AN INVESTIGATION OF NEW QUICKER IMPLICIT ITERATIONS IN HYPERBOLIC SPACES

by

Ahmed Mohamed ZIDAN [a,c*], ***Ahmed H. SOLIMAN*** [a,c], ***Tamer NABIL*** [a,d], ***and Mohamed Ahmed BARAKAT*** [b,c]

[a] Department of Mathematics, College of Science, King Khalid University, Abha, Saudi Arabia
[b] Department of Mathematics, College of Al Wajh, University of Tabuk, Saudi Arabia
[c] Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut Branch, Egypt
[d] Department of Basic Science, Faculty of Computers and Informatics, Suez Canal University, Ismailia, Egypt

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The present paper investigates the convergence of some new implicit iterations of coupled fixed point for non-linear contractive like functions on W-hyperbolic metric spaces. Moreover, we provide a theoretical comparison of our new iterations to illustrate the fastest iteration the coupled fixed point.

Key words: *implicit iterations, φ-contractive mappings, convergence rate*

Introduction

The established Banach Contraction Principle is the most significant outcome in spaces identified with metric spaces. It is successfully used to discover the solutions of some non-linear equations, for example, Volterra integral equations, and non-linear integral differential equations in Banach spaces. In addition, it supports the assembly of calculations in computational mathematics. Various expansions of Banach's Principle have been done, for the most part by generalizing the contraction operator, and sometimes by expanding the necessity of completeness or even both.

Guo *et al*. [1] presented some results about coupled fixed point followed by Bhaskar and Lakshmikantham [2]. In 2009, Lakshmikantham and Ćirić [3] defined the mixed g-monotone mappings and introduced coupled common fixed point results for non-linear contractive mappings in a metric space with a relation ordering, which also extended the fixed point results due to Bhaskar and Lakshmikantham [2].

On the other hand, there exists one more approach to respect the idea of convexity. Convexity in a vector space is characterized utilizing lines between points. In Euclidean spaces or more generally in Banach spaces, there is a line of most brief length that joins two points, and the length of this line is the line segment between its two endpoints. However, metric spaces do not normally have this convex structure. Takahashi [4] defined the convexity in distance spaces and gave some theorems of fixed point for non-expansive properties in such spaces. A convex distance space is more generalized than normed space and cone Banach space [4]. Later, diverse raised convex concepts have been presented on metric spaces. Kohlenbach [5] presented the concept of W-hyperbolic spaces which represents a consolidated approach for linear and non-linear concepts at the same time.

* Corresponding author, e-mail: zidan.math90@azhar.edu.eg

Iterative fixed points procedures in convex distance spaces have been obtained by many researchers, see, *e. g*., [6-9], using implicit iterative procedures which are incredible significance from numerical angle as they give precise estimate

The following forms give a metric version of implicit Mann and Ishikawa iteration schems defined by Ciric *et al*. [10, 11] in the background of W-hyperbolic spaces,

Consider G a mapping from a convex subset E of a W-hyperbolic space T into itself. Choose $t_0 \in E$ and give the sequence $\{t_n\}$:

$$t_n = W\left(t_{n-1}, Gz_n, \alpha_n\right), \quad z_n = W\left(t_n, Gt_n, \beta_n\right), \quad n \in \mathbb{N} \tag{1}$$

and

$$u_n = W\left(t_{n-1}, Gt_n, \alpha_n\right), \quad n \in \mathbb{N} \tag{2}$$

Recently, Yildirim and Abbas [12] have introduced the following implicit S-iteration procedure and showed that the convergence rate of this iteration is higher than the iterations (2) and (1).

Initiated with $t_n \in E$, we get the following sequence $\{t_n\}$:

$$t_n = W\left(Gt_{n-1}, Gz_n, \alpha_n\right), \quad z_n = W\left(Gt_n, Gt_n, \beta_n\right), \quad n \in \mathbb{N} \tag{3}$$

where (α_n) and (β_n) are certain real sequences in [0, 1]. After that, many fixed point theorems in some spaces established, see, *e. g*., [13, 14].

The present paper aims to give three new implicit iteration procedures, explore their convergence to coupled fixed point in the W-hyperbolic spaces, as well as to present a theoretical comparison between them to show the faster iteration the coupled fixed point.

