1(t)\Diamond^\chi F_3(t)}{1+2a_1a_2F_2(t)\Diamond^\chi F_3(t)}}\cdot$$

$$\cdot\Diamond^\chi\left(\sec^{\Diamond^\chi}\left(\Xi_7(x_1,y_1,z_1,t)\right)\Diamond^\chi\csc^{\Diamond^\chi}\left(\Xi_7(x_1,y_1,z_1,t)\right)\right) \tag{27}$$

$$U_6(x_1,y_1,z_1,t)=\pm3\sqrt{\frac{3a_2a_3F_1(t)}{9a_1a_2F_2(t)\Diamond^\chi F_3(t)-2}}\pm$$

$$\pm4\sqrt{\frac{3a_2a_3F_1(t)\Diamond^\chi F_3(t)}{2+15a_1a_2F_2(t)\Diamond^\chi F_3(t)}}\Diamond^\chi(\cot^{\Diamond^\chi}\left(2\Xi_8(x_1,y_1,z_1,\mathrm{t})\right)\pm$$

$$\pm i\frac{\sqrt{3a_3F_1(t)}}{2\sqrt{a_1F_2(t)\Diamond^\chi F_3(t)}\Diamond^\chi((\cot^{\Diamond^\chi}\left(2\Xi_8(x_1,y_1,z_1,\mathrm{t})\right)} \tag{28}$$

$$U_7\left(x_1,y_1,z_1,t\right)=\pm 3\sqrt{\frac{3a_2a_3F_1\left(t\right)}{9a_1a_2F_2\left(t\right)\Diamond^{\chi}F_3\left(t\right)-2}}\pm$$

$$\pm 4\sqrt{\frac{3a_2a_3F_1\left(t\right)\Diamond^{\chi}F_3\left(t\right)}{2+15a_1a_2F_2\left(t\right)\Diamond^{\chi}F_3\left(t\right)}}\Diamond^{\chi}(\tan^{\Diamond^{\chi}}\left(2\Xi_8\left(x_1,y_1,z_1,\ \mathrm{t}\right)\right)$$

$$\left[\pm i\frac{\sqrt{3a_3F_1\left(t\right)}}{2\sqrt{a_1F_2\left(t\right)\Diamond^{\chi}F_3\left(t\right)\Diamond^{\chi}(\tan^{\Diamond^{\chi}}\left(2\Xi_8\left(x_1,y_1,z_1,\ \mathrm{t}\right)\right)}}\right] \tag{29}$$

$$V_4\left(x_1,y_1,z_1,t\right)=V_5\ \left(x_1,y_1,z_1,t\right)=$$

$$=\pm\sqrt{\frac{3b_2b_3F_4\left(t\right)}{1-b_1b_2F_5\left(t\right)\Diamond^{\chi}F_6\left(t\right)}}\pm\sqrt{\frac{6b_2b_3F_4\left(t\right)\Diamond^{\chi}F_6\left(t\right)}{1+2b_1b_2F_5\left(t\right)\Diamond^{\chi}F_6\left(t\right)}}\cdot$$

$$\cdot\Diamond^{\chi}\left(\sec^{\Diamond^{\chi}}\left(\Xi_9\left(x_1,y_1,z_1,t\right)\right)\pm\sec^{\Diamond^{\chi}}\left(\Xi_9\left(x_1,y_1,z_1,t\right)\right)\right) \tag{30}$$

$$V_6\left(x_1,y_1,z_1,t\right)=\pm 3\sqrt{\frac{3b_2b_3F_4\left(t\right)}{9b_1b_2F_5\left(t\right)\Diamond^{\chi}F_6\left(t\right)-2}}\pm$$

$$\pm 4\sqrt{\frac{3b_2b_3F_4\left(t\right)\Diamond^{\chi}F_6\left(t\right)}{2+15b_1b_2F_5\left(t\right)\Diamond^{\chi}F_6\left(t\right)}}\Diamond^{\chi}(\cot^{\Diamond^{\chi}}\left(2\ \Xi_{10}\left(x_1,y_1,z_1,t\right)\right)\pm$$

