THE GENERALIZED GEGENBAUER-HUMBERTS WAVELET FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS

by

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In this paper we present a new method of wavelets, based on generalized Gegenbauer-Humberts polynomials, named generalized Gegenbauer-Humberts wavelets. The operational matrix of integration are derived. By using the proposed method converted linear and non-linear fractional differential equation a system of algebraic equations. In addition, discussed some examples to explain the efficiency and accuracy of the presented method.

Key words: *block-pulse functions, operational matrix or integration, the generalized Gegenbauer-Humberts polynomial, fractional calculus, orthogonal polynomials*

Introduction

Many issues in the physics, engineering, and sciences such as fluid-dynamic traffic, electrochemical processes, economics, electromagnetism, viscoelasticity, biosciences, control, diffusion and neurology can be modelling mathematically by fractional differential equations (FDE) [1]. Using the numerical methods usually solves most of the FDE to find the approximate solutions. There are many techniques and approaches to solve the FDE like: fractional difference method [2], differential transform method [3], Fourier transforms [4], Sumudu transform [5], Adomian decomposition method [6], variational iteration method [7], Bernstein operational matrix method [8], fractional differential transform method [9] and homotopy analysis method [10, 11].

One of the most coming techniques that is used in different sciences and engineering is the orthogonal functions [12, 13]. Many sets of functions are frequently used such as the sin-cos functions, block-pulse functions, Legendre, Laguerre and Chebyshev orthonormal. In the field of sciences and engineering, the orthogonal functions have shown many successes to solve the FDE such as wavelets method. Wavelet basis is transformed the underlying problem to a system algebraic equations by evaluating the integrals using operational matrices [14, 15]. Haar wavelet was constructed by Haar in 1909 is the modest of the orthogonal wavelets, Chen *et al*. [16] was the first who derived the operational matrix of Haar wavelet of fractional integration and used to solve the differential equations. The Legendre and Chebyshev wavelets gained more attractive from a lot of researchers too. The generalize of Legendre, Chebyshev and other polynomials is Gegenbauer (ultraspherical) polynomials [17] which

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are orthogonal on the interval [–1, 1]. To obtain the operational matrix for the Gegenbauer wavelet method, Rehman and Saeed [18] did the main role to investigate it. Also, Srivastava *et al*. [19] applied the Gegenbauer wavelet to find the solution of the fractional Bagley-Torvik equation.

Due to the amount of applications in different physical problems and sciences, it brings more attention find the optimal solutions of FDE. Therefore, this paper develops a new algorithm of wavelets by assembling some of wavelets methods in one formula. A generalize Gegenbauer wavelet method is considered to find the solution of fractional differential problems called the generalized Gegenbauer-Humberts wavelet (GHW). Operational matrix are derived and utilized for solving linear and non-linear FDE. The proposed method depends on the generalized Gegenbauer-Humberts polynomials to find the FDE solutions, and the provided results proven that the new scheme is effective to find the solutions of FDE.

Materials and methods

The first definition of fractional derivatives by the Riemann-Liouville, the Grunwald-Letnikov, the Hadamard, the Erde'lyi-Kober, and so on. The solution for a lot of physical problems in the life being easier when using the Caputo formula. Therefore, the Liouville-Caputo got more attention for initial value problems whatever the initial conditions are given, which is the most cases of physical processes.

Definitions of fractional derivative and integral

We review basic definitions of fractional differentiation and fractional integration [20]:

Definition 1. The Riemann-Liouville fractional integration operator of order $\alpha \geq 0$ of a function $u(t)$ is defined:

$$\left(I^{\alpha}u\right)(t)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{0}^{t}(t-\tau)^{\alpha-1}u(\tau)\,\mathrm{d}\tau, & \alpha>0\\ u(t), & \alpha=0\end{cases} \tag{1}$$

Definition 2. The Caputo fractional derivative operator of order $\alpha \geq 0$ of a function $u(t)$ is defined:

$$\left(D_t^{\alpha}u\right)(t)=\begin{cases}\dfrac{\mathrm{d}^n u(t)}{\mathrm{d}t^n}, & \alpha=n\in\mathbb{N}\\ \left[I^{n-\alpha}\left(\dfrac{\mathrm{d}}{\mathrm{d}t}\right)^{n}u\right](t), & n-1<\alpha<n\end{cases} \tag{2}$$

where $n = \lceil\alpha\rceil$ and $t > 0$. This is useful relation between the Riemann-Liouville and Caputo operators, which is given:

$$\left(I^{\alpha}D_t^{\alpha}u\right)(t)=u(t)-\sum_{k=0}^{n-1}u^{(k)}(0^{+})\frac{t^k}{k!}$$

where $n = \lceil\alpha\rceil$ and $t > 0$. The $(D^{\alpha}I^{\alpha})u(t) = u(t)$, $D^{\alpha}\beta = 0$, where β is a constant.

