ACCELERATED COMPETING RISKS MODEL FROM GOMPERTZ LIFETIME DISTRIBUTIONS WITH TYPE-II CENSORING SCHEME

by

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Original scientific paper
https://doi.org/10.2298/TSCI20S1165A

Time to failure under normal stress conditions may take a long period of time and statistical inferences under this condition is more serious. Then, the experiment is loaded under stress higher than normal one which is defined as accelerated life tests. This problem in this paper is discussed in the form of partially step-stress accelerated life test model when the lifetime of the product has Gompertz lifetime distribution and unites are fails under the two independent risks. The maximum likelihood method under type-II censoring scheme is used to formulate the point and asymptotic confidence interval estimators of model parameters. The two bootstrap methods are also used to formulate the point and approximate interval estimators. The numerical results are adopted in the form of Monte Carlo studying to illustrate, assess and compare all of the theoretical results. Finally, results are discussed in points to clarify results validity.

Key words: *Gompertz distribution, accelerated model, computing risks model, estimation with maximum likelihood and bootstrap method*

Introduction

The modern technology in a lifetime products has presented products with long lifetime and the problem of statistical inferences became more difficultly. Then, the problem of obtaining a sufficient information about the life products need to a higher stress level than the normal level which known with accelerated lufe tests (ALT), the key reference of ALT are presented in Nelson [1]. This problem has widely used with different authors see [2, 3]. The ALT are defined in different three types described as follows. Constant stress ALT, in which experiment is loaded under constant stress until the final point of the experiments [4]. Step stress ALT, in which the experiment is running at different stress levels and changing at prefixed time or number [5, 6]. Progressive stress ALT, in which the stress is kept with continuously increasing at all experiment steps [7, 8].

In cases which, the experiment is running at the normal and stress levels are defined with partially ALT. The type at which the experiment is running firstly at normal stress level and after pre-fixed time or number is called partially step-stress ALT. The commonly problem in life testing experiment or reliability studying, the units failure under different causes of failure. The risk of one causes of failure respected to the other causes is presented as a competing risks problem [9, 10]. Under accelerate life test model, this problem and its properties discussed by [11, 12]. The collected data obtaining from the experiments under partially step-stress ALT mod-

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el when the units are fails do to only one of two independent causes of failure are used in this paper to construct statistical inference for Gompertz lifetime distributions, more detail see [13].

For the type-II competing risks model, suppose n identical and independent units are used in a life testing experiments, under given prior number m. The time at which the first failure $T_{1;n}$ as well as its caused of failure ρ_1 are recorded, say $(t_{1;n}, \rho_1)$ is observed. The second failure and its caused of failure are recorded, say $(t_{2;n}, \rho_2)$ is observed. This operation is continued until number m of failure and its caused of failure are record, say $(t_{m;n}, \rho_\mathrm{m})$ is observed, where $\rho_i \in \{1, 2\}$, $i = 1, 2,..., m$. The set of data $(T_{1;n}, \rho_1) < (T_{2;n}, \rho_2) <...< (T_{m;n}, \rho_\mathrm{m})$ is denoted to type-II competing risk sample.

The likelihood function of observed type-II competing risk sample:

$$\underline{t} = \{(t_{m;n}, \rho_\mathrm{m}), (t_{2;n}, \rho_2),..., (t_{m;n}, \rho_\mathrm{m})\}$$

described

$$L\left(\underline{t} \mid \theta\right) = \frac{n!\left[S_1\left(t_m\right)S_2\left(t_m\right)\right]^{(n-m)}}{\left(n-m\right)!}\prod_{i=1}^{m}\left[h_1\left(t_i\right)\right]^{\delta(\rho_i=1)}\left[h_2\left(t_i\right)\right]^{\delta(\rho_i=2)} S_1\left(t_i\right)S_2\left(t_i\right) \tag{1}$$

where

$$1-F_i\left(.\right)=S_i\left(.\right) \text{ and } \begin{cases}\delta \text{ take the value, 1 at } \rho \\ \delta \text{ take the value, 0 at } \rho \neq\end{cases}, \ k=1,2 \tag{2}$$

and $t_i = t_{i;n}$ for $0 < t_1 < t_2 < ...<t_m < \infty$.

The problem of studying the lifetime model and its properties have considered in statistical field, more detail see [14]. The models of partially step-stress ALT is applied with type-II competing risks model to analyzed lifetime Gompertz samples, then through the paper, we formulated this models under Gompertz lifetime distribution. Also, parameter estimation with maximum likelihood and bootstrap methods are developed. Different tolls are used in numerical computation in the problem of point and interval estimation of model parameters. Under the tolls, mean (ME) and mean squared error (MSE) the point estimate is assessed and mean interval length (ML) and probability coverage (PC) the intervals estimate are assessed.