We need the following definitions and lemma in our main results.

Definition 1. [5] A W-hyperbolic space (T, d, W) is a distance space (T, d) endowed with a convexity function $W : T^2 \times [0, 1] \rightarrow T$ Which fulfills the next items:

– $d[a, W(b, c, \alpha)] \leq (1 - \alpha)d(a, b) + \alpha d(a, c)$,
– $d[a, W(b, c, \alpha), W(b, c, \beta)] = \|\alpha - \beta\| d(b, c)$,
– $W(b, c, \alpha), W(c, b, 1 - \alpha)$, and
– $d[W(b, e, \alpha), W(c, \omega, \alpha)] \leq (1 - \alpha)d(b, c) + \alpha d(e, \omega)$.

Definition 2. [12] A map $G : T \rightarrow T$ is called a contractive like function if there exists a continuous, strictly increasing function $\varphi : [0, 1) \rightarrow [0, 1)$ with $\varphi_0 = 0$ and a constant $\delta \in [0, 1)$ such that for any $a, b \in T$, we have:

$$d\left(Ga, Gb\right) \leq \delta d\left(a, b\right) + \varphi\left[d\left(a, Ga\right)\right] \tag{4}$$

Definition 3. [1] We say that (a, b) is a coupled fixed point of the function $G : T \times T \rightarrow X$:

$$G(a, b) = a \quad \text{and} \quad G(b, a) = b$$

where $a, b \in X$.

Lemma 1. Let $\{c_n\}_{n=1}^{\infty} \subset [0, \infty)$. If there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have:

$$c_{n+1} \leq \left(1 + \eta_n\right)c_n + \eta_n \theta_n$$

where

$$\eta_n \in (0,1), \quad \sum_{n=1}^{\infty} \eta = \infty \quad \text{and} \quad \theta n \geq 0$$

Then, the next holds:

$$0 \leq \lim_{n\to\infty} sup\ c_n \leq \lim_{n\to\infty} sup\ \theta_n$$

for all $n \in \mathbb{N}$.

Main convergent results

Definition 4. Let Z be a convex subset of T and $G : Z \times Z \to Z$. Choose t_0, $z_0 \in E$ and define the next two sequences $\{t_n\}$, $\{z_n\}$:

$$t_n = W\left[G^n\left(t_{n-1}, z_{n-1}\right), G\left(u_n, v_n\right), \alpha_n\right],\ z_n = W\left[G^n\left(z_{n-1}, t_{n-1}\right), G\left(v_n, u_n\right), \alpha_n\right]$$
$$u_n = W\left[t_n, G\left(t_n, z_n\right), \beta_n\right],\ v_n = W\left[z_n, G\left(z_n, t_n\right), \beta_n\right],\quad n \in \mathbb{N} \tag{5}$$

where $G^n(t, z) = G[G^{n-1}) (t, z), G^{n-1}) (z, t)]$ and (α_n) and (β_n) are certain real sequences in [0, 1].

Definition 5. Consider the assumptions of *Definition 4* and give the two sequences $\{t_n\},\{z_n\}$:

$$t_n = W\left[G\left(t_{n-1}, z_{n-1}\right), G\left(u_n, v_n\right), \alpha_n\right],\quad z_n = W\left[G\left(z_{n-1}, t_{n-1}\right), G\left(v_n, u_n\right), \alpha_n\right]$$
$$u_n = W\left[t_n, G\left(t_n, z_n\right), \beta_n\right],\quad v_n = W\left[z_n, G\left(z_n, t_n\right), \beta_n\right],\quad n \in \mathbb{N} \tag{6}$$

where (α_n) and (β_n) are certain real sequences in [0, 1].

Definition 6. Consider the assumptions of *Definition 4* and give the next two sequences $\{t_n\},\{z_n\}$:

$$t_n = W\left[t_{n-1}, G\left(u_n, v_n\right), \alpha_n\right],\ z_n = W\left[z_{n-1}, G\left(v_n, u_n\right), \alpha_n\right]$$
$$u_n = W\left[t_n, G\left(t_n, z_n\right), \beta_n\right],\ v_n = W\left[z_n, G\left(z_n, t_n\right), \beta_n\right],\quad n \in \mathbb{N} \tag{7}$$

where (α_n) and (β_n) are certain real sequences in [0, 1].