$$\pm i\frac{\sqrt{3b_3F_4\left(t\right)}}{2\sqrt{b_1F_5\left(t\right)\Diamond^{\chi}F_6\left(t\right)}\Diamond^{\chi}(\cot^{\Diamond^{\chi}}\left(2\Xi_{10}\left(x_1,y_1,z_1,t\right)\right)} \tag{31}$$

$$V_7\left(x_1,y_1,z_1,t\right)=\pm 3\sqrt{\frac{3b_2b_3F_4\left(t\right)}{9b_1b_2F_5\left(t\right)\Diamond^{\chi}F_6\left(t\right)-2}}\pm$$

$$\pm 4\sqrt{\frac{3b_2b_3F_4\left(t\right)\Diamond^{\chi}F_6\left(t\right)}{2+15b_1b_2F_5\left(t\right)\Diamond^{\chi}F_6\left(t\right)}}\Diamond^{\chi}(tan^{\Diamond^{\chi}}\left(2\ \Xi_{10}\left(x_1,y_1,z_1,t\right)\right)\pm$$

$$\pm i\frac{\sqrt{3b_3F_4\left(t\right)}}{2\sqrt{b_1F_5\left(t\right)\Diamond^{\chi}F_6\left(t\right)}\Diamond^{\chi}(tan^{\Diamond^{\chi}}\left(2\Xi_{10}\left(x_1,y_1,z_1,t\right)\right)} \tag{32}$$

$$W_4\left(x_1,y_1,z_1,t\right)=W_5\left(x_1,y_1,z_1,t\right)=\pm$$

$$\pm\sqrt{\frac{3c_1c_2F_7\left(t\right)}{1-c_1c_2F_8\left(t\right)\Diamond^{\chi}F_9\left(t\right)}}\pm\sqrt{\frac{6c_1c_2F_7\left(t\right)\Diamond^{\chi}F_9\left(t\right)}{1+2c_1c_2F_8\left(t\right)\Diamond^{\chi}F_9\left(t\right)}}\cdot$$

$$\cdot\Diamond^{\chi}\left(\sec^{\Diamond^{\chi}}\left(\Xi_{11}\left(x_1,y_1,z_1,t\right)\right)\Diamond^{\chi}(\mathrm{cec}^{\Diamond^{\chi}}(\Xi_{11}\left(x_1,y_1,z_1,t\right)\right) \tag{33}$$

$$W_6(x_1,y_1,z_1,t)=\pm 3\sqrt{\frac{3C_1C_2F_7(t)}{9c_1c_2F_8(t)\Diamond^{\chi}F_9(t)-2}}\pm$$

$$\pm 4\sqrt{\frac{3c_1c_2F_7(t)\Diamond^{\chi}F_9(t)}{2+15c_1c_2F_8(t)\Diamond^{\chi}F_9(t)}}\Diamond^{\chi}(\cot^{\Diamond^{\chi}}\left(2\Xi_{12}(x_1,y_1,z_1,t)\right)\pm$$

$$\pm i\frac{\sqrt{3c_1F_7(t)}}{2\sqrt{c_1F_8(t)\Diamond^{\chi}F_9(t)}\Diamond^{\chi}(\cot^{\Diamond^{\chi}}\left(2\Xi_{12}(x_1,y_1,z_1,t)\right)} \tag{34}$$

$$W_7(x_1,y_1,z_1,t)=\pm 3\sqrt{\frac{3c_1c_2F_7(t)}{9c_1c_2F_8(t)\Diamond^{\chi}F_9(t)-2}}\pm$$

$$\pm 4\sqrt{\frac{3c_1c_2F_7(t)^{\chi}F_9(t)}{2+15c_1c_2F_8(t)\Diamond^{\chi}F_9(t)}}\Diamond^{\chi}(tan^{\Diamond^{\chi}}\left(2\Xi_{12}(x_1,y_1,z_1,t)\right)\pm$$