Suppose that, at time zero n identical units with normal stress level are putted under lifetime testing. The prior integer, m and change stress time τ are considered. The test is running under normal stress level until time, τ, is reached, then the test is loaded to stress level higher than normal one and the test is running until m failure is observed. If m or 0 failure is observed before the time, τ, is reached the test is running under normal or stress level only, respectively and the test is removed at the time, T_m. Let the random integers r and $(m - r)$ is denoted to number of units fails under normal and stress condition, respectively where $0 \le r \le m$. Let us consider only two causes of failure are available and the failure time and it is caused are recorded. Then, the type-II competing risks random sample is described:

$$(T_{1;n}, \rho_1) < (T_{2;n}, \rho_2) < ...< (T_{r;n}, \rho_r) < \tau(T_{r+1;n}, \rho_{r+1})< ... < (T_{m;n}, \rho_m)$$

For the number of units fails under cause j is denoted by m_j, $j = 1, 2$. Then, we propose that the PDF of the test unit at two level of stress and two causes of failure satisfies the following assumption:

– The Gompertz lifetime distribution with common shape parameters λ and different another shape parameter β_j, $j = 1, 2$ for the random variable T_{ij} and $i = 1, 2,..., m$. Then, PDF and CDF, are given respectively:

$$f_j(t) = \beta_j \exp\left\{\lambda t - \frac{\beta_j}{\lambda}\left[\exp(\lambda t) - 1\right]\right\},\ t > 0,\ (\lambda > 0,\ \beta_j > 0) \tag{3}$$

$$F_j(t) = 1 - \exp\left\{\frac{-\beta_j}{\lambda}\left[\exp(\lambda t) - 1\right]\right\} \tag{4}$$

The corresponding SF and FRF of Gompertz life distribution for given some t, are given, respectively:

$$S_j(t) = \exp\left\{\frac{-\beta_j}{\lambda}\left[\exp(\lambda t) - 1\right]\right\},\ t > 0 \tag{5}$$

$$H_j(t) = \beta_j \exp(\lambda t),\ t > 0 \tag{6}$$

– The two times of failure T_i and T_{ij} i = 1, 2,..., m, are satisfied $T_i = \min\{T_{i1}, T_{i2}\}$ and T_{ij} has Gompertz life distribution with parameters λ and β_j, j = 1, 2. Hence, T_i distributed with Gompertz distribution with $(\beta_1 + \beta_2)$ and λ shape parameters, respectively. For the value of $\rho_i = 1$, we have number of units fails with cause one equal m_1 and for $\rho_i = 2$, number of units fails with second case equal m_2. The consideration that independence of the failure times T_{i1} and T_{i2}, i = 1,..., m and the two probability $P(T_{i1} < T_{i2}) = \beta_2 / (\beta_1 + \beta_2)$ and $P(T_{i2} < T_{i1}) = \beta_1 / (\beta_1 + \beta_2)$, the integer m_1 and m_2 has binomial distribution with sample size m and probability of succeed given by $\beta_2 / (\beta_1 + \beta_2)$ and $\beta_1 / (\beta_1 + \beta_2)$, where $m = m_1 + m_2$, respectively.

– The model of partially step-stress ALT has describe the total lifetime:

$$T = \begin{cases} X, & X < \tau \\ \tau + \theta^{-1}(X - \tau), & X > \tau \end{cases} \tag{7}$$

where X present the life under normal condition. Then, the Gompertz lifetime distribution with parameters, λ and β_j, j = 1, 2 and accelarted factor θ have the PDF decribed:

$$f(t) = \begin{cases} \text{at } t \ge \tau, & f_{2j}(t) \\ \text{at } 0 \le t < \tau, & f_{1j}(t) \\ \text{at } t < 0, & 0 \end{cases} \tag{8}$$

with $f_{1i}(t)$, is presented by eq. (3) and $f_{2i}(t)$ decribed:

$$f_{2j}(t) = \theta\beta_j \exp\left(\lambda\left[\theta(t-\tau) + \tau\right] - \frac{\beta_j}{\lambda}\left(\exp\left(\lambda\left[\theta(t-\tau) + \tau\right]\right) - 1\right)\right) \tag{9}$$

Also, CDF, SF, and HRFs are given:

$$F_{2j}(t) = 1 - \exp\left[-\frac{\beta_j}{\lambda}\left(\exp\left\{\lambda\left[\theta(t-\tau) + \tau\right]\right\} - 1\right)\right] \tag{10}$$

$$R_{2j}(t) = \exp\left[-\frac{\beta_j}{\lambda}\left(\exp\left\{\lambda\left[\theta(t-\tau) + \tau\right]\right\} - 1\right)\right] \tag{11}$$

and

$$h_{2j}(t) = \theta\beta_j \exp\left\{\lambda\left[\theta(t-\tau) + \tau\right]\right\} \tag{12}$$

The joint likelihood function of observed type-II competing risk sample

$$\underline{t} = (t_{1;n}, \rho_1) < (t_{2;n}, \rho_2) < ...< (t_{r;n}, \rho_r) < \tau(t_{r+1;n}, \rho_{r+1})< ... < (t_{m;n}, \rho_m)$$

with r and $(m - r)$ are number of units fails under the normal and accelerated conditions, respectively is given:

$$L(\underline{t}) \propto \left[S_{21}(t_m)S_{22}(t_m)\right]^{(n-m)} \prod_{i=1}^{r}\left[h_{11}(t_i)\right]^{\delta(\rho_i=1)}\left[h_{12}(t_i)\right]^{\delta(\rho_i=2)} S_{11}(t_i)S_{12}(t_i)\cdot$$

$$\cdot\prod_{i=r+1}^{m}\left[h_{21}(t_i)\right]^{\delta(\rho_i=1)}\left[h_{22}(t_i)\right]^{\delta(\rho_i=1)} S_{21}(t_i)S_{22}(t_i)$$

where $t_i = t_{i;n}$ and

$$0<(t_1,\rho_1)<(t_2,\rho_2)<...<(t_r,\rho_r)<\tau<(t_{r+1},\rho_{r+1})<...<(t_m,\rho_m)<\infty$$

Maximum likelihood estimation

In this section, we discussed the point and asymptotic confidence intervals with MLE of model parameters see [15, 16] under consideration that the units test under normal stress level until the prior fixed time, τ, is reached, then running under stress conditions until prior integer m is observed. Also, units are fails under the only two independent causes of failure.