Definition 7. Consider the assumptions of *Definition 4* and give the next two sequences $\{t_n\},\{z_n\}$:

$$t_n = W\left[t_{n-1}, G\left(t_n, z_n\right), \alpha_n\right],\ z_n = W\left[z_{n-1}, G\left(z_n, t_n\right), \alpha_n\right],\ n \in \mathbb{N} \tag{8}$$

where (α_n) and (β_n) are certain real sequences in [0, 1].

Definition 8. Consider all assumptions of *Definition 4* a mapping G on $T\times T$ to T is said to be a contractive like mapping for any t, z $n \in T$, we get:

$$d\left[G(t,z), G(u,v)\right] \le \frac{\delta}{2}\left[d(t,u)\right] + d(z,v)\Big] + \varphi\Big\{d\left[t, G(t,z)\right] + d\left[z, G(z,t)\right]\Big\} \tag{9}$$

Theorem 1. Let $G : E\times E \to E$ be a contractive-like mapping defined in on a non-empty closed convex subset E of a W-hyperbolic metric space (T, d, W) with $F(G) \neq \phi$ (where $F(G)$ is the set of coupled fixed point of G). Then, $\{t_n\}$and $\{z_n\}$ given in eq. (5), we have $\lim_{n\to\infty} t_n = p$ and $\lim_{n\to\infty} z_n = s$.

Proof. Assume that p, $s \in F(G)$. By eqs. (5) and (9), we get:

$$d\left(t_n, p\right) + d\left(z_n, s\right) = d\Big(W\left(G^n\left(t_{n-1}, z_{n-1}\right), G\left(u_n, v_n\right), \alpha_n\right), p\Big) +$$
$$+ d\Big(W\left(G^n\left(z_{n-1}, t_{n-1}\right), G\left(v_n, u_n\right), \alpha_n\right), s\Big) \le$$
$$\le \alpha_n d\Big(G\left(G^{n-1}\left(t_{n-1}, z_{n-1}\right), G^{n-1}\left(z_{n-1}, t_{n-1}\right)\right), G(p,s)\Big) + \left(1-\alpha_n\right) d\Big(G\left(u_n, v_n\right), G(p,s)\Big) +$$
$$+ \alpha_n d\Big(G\left(G^{n-1}\left(z_{n-1}, t_{n-1}\right), G^{n-1}\left(t_{n-1}, z_{n-1}\right)\right), G(s,p)\Big) + \left(1-\alpha_n\right) d\Big(G\left(v_n, u_n\right), G(s,p)\Big) \le$$
$$\le \ldots \le \alpha_n \delta^n\left[d\left(t_{n-1}, p\right) + d\left(z_{n-1}, s\right)\right] + \left(1-\alpha_n\right)\delta\left[d\left(u_n, p\right) + d\left(v_n, s\right)\right] \tag{10}$$

and

$$d(u_n,p)+d(v_n,s)=d\Big(W\Big(G^n(t_n,z_n),G(t_n,z_n),\beta_n\Big),p\Big)+$$
$$+d\Big(W\Big(G^n(z_n,t_n),G(z_n,t_n),\beta_n\Big),s\Big)\le\beta_n d\Big(G^n(z_n,t_n),G(p,s)\Big)+$$
$$+(1-\beta_n)d\Big(G(t_n,z_n),G(p,s)\Big)+\beta_n d\Big(G^n(y_n,x_n),G(s,p)\Big)+$$
$$+(1-\beta_n)d\Big(G(z_n,t_n),G(s,p)\Big)\le\delta\big[d(t_n,p)+d(z_n,s)\big] \qquad (11)$$