$$\pm i\frac{\sqrt{3C_1F_7(t)}}{2\sqrt{c_1F_8(t)\Diamond^{\chi}F_9(t)}\Diamond^{\chi}(tan^{\Diamond^{\chi}}\left(2\Xi_{12}(x_1,y_1,z_1,t)\right)} \tag{35}$$

where

$$\Xi_7(x_1,y_1,z_1,t)=a_1x_1+a_2y_1+a_3z_1-a_3\int_0^t\frac{F_1(\tau)}{1+2a_1a_2F_3(\tau)}\mathrm{d}\tau$$

$$\Xi_8(x_1,y_1,z_1,t)=a_1x_1+a_2y_1+a_3z_1+2a_3\int_0^t\frac{F_1(\tau)}{9a_1a_2F_3(\tau)-2}\mathrm{d}\tau$$

$$\Xi_9(x_1,y_1,z_1,t)=b_1x_1+b_2y_1+b_3z_1+b_3\int_0^t\frac{F_4(\tau)}{1+2b_1b_2F_6(\tau)}\mathrm{d}\tau$$

$$\Xi_{10}(x_1,y_1,z_1,t)=b_1x_1+b_2y_1+b_3z_1+2b_3\int_0^t\frac{F_4(\tau)}{9b_1b_2F_6(\tau)-2}\mathrm{d}\tau$$

$$\Xi_{11}(x_1,y_1,z_1,t)=c_1x_1+c_2y_1+c_3z_1-c_1\int_0^t\frac{F_7(\tau)}{1+2c_1c_2F_9(\tau)}\mathrm{d}\tau$$

$$\Xi_{12}(x_1,y_1,z_1,t)=c_1x_1+c_2y_1+c_3z_1+2c_1\int_0^t\frac{F_7(\tau)}{9c_1c_2F_9(\tau)-2}\mathrm{d}\tau$$

where a_i, b_i, and c_i ($i = 1, 2, 3$) are arbitrary constants. For the other two forms of the χ-Wick-type (3+1)-Dimensional modified BBM equations in the system (1), we can use the same breathing technique. Therefore, we simply list the non-Gaussian stochastic travelling wave solution for each specific form.

Example

Assume that $F_2(t) = \sigma_1 F_1(t)$, $F_3(t) = \sigma_2 F_1(t)$, and $F_1(t) = g(t) + \sigma_3 W^\chi(t)$, where σ_1, σ_2, σ_3 and are arbitrary constants, σ_1, σ_2, $\sigma_3 \neq 0$ and $g(t)$ is a measurable bounded function on R_+, $W^\chi(t) = \dot{B}^\chi(t)$ is the non-Gaussian white noise and $B^\chi(t)$ is χ-Brownian motion. We have the χ-Hermite transform:

$$\tilde{W}^\chi(t,s) = \sum_{n=1}^{\infty} S_n \int_0^t \chi_n(\tau)\mathrm{d}\tau \quad \text{and} \quad \exp^{\Diamond\chi} B^\chi(t) = \left[B^\chi(t) - \frac{t^2}{2}\right]$$

[16, 23]. Using the definition of $\tilde{W}^\chi(t,s)$ in the aforementioned section, we obtain the non-Gaussian white noise functional solution of eq. (1):

$$\Pi_{w1}(x_1,y_1,z_1,t) = \pm i \frac{\sqrt{3a_3(a_1a_2\sigma_1\sigma_2\left(g(t)+\sigma_3W^\chi(t)\right)^2-2)}\left(\exp\left(\psi_1(x_1,y_1,z_1,t)\right)-1\right)}{\sqrt{a_1\sigma_1(a_1a_2\sigma_1\sigma_2\left(g(t)+\sigma_3W^\chi(t)\right)^2-2)}\left(\exp\left(\psi_1(x_1,y_1,z_1,t)\right)-1\right)} \pm$$
$$\pm \frac{\sqrt{3a_1a_2a_3\sigma_1}\left(g(t)+\sigma_3W^\chi(t)\right)}{\sqrt{a_1\sigma_1(a_1a_2\sigma_1\sigma_2\left(g(t)+\sigma_3W^\chi(t)\right)^2-2)}\left(\exp\left(\psi_1(x_1,y_1,z_1,t)\right)-1\right)} \tag{36}$$