The point estimators

Let $\underline{t} = (t_{1;n}, \rho_1) < (t_{2;n}, \rho_2) < ...< (t_{r;n}, \rho_r) < \tau(t_{r+1;n}, \rho_{r+1})< ... < (t_{m;n}, \rho_m)$ be the observed type-II competing risks sample from Gompertz lifetime distribution, then the joint likelihood eq. (13) with distribution given by eqs. (3) and (9) is reduced:

$$L(\lambda,\beta_1,\beta_2,\theta\mid\underline{t}) = \beta_1^{m_1}\beta_2^{m_2}\theta^{(m-r)}\exp\left\{\lambda\sum_{i=1}^{r}t_i - \frac{(\beta_1+\beta_2)}{\lambda}\sum_{i=1}^{r}\exp(\lambda t_i) + \right.$$

$$+\lambda\sum_{i=r+1}^{m}\left[\theta(t_i-\tau)+\tau\right] - \frac{(\beta_1+\beta_2)}{\lambda}\sum_{i=r+1}^{m}\exp\left(\lambda\left[\theta(t_i-\tau)+\tau\right]\right) -$$

$$\left. -\frac{(n-m)(\beta_1+\beta_2)}{\lambda}\exp\left(\lambda\left[\theta(t_m-\tau)+\tau\right]\right) + \frac{n(\beta_1+\beta_2)}{\lambda}\right\}$$

Then the natural logarithm of likelihood eq. (14) is reduced:

$$\ell(\lambda,\beta_1,\beta_2,\theta\mid\underline{t}) = m_1\log\beta_1 + m_2\log\beta_2 + (m-r)\log\theta + \frac{n(\beta_1+\beta_2)}{\lambda} + \lambda\sum_{i=1}^{r}t_i -$$

$$-\frac{\beta_1+\beta_2}{\lambda}\sum_{i=1}^{r}\exp(\lambda t_i) + \lambda\sum_{i=r+1}^{m}\left[\theta(t_i-\tau)+\tau\right] - \frac{\beta_1+\beta_2}{\lambda}\sum_{i=r+1}^{m}\cdot$$

$$\cdot\exp\left\{\lambda\left[\theta(t_i-\tau)+\tau\right]\right\} - \frac{(n-m)(\beta_1+\beta_2)}{\lambda}\exp\left\{\lambda\left[\theta(t_m-\tau)+\tau\right]\right\}$$

The zero value of partial derivatives of eq. (15) are reduced to likelihood equations which is used to present the estimate of the model parameters:

$$\beta_1=\frac{\lambda m_1}{\sum_{i=1}^{r}\exp(\lambda t_i)+\sum_{i=r+1}^{m}\exp\left\{\lambda\left[\theta(t_i-\tau)+\tau\right]\right\}+(n-m)\exp\left\{\lambda\left[\theta(t_m-\tau)+\tau\right]\right\}-n} \tag{16}$$

$$\beta_2=\frac{\lambda m_2}{\sum_{i=1}^{r}\exp(\lambda t_i)+\sum_{i=r+1}^{m}\exp\left\{\lambda\left[\theta(t_i-\tau)+\tau\right]\right\}+(n-m)\exp\left\{\lambda\left[\theta(t_m-\tau)+\tau\right]\right\}-n} \tag{17}$$

$$\begin{gathered}\frac{(\beta_1+\beta_2)}{\lambda^2}\left\{\sum_{i=1}^{r}\exp(\lambda t_i)+\sum_{i=r+1}^{m}\exp\left(\lambda\left[\theta(t_i-\tau)+\tau\right]\right)+(n-m)\exp\left(\lambda\left[\theta(t_m-\tau)+\tau\right]\right)-n\right\}-\\-\frac{(\beta_1+\beta_2)}{\lambda}\left\{\sum_{i=1}^{m}\exp(\lambda t_i)+\sum_{i=r+1}^{m}\exp\left[\theta(t_i-\tau)+\tau\right]\exp\left(\lambda\left[\theta(t_i-\tau)+\tau\right]\right)+(n-m)\cdot\right.\\\left.\cdot\left[\theta(t_m-\tau)+\tau\right]\exp\left(\lambda\left[\theta(t_m-\tau)+\tau\right]\right)\right\}+\sum_{i=1}^{r}t_i+\sum_{i=r+1}^{m}\left[\theta(t_i-\tau)+\tau\right]=0\end{gathered} \tag{18}$$

and

$$\begin{gathered}\frac{(m-r)}{\theta}+\lambda\sum_{i=r+1}^{m}\exp(t_i-\tau)-(\beta_1+\beta_2)\sum_{i=r+1}^{m}\exp(t_i-\tau)\left\{\lambda\left[\theta(t_i-\tau)+\tau\right]\right\}-\\-(n-m)(\beta_1+\beta_2)(t_m-\tau)\exp(\lambda\left[\theta(t_m-\tau)+\tau\right]=0\end{gathered} \tag{19}$$

After replaced β_1 and β_2 from eqs. (16) and (17), the likelihood eqs. (18) to (19) are formulated in the form of two non-linear equations of the parameters λ and θ which need to numerical method such as Netwon Raphson solve. Also, after obtained $\hat{\lambda}$ and $\hat{\theta}$ the MLE estimate $\hat{\beta}_1$ and $\hat{\beta}_2$ are obtained from (16) and (17).