Generalized Gegenbauer-Humberts polynomials and the generalized Gegenbauer-Humberts wavelets

The generalized Gegenbauer-Humberts polynomials $P_m^{\lambda,y,c}(x)$, $m \geq 0$, which are defined by the generation function [21]:

$$\Phi(t) = \left(c - 2xt + yt^2\right)^{-\lambda} = \sum_{m\geq 0} P_m^{\lambda,y,c}(x)t^m \tag{3}$$

where $\lambda = 0$, y, and $c \neq 0$ are real number. As a special cases of eq. (3) we consider $P_m^{\lambda,y,c}(x)$ as: 2nd kind of Chebyshev polynomial $P_m^{1,1,1}(x) = U_m(x)$, Legendre polynomial $P_m^{1/2,1,1}(x) = \psi_m(x)$, Morgan-Voyc polynomial $P_m^{1,1,1}(x/2 + 1) = B_m(x)$, 1st kind of Fermat polynomial $P_m^{1,2,1}(x/2) = \phi_{m+1}(x)$, Dickson polynomial when $a > 0$ as $P_m^{1,2a,2}(x) = D_m(x, a)$ where a is a real parameter and Gegenbauer polynomial if $y = c = 1$.

The class of the generalized Gegenbauer-Humberts polynomial sequences satisfy the following recurrence relation [21]:

$$P_m^{\lambda,y,c}(x) = 2x\frac{\lambda+m-1}{c_m}P_{m-1}^{\lambda,y,c}(x) - y\frac{2\lambda+m-2}{c_m}P_{m-2}^{\lambda,y,c}(x), \ \forall m \geq 2 \tag{4}$$

with initial conditions:

$$P_0^{\lambda,y,c}(x) = \Phi(0) = c^{-\lambda}, \ P_1^{\lambda,y,c}(x) = \Phi'(0) = 2\lambda x c^{-\lambda-1}$$

The generalized Gegenbauer-Humberts polynomial sequence in eq. (4) is an orthogonal polynomial if $yc > 0$ [21]:

$$h_m = \int_s \left[P_m^{\lambda,y,c}(x)\right]^2 \mathrm{d}\mu(x), \ \forall m \geq 1$$

$$= \left(\frac{y}{c}\right)^m \frac{(\lambda+m-1)^{\underline{m}} \ (2\lambda+m-1)^{\underline{m}}}{m! \ (\lambda+m)^{\underline{m}}} h_0$$

where h_m is the normalization factor defined:

$$h_m = \left(\frac{y}{c}\right)^m c^{-\lambda} \sqrt{cy} \frac{\sqrt{\pi}\, 2^{(2-2\lambda)} \Gamma(2\lambda+m)\Gamma(\lambda+1)}{m!(\lambda+m)\left[\Gamma(\lambda)\right]^2 \Gamma\left(\lambda+\frac{1}{2}\right)} \tag{5}$$

where the falling fractional rotation $x^{\underline{n}}$, some times also denoted $(x)_r$ is defined:

$$x^{\underline{r}} = x(x-1)^{\underline{r-1}}, \ (r \geq 1), x^{\underline{0}} = 1$$

The family of discrete wavelets defined:

$$\psi_{k,n}(x) = r_0^{k/2}\psi\left(r_0^k x - ns_0\right)$$

In particular, when $r_0 = 2$, $s_0 = 1$ the $\psi(x)$ forms an orthogonal basis:

$$\psi_{n,m}^{y,c}(x) = \begin{cases} \dfrac{1}{\sqrt{h_m}} 2^{k/2} P_m^{\lambda,y,c}\left(2^k x - 2n + 1\right), & \dfrac{2n-2}{2^k} \leq x \leq \dfrac{2n}{2^k} \\ 0, & o.w. \end{cases} \tag{6}$$

where $k = 1, 2...$ is the level of resolution, $n = 1, 2,...,2^{k-1}$ is the translation parameter, $m = 0, 1,..,$ $M-1$ is the order of the generalized Gegenbauer-Humberts polynomial, $M > 0$, $yc > 0$. Corresponding to each λ, y, and c, we have a different family of wavelets.

Function approximations and the generalized Gegenbauer-Humberts wavelets matrix

We can expand any function $f(x) \in L_2[0,1]$ into truncated generalized GHW series:

$$f(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\ \psi_{n,m}^{y,c}(x) = C^T\ \psi^{y,c}(x)$$

where C and $\Psi^{y,c}(x)$ are $2^{k-1}M \times 1$ matrices given:

$$C = \left[c_{10}, c_{11}, ..., c_{1M-1}, c_{20}, c_{21}, ..., c_{2M-1}, ..., c_{2^{k-1}0}, c_{2^{k-1}1}, ..., c_{2^{k-1}M-1}\right]^T$$

$$\Psi^{y,c}(x) = \left[\psi_{10}^{y,c}(x), \psi_{11}^{y,c}(x), ..., \psi_{1M-1}^{y,c}(x), \psi_{20}^{y,c}(x), \psi_{21}^{y,c}(x), ..., \psi_{2M-1}^{y,c}(x), ...,\right.$$
$$\left. ..., \psi_{2^{k-1}0}^{y,c}(x), \psi_{2^{k-1}1}^{y,c}(x), ..., \psi_{2^{k-1}M-1}^{y,c}(x)\right]^T$$