$$\Pi_{w2}(x_1,y_1,z_1,t) = \pm\sqrt{\frac{a_1a_2\sigma_2(g(t)+\sigma_3W^\chi(t)}{\sigma_1(2-a_1a_2\sigma_2\left(g(t)+\sigma_3W^\chi(t)\right)}}\cdot$$
$$\cdot(\coth\left(\psi_1(x_1,y_1,z_1,t)\right)\ \mathrm{csch}\left(\psi_2(x_1,y_1,z_1,t)\right)\pm$$
$$\pm i\frac{\sqrt{3ba_2}}{\sqrt{2\sigma_1\sigma_2\left(g(t)+\sigma_3W^\chi(t)\right)}\left(\coth\left(\psi_2(x_1,y_1,z_1,t)\right)\pm\mathrm{csch}\left(\psi_2(x_1,y_1,z_1,t)\right)\right)} \tag{37}$$

$$\Pi_{w3}(x_1,y,z,t) = \pm\sqrt{\frac{3a_2a_3\sigma_2\left(g(t)+\sigma_3W^\chi(t)\right)}{\sigma_1(2-a_1a_2\sigma_2\left(g(t)+\sigma_3W^\chi(t)\right)}}\cdot$$
$$\cdot\left(\tanh\left(\psi_2(x_1,y_1,z_1,t)\right)\pm i\ \mathrm{sech}\left(\psi_2(x_1,y_1,z_1,t)\right)\right)\pm$$
$$=\pm i\frac{\sqrt{3ba_2}}{\sqrt{2\sigma_1\sigma_2\left(g(t)+\sigma_3W^\chi(t)\right)}\left(\tanh\left(\psi_2(x_1,y_1,z_1,t)\right)\pm i\ \mathrm{sech}\left(\psi_2(x_1,y_1,z_1,t)\right)\right)} \tag{38}$$

$$\Pi_{w4}(x_1,y_1,z_1,t) = \Pi_{w5}(x_1,y_1,z_1,t) =$$
$$=\pm\sqrt{\frac{3a_2a_3\sigma_2\left(g(t)+\sigma_3W^\chi(t)\right)}{1-(a_1a_2\sigma_1\sigma_2\ \left(g(t)+\sigma_3W^\chi(t)\right)^2}}\pm\left(g(t)+\sigma_3W^\chi(t)\right)\cdot$$
$$\cdot\sqrt{\frac{6a_2a_3\sigma_2}{1+2a_2a_3\sigma_1\sigma_2(g(t)+\sigma_3W^\chi(t))^2}}\ \ (\sec\left(\psi_3(x_1,y_1,z_1,t)\csc\left(\psi_3(x_1,y_1,z_1,t)\right)\right) \tag{39}$$

$$\Pi_{w6}\left(x_1,y_1,z_1,t\right)=\pm 3\sqrt{\frac{3a_2a_3\left(g\left(t\right)+\sigma_3W^{\chi}\left(t\right)\right)}{9a_1a_2\sigma_1\sigma_2(g\left(t\right)+\sigma_3W^{\chi}\left(t\right))^2-2}}\pm$$

$$\pm\left(g\left(t\right)+\sigma_3W^{\chi}\left(t\right)\right)\sqrt{\frac{3a_2a_3\sigma_2}{2+15a_2a_3\sigma_1\sigma_2(g\left(t\right)+\sigma_3W^{\chi}\left(t\right))^2}}\cdot$$

$$\cdot(\tan\left(2\psi_4\left(x_1,y_1,z_1,t\right)\right)\pm i\frac{\sqrt{3a_3}}{2\sqrt{a_{1\sigma_1\sigma_2}}\;(\cot\left(2\psi_4\left(x_1,y_1,z_1,t\right)\right)} \tag{40}$$