The asymptotic confidence intervals

From the log-likelihood function given by (15) by taken the second partially derivative respected to the model parameters $\lambda, \beta_1, \beta_2, \theta$ and compute the the minus expectation of this derivatives, the Fisher information matrix is constructed. In this sense the Fisher information matrix has different applications in quantum physics and information theory [17-19].

The confidence intervals of parameters is building from Fisher information but, practice, the operation of computation the minus expectation second partially derivative is more difficult to obtain. Hence, we use the approximate form of information matrix. If, $\Omega(\lambda, \beta_1, \beta_2, \theta)$ denoted to the Fisher information matrix, the we use to $\Omega_0(\hat{\lambda}, \hat{\beta}_1, \hat{\beta}_2, \hat{\theta})$ present the approximate matrix:

$$\Omega_0\left(\hat{\lambda},\hat{\beta}_1,\hat{\beta}_2,\hat{\theta}\right)=\left[-\frac{\partial^2\ell(\lambda,\beta_1,\beta_2,\theta\mid \underline{t})}{\partial\phi_i\partial\phi_j}\right]_{(\hat{\lambda},\hat{\beta}_1,\hat{\beta}_2,\hat{\theta})},\quad i,j=1,2,3,4 \tag{20}$$

where $\phi=\{\lambda, \beta_1, \beta_2, \theta\}$. From the propertry of asymptotic normality distribution of $\hat{\lambda}$, $\hat{\beta}_1$, $\hat{\beta}_2$, and $\hat{\theta}$ with mean $(\lambda, \beta_1, \beta_2, \theta)$ and variance covariance matrix $\Omega_0^{-1}(\hat{\lambda}, \hat{\beta}_1, \hat{\beta}_2, \hat{\theta})$:

$$\left(\hat{\lambda},\hat{\beta}_1,\hat{\beta}_2,\hat{\theta}\right)\sim N\left[(\lambda,\beta_1,\beta_2,\theta),\Omega_0^{-1}\left(\hat{\lambda},\hat{\beta}_1,\hat{\beta}_2,\hat{\theta}\right)\right] \tag{21}$$

And the second partially derivative of log-likelihood function (15) can be easily obtain. Then, the approximate 100(1 – 2α)% confidence intervals of model parameters λ, β_1, β_2, and θ and the variances e_1, e_2, e_3, and e_4 computed from diagonal of $\Omega_0^{-1}(\hat{\lambda}, \hat{\beta}_1, \hat{\beta}_2, \hat{\theta})$ is presented:

$$\hat{\lambda} \mp z_\alpha \sqrt{e_1}, \hat{\beta}_1 \mp z_\alpha \sqrt{e_2}, \hat{\beta}_2 \mp z_\alpha \sqrt{e_3}, \hat{\theta} \mp z_\alpha \sqrt{e_4} \tag{22}$$

under standard normal probability with tailed given by α.

Bootstrap confidence intervals

The problem of determination, point and interval estimators, bias and variance of an estimators and also, calibrate hypothesis tests, bootstrap algorithms presented very important method. Practice, the bootstrap algorithms are available in a different types, as given in [20] parametric bootstrap algorithm and in [21] non-parametric bootstrap algorithm. In this section, we adopted the parametric technique in formulation percentile bootstrap confidence interval [22] and bootstrap-t confidence interval [23], the two algorithms are described:

– The estimators $\hat{\lambda}$, $\hat{\beta}_1$, $\hat{\beta}_2$, and $\hat{\theta}$ are calculated under observed type-II competing risk sample $\underline{t} = \{(t_{1;n}, \rho_1), (t_{2;n}, \rho_2)...(t_{r;n}, \rho_r), (t_{r+1;n}, \rho_{r+1}),..., (t_{m;n}, \rho_m)$ from eqs. (16)-(19).
– Under the same values of m and τ generate a sample of size m from Gompertz distribution with $(\hat{\beta}_1 + \hat{\beta}_2,)$ and $\hat{\lambda}$ shape parameters, then the value of r satisfies $t_i < \tau$ is determined. Under consideration the estimate value $\hat{\theta}$ and the transformation given by eq. (7) the sample of size $(m - r)$ from accelerated Gompertz distribution is obtained. The two integers m_1, m_2 are generated from binomial distribution with probability $\beta_2 / (\beta_1 + \beta_2)$ and $\beta_1 / (\beta_1 + \beta_2)$, respectively.
– Under type-II competing risk bootstrap random sample $\underline{t}^* = \{t^*_{1;n}, t^*_{2;n},...,t^*_{r;n}, t^*_{r+1;n},..., t^*_{m;n}\}$ the bootstrap sample estimates $\hat{\lambda}^*$, $\hat{\beta}_1^*$, $\hat{\beta}_2^*$, and $\hat{\theta}^*$.
– Repeated integer S times for Steps 2 and 3.
– The vector of estimate values $\hat{\varphi}^* = (\hat{\lambda}^*, \hat{\beta}_1^*, \hat{\beta}_2^*, \hat{\theta}^*)$ are putting in an ascending order to be $\varphi^{*[i]} = (\hat{\lambda}^{*[i]}, \hat{\beta}_1^{*[i]}, \hat{\beta}_2^{*[i]}, \hat{\theta}^{*[i]})$ $i = 1, 2,..., S$.
– The point bootstrap estimates is calculated:

$$\hat{\varphi}_{\text{Boot}} = \frac{1}{S}\sum_{i=1}^{r} \hat{\varphi}^{*[i]} \tag{23}$$