By eqs. (10) and (11), we have:

$$d(t_n,p)+d(z_n,s)\le\alpha_n\delta^n\big[d(t_{n-1},p)+d(z_{n-1},s)\big]+(1-\alpha_n)\delta^2\big(\beta_n\delta+(1-\beta_n)\delta\big)\cdot$$
$$\cdot\big[d(t_n,p)+d(z_n,p)\big]<\alpha_n\delta^n\big[d(t_{n-1},p)+d(z_{n-1},s)\big]+(1-\alpha_n)\delta\big[d(t_n,p)+d(z_n,p)\big] \qquad (12)$$

which implies that:

$$d(t_n,p)+d(z_n,s)<\frac{\alpha_n\delta^n}{1-(1-\alpha_n)\delta}\big[d(t_{n-1},p)+d(z_{n-1},s)\big] \qquad (13)$$

we put:

$$A_n=\frac{\alpha_n\delta^n}{1-(1-\alpha_n)\delta}$$

then

$$1-A_n=1-\frac{\alpha_n\delta^n}{1-(1-\alpha_n)\delta}=\frac{1-(1-\alpha_n)\delta-\alpha_n\delta^n}{1-(1-\alpha_n)\delta}\ge1-(1-\alpha_n)\delta-\alpha_n\delta^n$$

implies that:

$$A_n\le(1-\alpha_n)\delta+\alpha_n\delta^n=\delta \qquad (14)$$

From eqs. (13) and (14), we have:

$$d(t_n,p)+d(z_n,s)<\delta\big[d(t_{n-1},p)+d(z_{n-1},s)\big]<(\delta)^2\big[d(t_{n-2},p)+d(z_{n-2},s)\big]<$$
$$\vdots$$
$$<(\delta)^n\big[d(t_0,p)+d(z_0,s)\big] \qquad (15)$$

Using the fact that $0\le\delta<1$, we conclude that $\lim_{n\to\infty}[d(x_n,p)+d(y_n,s)=0$.

Theorem 2. Consider the assumptions of eq. (9) and let the iteration eq. (6), we have $\lim_{n\to\infty}t_n=p$ and $\lim_{n\to\infty}z_n=s$

Proof. Assume that $p, s\in F(G)$. Using eq. (6) and eq. (9), we get:

$$d(t_n,p)+d(z_n,s)=d\Big(W\Big(G(t_{n-1},z_{n-1}),G(u_n,v_n),\alpha_n\Big),p\Big)+$$
$$+d\Big(W\Big(G(z_{n-1},t_{n-1}),G(v_n,u_n),\alpha_n\Big),s\Big)\le$$
$$\le\alpha_n d\Big(G(t_{n-1},z_{n-1}),G(p,s)\Big)+(1-\alpha_n)d\Big(G(u_n,v_n),G(p,s)\Big)+$$
$$+\alpha_n d\Big(G(z_{n-1},t_{n-1}),G(s,p)\Big)+(1-\alpha_n)d\Big(G(v_n,u_n),G(s,p)\Big)\le$$
$$\le\alpha_n\delta\big[d(t_{n-1},p)+d(z_{n-1},s)\big]+\alpha_n\varphi\Big(d\big(p,G(p,s)\big)+d\big(s,G(s,p)\big)\Big)+$$
$$+(1-\alpha_n)\delta\big[d(u_n,p)+d(v_n,s)\big]+(1-\alpha_n)\varphi\Big(d\big(p,G(p,s)\big)+d\big(s,G(s,p)\big)\Big)\le$$
$$\le\alpha_n\delta\big[d(t_{n-1},p)+d(z_{n-1},s)\big]+(1-\alpha_n)\delta\big[d(u_n,p)+d(v_n,s)\big] \qquad (16)$$

and

$$d(u_n,p)+d(v_n,s)=d\Big(W\big(G(t_n,z_n),G(t_n,z_n),\beta_n\big),p\Big)+$$
$$+d\Big(W\big(G(z_n,t_n),G(z_n,t_n),\beta_n\big),s\Big)\le$$
$$\le\delta\big[d(t_n,p)+d(z_n,s)\big] \tag{17}$$

therefore:

$$d(t_n,p)+d(z_n,s)\le\alpha_n\delta\big[d(t_{n-1},p)+d(z_{n-1},s)\big]+$$
$$+(1-\alpha_n)\delta\big[d(t_n,p)+d(z_n,s)\big]<$$
$$<\alpha_n\delta\big[d(t_{n-1},p)+d(z_{n-1},s)\big]+(1-\alpha_n)\delta\big[d(t_n,p)+d(z_n,p)\big] \tag{18}$$