The collection points of the generalized GHW are taken as $x_i = (2i-1)/(2^k M)$, where $i = 1, 2,..., 2^{k-1}M$. The GHW matrix is given:

$$\Psi_{2^{k-1}M \times 2^{k-1}M}^{y,c} = \left[\Psi^{y,c}\left(\frac{1}{2^k M}\right), \Psi^{y,c}\left(\frac{3}{2^k M}\right), \cdots, \Psi^{y,c}\left(\frac{2^k M - 1}{2^k M}\right)\right] \tag{7}$$

In particular, we fix $k = 2$, $M = 3$, we have $n = 1, 2$ and $m = 0, 1, 2$, for fix value of $y = 3$, $c = 1$, and $\lambda = 12$ the GHW matrix is given:

$$\Psi_{6\times 6}^{3,1} = \begin{bmatrix} 1.074567 & 1.074567 & 1.074567 & 0. & 0. & 0. \\ -2.108965 & 0. & 2.108965 & 0. & 0. & 0. \\ 2.293272 & -.804134 & 2.293272 & 0. & 0. & 0. \\ 0. & 0. & 0. & 1.074567 & 1.074567 & 1.074567 \\ 0. & 0. & 0. & -2.108965 & 0. & 2.108965 \\ 0. & 0. & 0. & 2.293272 & -.804134 & 2.293272 \end{bmatrix}$$

Similarly, we get different Gegenbauer wavelet matrices for different value of y, c, and λ.

The GHW operational matrix of fractional order integration

We write $f(x) \approx C^T\ \Psi^{y,c}(x)$, an arbitrary function $f \in L_2[0,1]$ can be expanded into a block-pulse functions:

$$f(x) \approx \sum_{i=0}^{m-1} f_i b_i(x) = f^T B(x),\ m = 2^{k-1}M$$

where f_i is the coefficients of the block -pulse function. The generalized GHW can be expanded into m-set of block-pulse functions:

$$\psi^{y,c}(x) = \Psi_{m\times m}^{y,c} B(x) \tag{8}$$

The fractional integral of the block-pulse function vector can be written as: $(I^{\alpha}B)(x) = F^{\alpha}_{m\times m}B(x)$ where $F^{\alpha}_{m\times m}$ is the block-pulse matrix of integration given [14]:

$$F^{\alpha}_{m\times m} = \frac{1}{m^{\alpha}\,\Gamma(\alpha+2)}\begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix} \tag{9}$$

$$\xi_i = (i+1)^{\alpha+1} - 2i^{\alpha+1} + (i-1)^{\alpha+1} \text{ with } P^{y,c,\alpha}_{m\times m} = \Psi^{y,c}_{m\times m}\ F^{\alpha}\ (\Psi^{y,c}_{m\times m})^{-1} \tag{10}$$

where $P^{y,c,\alpha}_{m\times m}$ is the GHW operational matrix of integration of fractional order α. In particular, for $k = 2$, $M = 3$, for fix value of $y = 3$, $c = 1$, $\lambda = 5$, and $\alpha = 0.5$ the GHW matrix is given:

$$P^{3,1,0.5}_{6\times 6} = \begin{bmatrix} 0.53680 & 0.15761 & -0.31336 & 0.43691 & -0.7547 & 0.26957 \\ -0.21066 & 0.22434 & 0.16149 & 0.85907 & -0.44957 & 0.24122 \\ 0.40907 & -0.37608 & 0.16046 & 0.75705 & -0.20247 & 0.10034 \\ 0. & 0. & 0. & 0.53680 & 0.15761 & -0.31336 \\ 0. & 0. & 0. & -0.21066 & 0.22434 & 0.16149 \\ 0. & 0. & 0. & 0.40907 & -0.37608 & 0.16046 \end{bmatrix}$$

Convergence of the GHW

Theorem. The series

$$f(x) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi^{y,c}_{n,m}(x)$$

converges to $f(x)$, when $2^{k-1} \to \infty$.

Proof. To prove this *Theorem*, we will use the fact that is every Cauchy sequence is convergent. Since the wavelet basis represent a family of orthonormal functions in the space $L_2(R)$, take the inner product of $f(x)$ and $\psi^{y,c}_{n,m}$, where $c_{nm} = \langle f(x), \psi^{y,c}_{n,m}(x)\rangle$ We assume that $\hat{l} = 2^{k-1}$, $l = 2^{a-1}$, and $\hat{d} = M$, and $d = N$, where k, a the resolutions level, and M, N the order of the generalized Gegenbauer-Humberts polynomials.