Percentile bootstrap confidence intervals:

Let $H(X) = P(\hat{\varphi}^* \leq x)$ is a CDF of $\hat{\varphi}^*$, then $\hat{\varphi}^*_{\text{Boot}} = H - 1(x)$ for given x. Then 100(1 – 2α)% bootstrap confidence interval of $\hat{\varphi}^*$ presented:

$$\left[\hat{\varphi}^*_{\text{Boot}}(\alpha), \hat{\varphi}^*_{\text{Boot}}(1-\alpha)\right] \tag{24}$$

Bootstrap-t confidence intervals.

Let us define statistic $\xi_\varphi^{*[1]} < \xi_\varphi^{*[2]} <...< \xi_\varphi^{*[3]}$, as:

$$\xi_\varphi^{*[j]} = \frac{\hat{\varphi}^{*[i]} - \hat{\varphi}}{\sqrt{\text{var}\left(\hat{\varphi}^{*[i]}\right)}}, j = 1, 2, ..., S, \ l = 1, 2 \tag{25}$$

where $\varphi = (\lambda, \beta_1, \beta_2, \theta)$. For given $H(x) = P(\xi_\varphi^* \leq x)$ be CDF of ξ_φ^*

Hence, for the given value x the estimate $\hat{\varphi}^*_{\text{Boot-}t}$:

$$\hat{\varphi}^*_{\text{Boot-}t} = \hat{\varphi}^* + \sqrt{\text{var}\left(\hat{\varphi}^*\right)} H^{-1}(x) \tag{26}$$

The100(1 – 2α)% bootstrap-t approximate confidence intervals of $\hat{\varphi}^*$ is given:

$$\left[\hat{\varphi}^*_{Boot-t}(\alpha),\ \hat{\varphi}^*_{Boot-t}(1-\alpha)\right] \tag{27}$$

Numerical study

The theoretical results are assessed in this section for different sample size n, different effective sample size m, different effective accelerated time, τ, and different parameters vector $\varphi = (\lambda, \beta_1, \beta_2, \theta)$. The results is reported for two sample of parameters values, φ = {(1.5, 0.1, 0.2, 2.0), (0.01, 0.01, 0.015, 2.5)} and the accelerated time τ = {(1.5, 2.0), (30, 70)}, respectively. In all results of simulation study the operation depend on 1000 type-II competing risks data generated from Gompertz distribution under $(\beta_1 + \beta_2)$ and λ shape parameters. The sample of size r generated under normal condition satisfies $t_i < \tau$ and sample of size $(m - r)$ generated under transformation (7) for accelerated condition. Number of failures under the two causes m_1, m_2 are generated from binomial distribution with probability $\beta_2 / (\beta_1 + \beta_2)$ and $\beta_1 / (\beta_1 + \beta_2)$, respectively. In our studying, we measure the effect of the change of sample and effective sample sizes (n, m), effect time τ and the parameter values.

Hence, we calculate the point MLE and two bootstrap estimates. The results in the point estimate are assessed with two tolls ME and MSE and the results are presented in tabs. 1 and 3. In the case of approximate confidence intervals are measure with ML and PC and results are reported in tabs. 2 and 4.

Table 1. The ME and MSE for (λ, β_1, β_2, θ) = (1.5, 0.1, 0.2, 2.0)

		MLE				Bootstrap			
(τ, n, m)		λ	β_1	β_2	θ	λ	β_1	β_2	θ
(1.5, 30, 15)	*ME*	1.3977	0.1554	0.2421	2.4124	1.7667	0.1001	0.2521	2.7451
	MSE	0.5925	0.1069	0.1305	0.7421	0.7337	0.0707	0.1412	0.9542
(1.5, 30, 25)	*ME*	1.4375	0.1400	0.1643	2.4116	1.6782	0.0968	0.2550	2.7541
	MSE	0.4272	0.0813	0.1137	0.6645	0.5091	0.0877	0.1392	0.9427
(1.5,50,25)	MEs	1.4221	0.1321	0.1702	2.4006	1.6612	0.0977	0.2440	2.7333
	MSE	0.4152	0.0801	0.1105	0.6547	0.4985	0.0811	0.1102	0.9108
(1.5, 50, 40)	*ME*	1.4000	0.1302	0.1812	2.3892	1.6547	0.1241	0.2321	2.6541
	MSE	0.4018	0.0788	0.0999	0.6208	0.4712	0.0799	0.1009	0.8412
(1.5, 70, 40)	*ME*	1.107	0.1284	0.1912	2.3800	1.6511	0.1211	0.2309	2.6331
	MSE	0.3912	0.0772	0.0987	0.6160	0.4310	0.0780	0.0998	0.8002
(1.5, 70, 60)	*ME*	1.4503	0.1203	0.1932	2.2554	1.5821	0.1191	0.2207	2.5332
	MSE	0.3001	0.0662	0.0907	0.5987	0.4007	0.0643	0.0878	0.7412
(2.0, 30, 15)	*ME*	1.4512	0.1545	0.2400	2.3985	1.7521	0.12131	0.2421	2.4215
	MSE	0.5842	0.0969	0.1274	0.7332	0.7247	0.0677	0.1372	0.9451
(20, 30, 25)	*ME*	1.6421	0.1530	0.2338	2.3310	1.6104	0.0968	0.2411	2.4120
	MSE	0.4145	0.0713	0.1100	0.6561	0.5001	0.0737	0.1300	0.9365
(2.0, 50, 25)	*ME*	1.4721	0.1501	0.2302	2.3310	1.6421	0.0992	0.2440	2.4000