which implies:

$$d(t_n,p)+d(z_n,s)<\frac{\alpha_n\delta}{1-(1-\alpha_n)\delta}\big[d(t_{n-1},p)+d(z_{n-1},s)\big] \tag{19}$$

we put:

$$A_n=\frac{\alpha_n\delta}{1-(1-\alpha_n)\delta}$$

then

$$1-A_n=1-\frac{\alpha_n\delta}{1-(1-\alpha_n)\delta}=\frac{1-\delta}{1-(1-\alpha_n)\delta}\ge1-\delta$$

implies that:

$$A_n\le\delta \tag{20}$$

Adopting the same argument of eq. (15) in the proof of *Theorem 1*, we conclude:

$$\lim_{n\to\infty}\big[d(t_n,p)+d(z_n,s)\big]=0 \tag{21}$$

Theorem 3 Consider the assumptions of eq. (9) and let the iteration eq. (7) with $\sum_n^\infty(1-\alpha_n)$ we have $\lim_{n\to\infty}t_n=p$ and $\lim_{n\to\infty}z_n=s$.

Proof. Assume that $p, s\in F(G)$. Using eq. (7) and eq. (9), we get:

$$d(t_n,p)+d(z_n,s)=d\Big(W\big(G(t_{n-1},z_{n-1}),G(u_n,v_n),\alpha_n\big),p\Big)+$$
$$+d\Big(W\big(G(z_{n-1},t_{n-1}),G(v_n,u_n),\alpha_n\big),s\Big)$$
$$\le\alpha_n\delta\big[d(t_{n-1},p)+d(z_{n-1},s)\big]+(1-\alpha_n)\delta\big[d(u_n,p)+d(v_n,s)\big] \tag{22}$$

and

$$d(u_n,p)+d(v_n,s)\le\delta\big[d(t_n,p)+d(z_n,p)\big] \tag{23}$$

therefore

$$d(t_n,p)+d(z_n,s)\le\alpha_n\delta\big[d(t_{n-1},p)+d(z_{n-1},s)\big]+(1-\alpha_n)\delta\big[d(t_n,p)+d(z_n,s)\big]<$$
$$<\alpha_n\delta\big[d(t_{n-1},p)+d(z_{n-1},s)\big]+(1-\alpha_n)\delta\big[d(t_n,p)+d(z_n,s)\big] \tag{24}$$

which implies:

$$d(t_n,p)+d(z_n,s)<\frac{\alpha_n\delta}{1-(1-\alpha_n)\delta}\big[d(t_{n-1},p)+d(z_{n-1},s)\big] \tag{25}$$

we put:

$$A_n = \frac{\alpha_n \delta}{1-(1-\alpha_n)\delta}$$

then

$$1-A_n = 1-\frac{\alpha_n \delta}{1-(1-\alpha_n)\delta} = \frac{1-\delta}{1-(1-\alpha_n)\delta} \geq 1-\delta$$

implies that

$$A_n \leq \delta \tag{26}$$

From eq. (25) and eq. (26), we have:

$$\begin{aligned} d(t_n, p) + d(z_n, s) &< \delta\left[d(t_{n-1}, p) + d(z_{n-1}, s)\right] < \\ &< \delta^n\left[d(t_0, p) + d(z_0, s)\right] \end{aligned} \tag{27}$$

Using the fact that $0 \leq \delta < 1$, we conclude that $\lim_{n\to\infty} [d(t_n, p) + d(z_n, s) = 0$.

Theorem 4. Consider the assumptions of eq. (9) and let the iteration eq. (8) with $\sum_n^\infty (1-\alpha_n) = \infty$ we have $\lim_{n\to\infty} t_n = p$ and $\lim_{n\to\infty} z_n = s$.