Let $B_{i\hat{d}}$ represent a sequence of partial sums of $c_{ij}\psi^{y,c}_{n,m}(x)$, we need to prove that $B_{i\hat{d}}$ is a Cauchy sequence converges to $f(x)$ when $\hat{l}$, $\hat{d} \to \infty$. Firstly, we prove that $B_{i\hat{d}}$ is a Cauchy sequence, suppose that $B_{i\hat{d}}$ be an arbitrary sums of $c_{ij}\psi^{y,c}_{i,j}(x)$ with $\hat{l} > l$, $\hat{d} >$ d:

$$\| B_{\hat{l},\hat{d}} - B_{l,d} \|^2 = \left\| \sum_{i=l+1}^{\hat{l}}\sum_{j=d}^{\hat{d}-1} c_{ij}\psi^{y,c}_{i,j}(x) \right\|^2 = \left\langle \sum_{i=l+1}^{\hat{l}}\sum_{j=d}^{\hat{d}-1} c_{ij}\psi^{y,c}_{i,j}(x), \sum_{s=l+1}^{\hat{l}}\sum_{r=d}^{\hat{d}-1} c_{sr}\psi^{y,c}_{s,r}(x) \right\rangle =$$
$$= \sum_{i=l+1}^{\hat{l}}\sum_{j=d}^{\hat{d}-1}\sum_{s=l+1}^{\hat{l}}\sum_{r=d}^{\hat{d}-1} c_{ij}c_{sr}\left\langle \psi^{y,c}_{i,j}(x), \psi^{y,c}_{s,r}(x)\right\rangle = \sum_{i=l+1}^{\hat{l}}\sum_{j=d}^{\hat{d}-1} | c_{ij} |^2 \tag{11}$$

As $\hat{l}$, $\hat{d} \to \infty$, by the definition of the Bessel's inequality, we have $\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}|c_{ij}|^2$ is convergent. This implies $B_{i\hat{d}}$ is a Cauchy sequence converes to say $y(x) \in L_2[0,1)$ Now, to show that $y(x) = f(x)$:

$$\begin{aligned}\left\langle y(x)-f(x),\psi_{i,j}^{y,c}(x)\right\rangle &= \left\langle y(x),\psi_{i,j}^{y,c}(x)\right\rangle - \left\langle f(x),\psi_{i,j}^{y,c}(x)\right\rangle \\ &= \lim_{\hat{l},\hat{d}\to\infty}\left\langle B_{\hat{l},\hat{d}},\psi_{i,j}^{y,c}(x)\right\rangle - c_{ij} = c_{ij}-c_{ij}=0\end{aligned} \tag{12}$$

This implies $\sum_{i=1}^{\hat{l}}\sum_{j=0}^{\hat{d}-1} c_{ij}\,\psi_{i,j}^{y,c}(x)$ converges to $f(x)$ as $\hat{l}$, $\hat{d} \to \infty$.

Description of the proposed method

In this section, we use the GHW operational matrix to solve non-linear Riccati fractional equation of the form:

$$D^{\alpha}u(t) = N(t)u^2 + Q(t)u + R(t),\;\; t>0,\;\; 0<\alpha\le 1 \tag{13}$$

with the initial condition $u(0) = h$. We suppose that the functions $D^{\alpha}u(t)$, $N(t)$, $Q(t)$, and $R(t)$ are approximated using GHW:

$$D^{\alpha}u(t) = U^T\,\Psi^{y,c}(t) \tag{14}$$

$$u(t) \approx U^T\,P^{y,c,\alpha}\,\Psi^{y,c}(t) + U_0^T\,\Psi^{y,c}(t) = C^T\,\Psi^{y,c}(t) \tag{15}$$

$$N(t) = V^T\,\Psi^{y,c}(t),\;\; Q(t) = W^T\,\Psi^{y,c}(t),\;\; R(t) = X^T\,\Psi^{y,c}(t) \tag{16}$$

Substituting eqs. (14)-(16) in eq. (13):

$$U^T\,\Psi^{y,c}(t) = V^T\,\Psi^{y,c}(t)\left[C^T\,\Psi^{y,c}(t)\right]^2 + W^T\,\Psi^{y,c}(t)C^T\,\Psi^{y,c}(t) + X^T\,\Psi^{y,c}(t) \tag{17}$$

Substituting eq. (9) into eq. (17):

$$C^T\,\Psi_{m\times m}^{y,c}(t) = V^T\left[C^T\,\Psi_{m\times m}^{y,c}(t)\right]^2 + W^T\,C^T\,\Psi_{m\times m}^{y,c}(t) + X^T \tag{18}$$

where C, V, W, and $\Psi_{m\times m}^{u,c}(t)$ are known, eq. (18) represents a system of a non-linear equations with unknown vector U. This system of non-linear equations can be solved approximately.

Algorithm: input: $M \in \mathbb{N}$, $k \in \mathbb{N}\cup\{0\}$, $\mu \in \mathbb{N}/\{1\}$, $0 < \alpha \le 1$, and the functions $N(t)$, $Q(t)$, $R(t)$, and h.