→

Table 1. Continuation

		MLE				Bootstrap			
(τ, n, m)		λ	β_1	β_2	θ	λ	β_1	β_2	θ
	MSE	0.4100	0.0701	0.1078	0.6426	0.4874	0.0721	0.1101	0.9002
(2.0, 50, 40)	*ME*	1.4708	0.1208	0.2212	2.2999	1.6327	0.1242	0.2321	2.3215
	MSE	0.4001	0.0682	0.0981	0.6100	0.4612	0.0654	0.1182	0.8364
(2.0, 70, 40)	*ME*	1.4737	0.1207	0.2212	2.2951	1.6330	0.1209	0.2229	2.3541
	MSE	0.3845	0.0654	0.0812	0.6077	0.4221	0.0632	0.0891	0.7985
(2.0, 70, 60)	*ME*	1.4821	0.1211	0.2112	2.2441	1.5991	0.1200	0.2187	2.3251
	MSE	0.2999	0.0600	0.0765	0.5845	0.3992	0.0581	0.0799	0.7362

Table 2. ML and PC for (λ, β_1, β_2, θ) = (1.5, 0.1, 0.2, 2.0)

		MLE				Bootstrap-*p*				Bootstrap-*t*			
(τ, n, m)		λ	β_1	β_2	θ	λ	β_1	β_2	θ	λ	β_1	β_2	θ
(1.5 ,30, 15)	*ML*	3.642	0.278	0.481	4.842	3.992	0.299	0.512	5.754	3.245	0.244	0.450	4.252
	PC	(0.88)	(0.88)	(0.89)	(0.90)	(0.87)	(0.88)	(0.90)	(0.90)	(0.91)	(0.88)	(0.91)	(0.92)
(1.5, 30, 25)	*ML*	3.584	0.255	0.445	4.658	3.881	0.290	0.494	5.369	3.131	0.240	0.439	4.211
	PC	(0.89)	(0.90)	(0.89)	(0.91)	(0.92)	(0.90)	(0.92)	(0.90)	(0.92)	(0.91)	(0.92)	(0.91)
(1.5, 50, 25)	*ML*	3.425	0.231	0.425	4.352	3.741	0.271	0.482	5.24	3.120	0.219	0.418	4.215
	PC	(0.90)	(0.90)	(0.90)	(0.91)	(0.92)	(0.90)	(0.9)	(0.91)	(0.92)	(0.92)	(0.92)	(0.93)
(1.5, 50, 40)	*ML*	3.352	0.222	0.419	3.999	3.215	0.258	0.461	4.718	2.999	0.210	0.411	4.090
	PC	(0.91)	(0.91)	(0.91)	(0.92)	(0.90)	(0.91)	(0.92)	(0.94)	(0.93)	(0.92)	(0.92)	(0.92)
(1.5, 70, 40)	*ML*	3.254	0.211	0.399	3.999	3.005	0.248	0.441	4.511	2.942	0.197	0.399	3.999
	PC	(0.92)	(0.92)	(0.92)	(0.93)	(0.92)	(0.92)	(0.91)	(0.92)	(0.95)	(0.93)	(0.95)	(0.96)
(1.5, 70, 60)	*ML*	3.224	0.171	0.387	3.741	2.952	0.232	0.425	4.311	2.824	0.174	0.382	3.842
	PC	(0.93)	(0.92)	(0.91)	(0.93)	(0.94)	(0.92)	(0.93)	(0.95)	(0.94)	(0.94)	(0.95)	(0.93)
(2.0, 30, 15)	*ML*	3.456	0.241	0.452	4.452	3.892	0.282	0.498	5.598	3.214	.2231	.4362	4.231
	PC	(0.90)	(0.89)	(0.89)	(0.92)	(0.90)	(0.88)	(0.90)	(0.98)	(0.91)	(0.88)	(0.91)	(0.92)
(2.0, 30, 25)	*ML*	3.356	0.232	0.430	4.445	3.752	0.271	0.482	5.123	3.110	0.222	0.422	4.001
	PC	(0.91)	(0.90)	(0.89)	(0.91)	(0.92)	(0.90)	(0.91)	(0.90)	(0.92)	(0.91)	(0.92)	(0.91)
(2.0, 50, 25)	*ML*	3.201	0.218	0.414	4.142	3.521	0.260	0.471	5.001	3.009	0.206	0.404	4.000
	PC	(0.91)	(0.91)	(0.91)	(0.92)	(0.92)	(0.91)	(0.92)	(0.90)	(0.92)	(0.93)	(0.92)	(0.92)
(2.0, 50, 40)	*ML*	3.123	0.201	0.400	3.825	3.005	0.240	0.442	4.501	2.809	0.199	0.398	3.990
	PC	(0.92)	(0.91)	(0.92)	(0.93)	(0.90)	(0.91)	(0.92)	(0.97)	(0.93)	(0.93)	(0.92)	(0.96)
(2.0, 70, 40)	*ML*	3.077	0.191	0.390	3.808	2.995	0.231	0.428	4.490	2.798	0.181	0.387	3.901
	PC	(0.92)	(0.93)	(0.97)	(0.93)	(0.92)	(0.92)	(0.92)	(0.97)	(0.95)	(0.93)	(0.95)	(0.96)
(2.0, 70, 60)	*ML*	3.020	0.156	0.377	3.583	2.745	0.218	0.414	4.291	2.651	0.166	0.374	3.600
	PC	(0.93)	(0.93)	(0.91)	(0.93)	(0.94)	(0.92)	(0.93)	(0.95)	(0.94)	(0.92)	(0.95)	(0.94)