Proof. Assume that $p, s \in F(G)$. Using eq. (8) and eq. (9), we get:

$$\begin{aligned} d(t_n, p) + d(z_n, s) &= d\Big(W\big(G(t_{n-1}, z_{n-1}), G(t_n, z_n), \alpha_n\big), p\Big) + \\ &+ d\Big(W\big(G(z_{n-1}, t_{n-1}), G(z_n, t_n), \alpha_n\big), s\Big) \leq \\ &\leq \alpha_n \delta\left[d(t_{n-1}, p) + d(z_{n-1}, s)\right] + (1-\alpha_n)\delta\left[d(t_n, p) + d(z_n, s)\right] \end{aligned} \tag{28}$$

which implies:

$$d(t_n, p) + d(z_n, s) < \frac{\alpha_n \delta}{1-(1-\alpha_n)\delta}\left[d(t_{n-1}, p) + d(z_{n-1}, s)\right] \tag{29}$$

we put:

$$A_n = \frac{\alpha_n \delta}{1-(1-\alpha_n)\delta}$$

then

$$1-A_n = 1-\frac{\alpha_n \delta}{1-(1-\alpha_n)\delta} = \frac{(1-\delta)}{1-(1-\alpha_n)\delta} \geq (1-\delta)$$

implies that:

$$A_n \leq \delta \tag{30}$$

From eq. (29) and eq. (30), we have:

$$\begin{aligned} d(t_n, p) + d(z_n, s) &< \delta\left[d(t_{n-1}, p) + d(z_{n-1}, s)\right] < \\ &< \delta^n\left[d(t_0, p) + d(z_0, s)\right] \end{aligned} \tag{31}$$

Using the fact that $0 \leq \delta < 1$, we conclude that $\lim_{n\to\infty} [d(t_n, p) + d(z_n, s) = 0$.

Theorem 5. Let $G : E \times E \to E$ be a contractive-like mapping in (9) on a non-empty closed convex subset E of a W-hyperbolic metric space (T, d, W) with $F(G) = \phi$. Then, for the sequences $\{t_n\},\{z_n\}$in eq. (5) with $\sum_n^\infty (1-\alpha_n) = \infty$ tends to the fixed point of T quicker than eqs. (6)-(8) iterations.

Proof. Let $p = G(p, s)$, $s = G(s, p)$ and by eq. (9), we have:

$$d(t_n,p)+d(z_n,s)<\frac{\alpha_n\delta}{1-(1-\alpha_n)\delta}\left[d(t_{n-1},p)+d(z_{n-1},s)\right] <\ldots< I_n \tag{32}$$

where

$$I_n=\left[\frac{\alpha_n\delta}{1-(1-\alpha_n)\delta}\right]^n\left[d(t_0,p)+d(z_0,s)\right] \tag{33}$$

Using implicit iteration eq. (7), we obtain:

$$d(t_n,p)+d(z_n,s)<\frac{\alpha_n}{1-(1-\alpha_n)\delta}\left[d(t_{n-1},p)+d(z_{n-1},s)\right] <\ldots< J_n \tag{34}$$

where

$$J_n=\left[\frac{\alpha_n}{1-(1-\alpha_n)\delta}\right]^n\left[d(t_0,p)+d(z_0,s)\right] \tag{35}$$

Using implicit iteration eq. (8), we obtain:

$$d(t_n,p)+d(z_n,s)<\frac{\alpha_n}{1-(1-\alpha_n)\delta}\left[d(t_{n-1},p)+d(z_{n-1},s)\right] <\ldots< K_n \tag{36}$$

where

$$K_n=\left[\frac{\alpha_n}{1-(1-\alpha_n)\delta}\right]^n\left[d(t_0,p)+d(z_0,s)\right] \tag{37}$$

Using iteration process eq. (5) and eq. (13), we have:

$$d(t_n,p)+d(z_n,s)<\frac{\alpha_n\delta^n}{1-(1-\alpha_n)\delta}\left[d(t_{n-1},p)+d(z_{n-1},s)\right] <\ldots< L_n \tag{38}$$

where

$$L_n=\left[\frac{\alpha_n\delta^n}{1-(1-\alpha_n)\delta}\right]^n\left[d(t_0,p)+d(z_0,s)\right] \tag{39}$$

From eqs. (33), (35), (37), and (39) one can note that:

$$\lim_{n\to\infty}\frac{L_n}{I_n}=0,\quad \lim_{n\to\infty}\frac{L_n}{J_n}=0,\quad \textit{and}\quad \lim_{n\to\infty}\frac{L_n}{K_n}=0$$

Remark. It will be interesting to find more applications to our current paper in other fields see [15-30].

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