Step 1: Define the basis function $\psi_{n.m}^{y,c}$ by eq. (6) and the vector $\Psi^{u,c}$ using:

$$\Psi^{y,c} = \left[\Psi_{10}^{y,c}(t),\cdots,\Psi_{1M-1}^{y,c}(t)\,|\,\Psi_{20}^{y,c}(t),\cdots,\Psi_{2M-1}^{y,c}(t)\right|\cdots\left|\Psi_{\mu^k\,0}^{y,c}(t),\cdots,\Psi_{\mu^k\,M-1}^{y,c}\right]^T$$

Step 2: Compute the GHW matrix $\psi_{m\times m}^{u,c}$ and by eq. (8).
Step 3: Compute the GHW operational matrix $P^{y,c,\alpha}$ and $P^{y,c,2\alpha}$ using eq. (10).
Step 4: Define the unknown matrix $U = [u_{ij}]_{m\times m}$ where $m = \mu^k M$.
Step 5: Compute the vectors V, W, C, X in eqs. (15) and (16).
Step 6: Solve the non-linear system in eq. (18) for the unknown vector U.
Output: The approximate solution: $u(t) \approx C^T\,\Psi^{u,c}(t)$.

Numerical simulations

In this section, we implement the GHW method to solve several examples of linear and non-linear fractional differential equation.

Example 1. Consider the equation:

$$D^{\alpha} y(t) + y(t) = \frac{\Gamma(3)}{\Gamma(3-\alpha)} t^{2+\alpha} + t^2 ,\ 0 < \alpha < 1 \tag{19}$$

The exact solution is given by $y(t) = t^2$. Now, we implement the GHW method:

$$y(t) = t^2 + t^{2+\alpha} \frac{\Gamma(3)}{\Gamma(\alpha+3)} - I^{\alpha} y(t) \tag{20}$$

Let:

$$y(t) = C^T \Psi^{y,c}(t) \tag{21}$$

then

$$I^{\alpha} y(t) = C^T I^{\alpha} \Psi^{y,c}(t) = C^T P_{m\times m}^{y,c,\alpha}\ \Psi^{y,c}(t) \tag{22}$$

Substituting eqs. (21) and (22) into eq. (20), we get the following system of algebraic equations:

$$C^T \Psi^{y,c}(t) = t^2 + t^{2+\alpha} \frac{\Gamma(3)}{\Gamma(\alpha+3)} - C^T P_{m\times m}^{y,c,\alpha}\ \Psi^{y,c}(t)$$

Solving the aforementioned system of linear equations forthe unknown vector C. When applying the presented method for $\alpha = 0.8$, $\lambda = 9$, $y = 3$, $c = 1$ with $k = 2$, $M = 3$, and $k = 2$, $M = 5$ we obtain the approximate solutions as in the tab. 1. For $\alpha = 0.8$, fig. 1(a) shown the results.

Table 1. Exact and approximate solution for different values of *k*, *λ*, *M* in *Example 1*

t	Exact solution	GHW method $k = 2$, $M = 3$	Absolute error	GHW method $k = 2$, $M = 5$	Absolute error
0	0	$-0.20 \cdot 10^{-4}$	$0.25335 \cdot 10^{-4}$	$0.13745 \cdot 10^{-4}$	$0.85887 \cdot 10^{-5}$
0.1	0.01	$0.9403 \cdot 10^{-2}$	$0.59943 \cdot 10^{-3}$	$0.97656 \cdot 10^{-2}$	$0.23730 \cdot 10^{-3}$
0.2	0.04	$0.38921 \cdot 10^{-1}$	$0.10793 \cdot 10^{-2}$	$0.39597 \cdot 10^{-1}$	$0.40507 \cdot 10^{-3}$
0.3	0.09	$0.88533 \cdot 10^{-1}$	$0.14648 \cdot 10^{-4}$	$0.89465 \cdot 10^{-1}$	$0.53493 \cdot 10^{-3}$
0.4	0.16	0.15824	$0.17561 \cdot 10^{-2}$	0.15934	$0.64695 \cdot 10^{-3}$
0.5	0.25	0.24799	$0.20097 \cdot 10^{-2}$	0.24929	$0.74516 \cdot 10^{-3}$
0.6	0.36	0.35778	$0.22201 \cdot 10^{-2}$	0.35920	$0.83500 \cdot 10^{-3}$
0.7	0.49	0.48758	$0.24133 \cdot 10^{-2}$	0.48909	$0.91970 \cdot 10^{-3}$
0.8	0.64	0.63740	$0.25892 \cdot 10^{-2}$	0.63899	$0.10010 \cdot 10^{-2}$
0.9	0.81	0.80724	$0.27479 \cdot 10^{-2}$	0.80890	$0.10801 \cdot 10^{-2}$