Table 3. The ME and MSE for (λ, β_1, β_2, θ) = (0.01, 0.01, 0.015, 2.5)

		MLE				Bootstrap			
(τ, n, m)		λ	β_1	β_2	θ	λ	β_1	β_2	θ
(30, 30, 15)	*ME*	0.0135	0.0129	0.0141	5.0153	0.0165	0.0142	0.0151	5.0521
	MSE	0.0177	0.0187	0.0192	0.9854	0.0212	0.0214	0.0241	0.9982
(30, 30, 25)	*ME*	0.0124	0.0118	0.0135	5.0047	0.0145	0.0130	0.0140	5.0221
	MSE	0.0155	0.0163	0.0171	0.9624	0.0200	0.0201	0.0230	0.9775
(30, 50, 25)	*ME*	0.0121	0.0117	0.0132	5.0043	0.0141	0.0124	0.0136	5.0218
	MSE	0.0150	0.0151	0.0158	0.9611	0.0192	0.0185	0.0219	0.9524
(30, 50, 40)	*ME*	0.0119	0.0115	0.0118	4.9025	0.0129	0.0111	0.0118	5.0100
	MSE	0.0115	0.0121	0.0130	0.9598	0.0140	0.0162	0.0201	0.9408
(30, 70, 40)	*ME*	0.0117	0.0112	0.0114	4.9021	0.0125	0.0108	0.0114	4.9901
	MSE	0.0100	0.0101	0.0111	0.9250	0.0118	0.0140	0.0185	0.9210
(30, 70, 60)	*ME*	0.0110	0.0108	0.0102	4.9003	0.0104	0.0101	0.0100	4.9621
	MSE	0.0092	0.0098	0.0100	0.7541	0.0101	0.0102	0.0141	0.8542
(70, 30, 15)	*ME*	0.0129	0.0122	0.0138	4.9148	0.0142	0.0140	0.0145	4.9991
	MSE	0.0165	0.0177	0.0180	0.9841	0.0201	0.0203	0.0232	0.9971
(70, 30, 25)	*ME*	0.0120	0.0119	0.0137	5.1100	0.0138	0.0132	0.0137	5.0215
	MSE	0.0144	0.0154	0.0162	0.9613	0.0190	0.0192	0.0218	0.9784
(70, 50, 25)	*ME*	0.0120	0.0114	0.0127	5.0036	0.0142	0.0117	0.0128	5.0207
	MSE	0.0141	0.0140	0.0147	0.9602	0.0190	0.0178	0.0211	0.9512
(20, 50, 40)	*ME*	0.0120	0.0111	0.0112	4.9022	0.0124	0.0102	0.0111	5.0110
	MSE	0.0107	0.0109	0.0122	0.9540	0.0129	0.0151	0.0190	0.9390
(70, 70, 40)	*ME*	0.0111	0.0111	0.0119	4.9028	0.0124	0.0102	0.0111	4.9890
	MSE	0.0092	0.0094	0.0100	0.9239	0.0108	0.0129	0.0160	0.9199
(70, 70, 60)	*ME*	0.0111	0.0107	0.0110	4.9014	0.0110	0.0102	0.0100	4.9608
	MSE	0.0081	0.0087	0.0091	0.7500	0.0098	0.0089	0.0136	0.8512

Table 4. The ML and PC for (λ, β_1, β_2, θ) = (0.01, 0.01, 0.015, 2.5)