$$\Pi_{w7}\left(x_1,y_1,z_1,t\right)=\pm 3\sqrt{\frac{3a_2a_3\left(g\left(t\right)+\sigma_3W^{\chi}\left(t\right)\right)}{9a_1a_2\sigma_1\sigma_2(g\left(t\right)+\sigma_3W^{\chi}\left(t\right))^2-2}}\pm$$

$$\pm 4\left(g\left(t\right)+\sigma_3W^{\chi}\left(t\right)\right)\sqrt{\frac{3a_2a_3\sigma_2}{2+15a_1a_2\sigma_1\sigma_2 g\left(t\right)+\sigma_3W^{\chi}\left(t\right))^2}}\cdot$$

$$\cdot(\tan\left(2\psi_4\left(x_1,y_1,z_1,t\right)\right)\pm i\frac{\sqrt{3a_3}}{2\sqrt{a_{1\sigma_1\sigma_2}}\left(\tan\left(2\psi_4\left(x_1,y_1,z_1,t\right)\right)\right)} \tag{41}$$

where

$$\psi_1\left(x_1,y_1,z_1,t\right)=a_1x_1+a_2y_1+a_3z_1+2a_3\int_0^t\frac{g\left(\tau\right)+\sigma_3W^{\chi}\left(\tau\right)}{a_1a_2\sigma_2\left(g\left(\tau\right)+\sigma_3W^{\chi}\left(\tau\right)\right)-2}\mathrm{d}\tau$$

$$\psi_2\left(x_1,y_1,z_1,t\right)=a_1x_1+a_2y_1+a_3z_1+2a_3\int_0^t\frac{g\left(\tau\right)+\sigma_3W^{\chi}\left(\tau\right)}{4+a_1a_2\sigma_2\left(g\left(\tau\right)+\sigma_3W^{\chi}\left(\tau\right)\right)}\mathrm{d}\tau$$

$$\psi_3(x_1,y_1,z_1,t)=a_1x_1+a_2y_1+a_3z_1+2a_3\int_0^t\frac{g\left(\tau\right)+\sigma_3W^{\chi}\left(\tau\right)}{1+2a_2a_3\sigma_2\left(g\left(\tau\right)+\sigma_3W^{\chi}\left(\tau\right)\right)}\mathrm{d}\tau$$

$$\psi_3(x_1,y_1,z_1,t)=a_1x_1+a_2y_1+a_3z_1+2a_3\int_0^t\frac{g\left(\tau\right)+\sigma_3W^{\chi}\left(\tau\right)}{9a_1a_2\sigma_2\left(g\left(\tau\right)+\sigma_3W^{\chi}\left(\tau\right)\right)-2}\mathrm{d}\tau$$

Conclusion

In this paper, we used a new and general version of the modified tanh-coth method for dealing with multi-dimensional non-linear PDE. Using this generalization of the modified tanh-coth method, transforming χ-Hermite, and white noise theory, we generated a new set of exact travelling non-Gaussian wave solutions for the (3+1)-dimensional modified BBM equations. If the problem is considered in a non-Gaussian stochastic environment, we can obtain non-Gaussian stochastic (3+1)-dimensional modified BBM equations. For this aim, we develop a non-Gaussian Wick calculus based on the theory of hypecomplex systems to get exact travelling wave solutions of (3+1)-dimensional modified BBM equations and non-Gaussian white noise functional solutions of χ-Wick-type stochastic (3+1)-dimensional modified BBM equations. We constructed a new set of exact travelling non-Gussian wave solutions for (3+1)-dimensional modified BBM equations with variable coefficients. This collection included exponentail, hyperbolic and trigonometric-type solutions. The deterministic (3+1)-dimensional non-linear modified equations with constant coefficients were solved by Wazwaz in [35]. Thus, for this model, our findings are more general than his. We obtained stochastic non-Gussian traveling wave solutions for stochastic (3+1)-dimensional modified BBM equations. With the help of inverse χ-Hermite transform, we also showed an example of how the stochastic solutions can be given as a functional solution for non-Gussian white noise. Note that in mathematical physics, the schema proposed in this paper can be used to solve several non-linear equations of evolution [10-15].

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