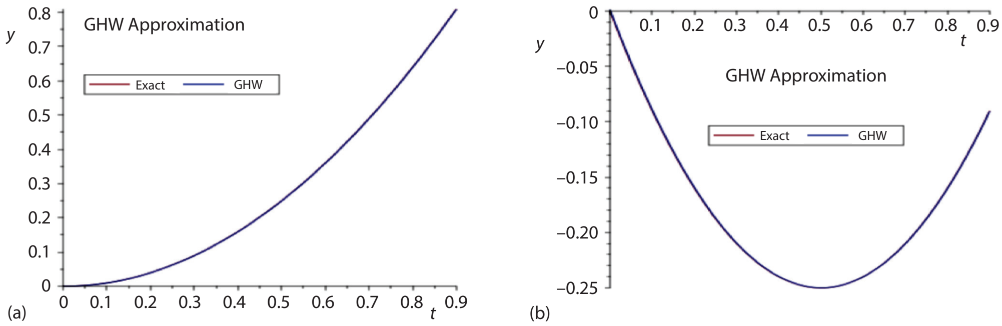

Figure 1. Exact and approximate solution when (a) *Example 1*: $\alpha = 1, \lambda = 9, K = 2$, and $M = 5$, (b) *Example 2*: $\alpha = 1, \lambda = 11, K = 2$, and $M = 5$

Example 2. Our example covers the inhomogeneous linear equation:

$$D^{\alpha} y(t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} - y(t) + t^2 - t,\ 0 < \alpha \le 1,\ t < 0 \tag{23}$$

with initial condition $y(0) = 0$.

We applied the GHW approach to solve eq. (23) for a different values of α. Using section *Description of the proposed method*, we convert eq. (23) to the system:

$$C^T \Psi^{y,c}(t) = t^2 + t^{2+\alpha} \frac{\Gamma(3)}{\Gamma(\alpha+3)} - t^{(1+\alpha)} \frac{\Gamma(2)}{\Gamma(\alpha+2)} - C^T P_{m\times m}^{y,c,\alpha} \Psi(t)$$

Solving the last system, we approach to the exact solution that is $y(t) = t^2 - t$, see fig. 1(b). Table 2 shows the absolute errors of a different values of α when $k = 2$, $M = 5$, $y = 3$, $c = 1$, and $\lambda = 11$.

Table 2. The absolute error of the approximate solution in *Example 2* for a different values of α

t	Exact solution	Absolute error $\alpha = 0.3$	Absolute error $\alpha = 0.7$	Absolute error $\alpha = 0.1$
0	0.	$0.55395 \cdot 10^{-2}$	$0.32650 \cdot 10^{-2}$	$0.12536 \cdot 10^{-2}$
0.1	0.01	$0.10153 \cdot 10^{-2}$	$0.12743 \cdot 10^{-2}$	$0.97553 \cdot 10^{-3}$
0.2	0.04	$0.21675 \cdot 10^{-4}$	$0.53432 \cdot 10^{-3}$	$0.72389 \cdot 10^{-3}$
0.3	0.09	$0.12233 \cdot 10^{-3}$	$0.19669 \cdot 10^{-3}$	$0.49622 \cdot 10^{-3}$
0.4	0.16	$0.36957 \cdot 10^{-3}$	$0.79304 \cdot 10^{-4}$	$0.29023 \cdot 10^{-3}$
0.5	0.25	$0.40140 \cdot 10^{-3}$	$0.24908 \cdot 10^{-3}$	$0.10385 \cdot 10^{-3}$
0.6	0.36	$0.47173 \cdot 10^{-3}$	$0.39335 \cdot 10^{-3}$	$0.64764 \cdot 10^{-4}$
0.7	0.49	$0.52497 \cdot 10^{-3}$	$0.51013 \cdot 10^{-3}$	$0.21733 \cdot 10^{-3}$
0.8	0.64	$0.56633 \cdot 10^{-3}$	$0.60549 \cdot 10^{-3}$	$0.35536 \cdot 10^{-3}$
0.9	0.81	$0.59949 \cdot 10^{-3}$	$0.68272 \cdot 10^{-3}$	$0.48025 \cdot 10^{-3}$

Example 3. Consider the following fractional order Riccati differential equation:

$$D^{\alpha} y(t) = 1 - y^2(t),\ 0 < \alpha \le 1 \tag{24}$$

with initial condition $y(0) = 0$. Exact solution for $\alpha = 1$ was found to be:

$$y(t)=\frac{e^{2t}-1}{e^{2t}+1}$$

The integral representation of eq. (24) and the initial condition are given:

$$y(t)=y(0)+\frac{t^{\alpha}}{\Gamma(\alpha+1)}-I^{\alpha}y^{2}(t) \tag{25}$$

let

$$y(t)=C^{T}\Psi^{y,c}(t) \tag{26}$$

then

$$I^{\alpha}y(t)=C^{T}I^{\alpha}\Psi^{y,c}(t)=C^{T}P_{m\times m}^{y,c,\alpha}\Psi^{y,c}(t) \tag{27}$$

By substituting eqs. (26) and (27) into eq. (25), we get the following system of algebraic equations:

$$C^{T}\Psi^{y,c}(t)=\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\begin{bmatrix} r_1^2 & r_2^2 & \cdots & r_{2^{k-1}M}^2 \end{bmatrix}$$

where

$$\begin{bmatrix} r_1^2 & r_2^2 & \cdots & r_{2^{k-1}M}^2 \end{bmatrix}=C^{T}P_{m\times m}^{y,c,\alpha}\Psi_{2^{k-1}M\times 2^{k-1}M}^{y,c}(t)$$

Solving the non-linear system for an unknown vector C using the Newton iteration method. By applying the presented method for $\alpha = 1$, $\lambda = 7$, $y = 3$, $c = 1$ with $k = 2$, $M = 3$, and $k = 4$, $M = 10$, we obtain the approximate solutions with the absolute error of a different values of α as in the tab. 3. For $\alpha = 1$, fig. 2(a) shown the results.