		MLE				Bootstrap-*p*				Bootstrap-*t*			
(τ, n, m)		λ	β_1	β_2	θ	λ	β_1	β_2	θ	λ	β_1	β_2	θ
(30, 30, 15)	*ML*	0.122	0.108	0.210	5.841	0.184	0.124	0.233	5.875	0.102	0.100	0.201	5.621
	PC	(0.90)	(0.89)	(0.89)	(0.90)	(0.88)	(0.89)	(0.88)	(0.88)	(0.91)	(0.89)	(0.90)	(0.92)
(30, 30, 25)	*ML*	0.103	0.099	0.129	5.548	0.115	0.110	0.150	5.621	0.098	0.097	0.112	5.500
	PC	(0.89)	(0.90)	(0.90)	(0.91)	(0.91)	(0.90)	(0.90)	(0.91)	(0.92)	(0.91)	(0.91)	(0.91)
(30, 50, 25)	*ML*	0.097	0.098	0.113	5.520	0.103	0.101	0.139	5.613	0.097	0.096	0.101	5.490
	PC	(0.91)	(0.90)	(0.90)	(0.90)	(0.92)	(0.90)	(0.9)	(0.91)	(0.92)	(0.92)	(0.94)	(0.92)
(30, 50, 40)	*ML*	0.088	0.096	0.101	5.501	0.098	0.095	0.122	5.600	0.096	0.094	0.097	5.413
	PC	(0.92)	(0.92)	(0.91)	(0.92)	(0.91)	(0.91)	(0.92)	(0.94)	(0.93)	(0.96)	(0.92)	(0.94)
(30, 70, 40)	*ML*	0.081	0.095	0.098	5.492	0.083	0.080	0.113	5.491	0.094	0.092	0.084	5.390
	PC	(0.92)	(0.92)	(0.92)	(0.93)	(0.92)	(0.92)	(0.91)	(0.92)	(0.96)	(0.93)	(0.92)	(0.97)

→

Table 4. Continuation

		MLE				Bootstrap-*p*				Bootstrap-*t*			
(τ, n, m)		λ	β_1	β_2	θ	λ	β_1	β_2	θ	λ	β_1	β_2	θ
(30, 70, 60)	*ML*	0.069	0.094	0.081	5.482	0.075	0.062	0.100	5.460	0.092	0.086	0.071	5.377
	PC	(0.93)	(0.95)	(0.91)	(0.93)	(0.94)	(0.92)	(0.95)	(0.95)	(0.94)	(0.94)	(0.95)	(0.97)
(70, 30, 15)	*ML*	0.118	0.096	0.201	5.833	0.175	0.118	0.225	5.865	0.095	0.099	0.192	5.600
	PC	(0.89)	(0.88)	(0.89)	(0.90)	(0.87)	(0.89)	(0.90)	(0.90)	(0.91)	(0.89)	(0.91)	(0.94)
(70, 30, 25)	*ML*	0.099	0.095	0.122	5.541	0.110	0.102	0.141	5.614	0.091	0.092	0.103	5.480
	PC	(0.89)	(0.91)	(0.89)	(0.91)	(0.92)	(0.90)	(0.92)	(0.91)	(0.92)	(0.91)	(0.92)	(0.91)
(70, 50, 25)	*ML*	0.091	0.0976	0.102	5.508	0.098	0.094	0.122	5.600	0.081	0.094	0.099	5.487
	PC	(0.90)	(0.90)	(0.90)	(0.91)	(0.92)	(0.90)	(0.9)	(0.91)	(0.95)	(0.92)	(0.92)	(0.93)
(20, 50, 40)	*ML*	0.086	0.093	0.097	5.492	0.093	0.091	0.114	5.589	0.091	0.090	0.090	5.400
	PC	(0.92)	(0.91)	(0.91)	(0.92)	(0.91)	(0.91)	(0.92)	(0.93)	(0.93)	(0.92)	(0.92)	(0.94)
(70, 70, 40)	*ML*	0.077	0.091	0.089	5.487	0.079	0.069	0.109	5.482	0.079	0.090	0.079	5.379
	PC	(0.92)	(0.92)	(0.92)	(0.93)	(0.92)	(0.92)	(0.91)	(0.92)	(0.95)	(0.93)	(0.95)	(0.96)
(70, 70, 60)	ML	0.068	0.093	0.080	5.478	0.066	0.061	0.095	5.448	0.085	0.078	0.064	5.342
	PC	(0.90)	(0.92)	(0.91)	(0.93)	(0.94)	(0.92)	(0.93)	(0.95)	(0.95)	(0.94)	(0.95)	(0.95)

Conclusions

The problem of ALT is adopted over the product has Gompertz lifetime distribution with independent two cause. Some important points are reported form the results of simulation study observed from tabs. 1-4. All the results in tabs. 1-4 has shown that, results is more acceptable. Also, we observe about accelerated time, τ, the results are more acceptable for the large value of τ. The results are getting better for increasing values of increasing (m/n). The bootstrap-*t* give more accurate results than the MLE and percentile bootstrap methods. The results are exactable for different choose of the parameters values:

Nomenclature

$F(.)$ – CDF of t_i
$F_j(.)$ – CDF of t_{ij}
$f(.)$ – PDF of t_i
$f_j(.)$ – PDF of t_{ij}
$R_j(.)$ – relability funtion (RF) of t_{ij}
t_i – i^{th} unit failure time
t_{ij} – i^{th} unit failure time under cause j

Greek symbols

ρ_j – The indicator value expresed to cause of failure of the i^{th} unit
τ – prior stress time

Acronyms

ALT – accelerated lufe tests
CDF – cumulative distribution function
FRF – failure rate function
HRF – hazard failure rate function
ME – mean
MLE – maximum likelihood estimate
ML – mean interval length
MSE – mean squared error
PC – probability coverage
PDF – probability density function
SF – survival function

Acknowledgment

This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. (KEP-PhD-61-130-38). The authors, therefore, acknowledge with thanks DSR for technical and financial support.

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Paper submitted: April 15, 2020
Paper revised: May 29, 2020
Paper accepted: June 10, 2020