Table 3. Exact and approximate solution for a different values of *k*, *M*, and *α* in Example 3

t	Exact solution	GHW method $k = 2$ $M = 3$	Absolute error $\alpha = 1$	GHW method $k = 4$ $M = 10$	Absolute error $\alpha = 0.5$	Absolute error $\alpha = 0.7$	Absolute error $\alpha = 1$
0	0	$-0.158918\cdot10^{-2}$	$0.15898\cdot10^{-2}$	$-0.9100\cdot10^{-9}$	$0.376174122\cdot10^{-1}$	$0.71337195\cdot10^{-2}$	$0.91000\cdot10^{-9}$
0.1	$0.9968\cdot10^{-1}$	0.100090	$0.422740\cdot10^{-3}$	$0.999748\cdot10^{-1}$	0.2474934483	0.1188956408	$0.30681\cdot10^{-3}$
0.2	0.197375	0.199739	$0.236404\cdot10^{-2}$	0.199599	0.2710640905	0.1523423472	$0.22245\cdot10^{-2}$
0.3	0.291312	0.297356	$0.604409\cdot10^{-2}$	0.297984	0.2513949822	0.1613220518	$0.66715\cdot10^{-2}$
0.4	0.379948	0.392942	$0.129938\cdot10^{-1}$	0.393678	0.2104880241	0.1540577586	$0.13729\cdot10^{-1}$
0.5	0.462117	0.480994	$0.188776\cdot10^{-1}$	0.484752	0.1604248546	0.1347100156	$0.22635\cdot10^{-1}$
0.6	0.537049	0.568240	$0.311910\cdot10^{-1}$	0.568943	$0.996134238\cdot10^{-1}$	0.1055419192	$0.31893\cdot10^{-1}$
0.7	0.604367	0.643129	$0.387615\cdot10^{-1}$	0.643852	$0.500931258\cdot10^{-1}$	$0.709009684\cdot10^{-1}$	$0.39484\cdot10^{-1}$
0.8	0.664036	0.705660	$0.416241\cdot10^{-1}$	0.707212	$0.101737943\cdot10^{-1}$	$0.138421720\cdot10^{-1}$	$0.43177\cdot10^{-1}$
0.9	0.716297	0.755835	$0.395376\cdot10^{-1}$	0.757152	0.1347159371	$0.443972316\cdot10^{-1}$	$0.40846\cdot10^{-1}$

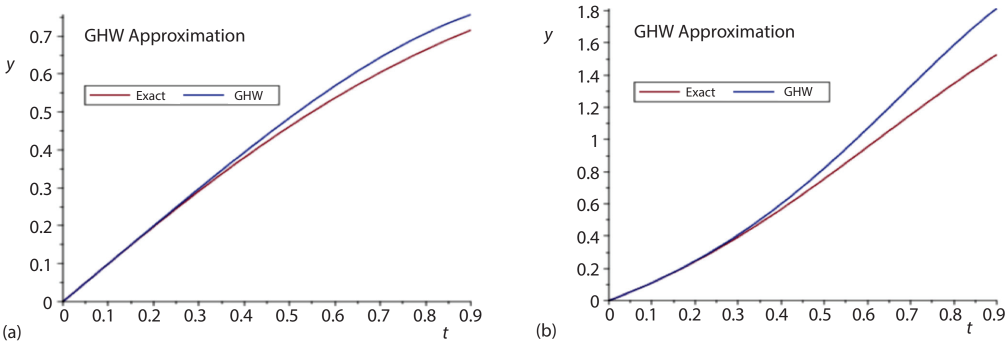

Figure 2. Exact and approximate solution when (a) *Example 3*: $\alpha = 1$, $\lambda = 7$, $K = 4$, and $M = 10$, (b) *Example 4*: $\alpha = 1$, $\lambda = 17$, $y = 2$, $c = 1$, $K = 4$, and $M = 5$

Table 4. Exact and approximate solution for the different values of *k* and *M* in *Example 4*

t	Exact solution	GHW method $k = 2$ $M = 5$	Absolute error	GHW method $k = 4$ $M = 5$	Absolute error
0	0	$0.222006 \cdot 10^{-2}$	$0.22200 \cdot 10^{-2}$	$0.156191 \cdot 10^{-3}$	$0.156192 \cdot 10^{-3}$
0.1	0.110295	0.114168	$0.38735 \cdot 10^{-2}$	0.110886	$0.591400 \cdot 10^{-3}$
0.2	0.241976	0.249936	$0.79599 \cdot 10^{-2}$	0.245629	$0.365301 \cdot 10^{-2}$
0.3	0.395104	0.413538	$0.18433 \cdot 10^{-1}$	0.407985	$0.128802 \cdot 10^{-1}$
0.4	0.567812	0.606961	$0.39149 \cdot 10^{-1}$	0.600243	$0.324313 \cdot 10^{-1}$
0.5	0.756014	0.830446	$0.74431 \cdot 10^{-1}$	0.822016	$0.660017 \cdot 10^{-1}$
0.6	0.953566	1.076700	0.123134	1.068512	0.114946
0.7	1.152948	1.336847	0.183898	1.328861	0.175912
0.8	1.346363	1.591813	0.245450	1.585271	0.238907
0.9	1.526911	1.818491	0.291579	1.814090	0.287179

Table 5. The absolute error of the approximate solution in *Example 4* for the different values of *y* and *c*

t	Absolute error $y = 1$ $c = 1$	Absolute error $y = 2$ $c = 1$	Absolute error $y = 2$ $c = 2$	Absolute error $y = 2$ $c = 3$
0	$0.1561919 \cdot 10^{-3}$	$0.1561919 \cdot 10^{-3}$	$0.1561919 \cdot 10^{-3}$	$0.1561919 \cdot 10^{-3}$
0.1	$0.5914011 \cdot 10^{-3}$	$0.5914011 \cdot 10^{-3}$	$0.5914011 \cdot 10^{-3}$	$0.5914011 \cdot 10^{-3}$
0.2	$0.3653015 \cdot 10^{-2}$	$0.3653015 \cdot 10^{-2}$	$0.3653015 \cdot 10^{-2}$	$0.3653015 \cdot 10^{-2}$
0.3	$0.1288026 \cdot 10^{-1}$	$0.1288026 \cdot 10^{-1}$	$0.1288026 \cdot 10^{-1}$	$0.1288026 \cdot 10^{-1}$
0.4	$0.3243133 \cdot 10^{-1}$	$0.3243133 \cdot 10^{-1}$	$0.3243133 \cdot 10^{-1}$	$0.3243133 \cdot 10^{-1}$
0.5	$0.6600171 \cdot 10^{-1}$	$0.6600171 \cdot 10^{-1}$	$0.6600171 \cdot 10^{-1}$	$0.6600171 \cdot 10^{-1}$
0.6	0.1149466	0.1149466	0.1149466	0.1149466
0.7	0.1759123	0.1759123	0.1759123	0.1759123
0.8	0.2389076	0.2389076	0.2389076	0.2389076
0.9	0.2871790	0.2871790	0.2871790	0.2871790

From the results in these tables, it is clear that the approximate solutions converge to the exact solution when we increase the values of k and M, also when we approach α to 1.

Example 4. Consider another fractional Riccati differential equation:

$$D^{\alpha} y(t) = 1 + 2y(t) - y^2(t), \quad 0 < \alpha \le 1 \tag{28}$$

with initial condition $y(0) = 0$. Exact solution for $\alpha = 1$ was found to be:

$$y(t) = 1 + \sqrt{2}\tanh\left[\sqrt{2t} + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$$

By appling the same procedure of *Example 3*, we get the following system:

$$C^T \Psi^{y,c}(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 2C^T P^{y,c,\alpha}_{m\times m} \Psi^{y,c}(t) - \begin{bmatrix} r_1^2 & r_2^2 & \cdots & r^2_{2^{k-1}M} \end{bmatrix}$$

where

$$\begin{bmatrix} r_1^2 & r_2^2 & \cdots & r^2_{2^{k-1}M} \end{bmatrix} = C^T P^{y,c,\alpha}_{m\times m} \Psi^{y,c}_{2^{k-1}M\times 2^{k-1}M}(t)$$

We can find the unknown vector C, by solving the aforementioned system of a non-linear equations. By applying the presented method for $\alpha = 1$, $\lambda = 17$, $y = 3$, and $c = 1$ with $k = 2$, $M = 5$, and $k = 4$, $M = 5$, we obtain the approximate solutions as in the tab. 4. While in tab. 5 we obtained the absolute error of a different values of y and c and these results obtained with $k = 4$, $M = 5$, and $\lambda = 17$, we can see the change of values of y and c there is no a big different of error. Figure 1 shown the results when $\alpha = 1$, $\lambda = 17$, $y = 2$, $c = 1$, $k = 4$, and $M = 5$.

Conclusion

In the present article, we proposed an algorithm based on generalized GHW to solve linear and non-linear problems with fractional order. The main impulse of this work is to summarize some wavelet methods by a one method. The operational matrix of fractional integration obtained for GHW, then applied it to convert the given problem to a system of algebraic equations can easy to solve it. We compared the outcomes of the proposed method with the existing results for a different values of y, c, and λ. The results showed that how effectively of the new algorithm to tackle the problems of fractional order.

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